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Measurability of a two variable function: a new proof

Medibilidad de una función de dos variables: una nueva demostración

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Abstract. Given a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is continuous in each variable separately, in this paper we prove that f is measurable by using a different approach than the one which is normally used.

***Key words*:** measurable functions, product spaces.

Resumen. Dada una función $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ la cual es continua en cada variable por separado, en este artículo demostramos que f es medible usando una técnica diferente a la que usualmente se utiliza.

***Palabras clave*:** funciones medibles, espacios producto.

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1 Introduction

Measurability (more precisely, Lebesgue-measurability) of a two variable function $f : \mathbb{R}^2 \to \mathbb{R}$ is an interesting problem which has been studied extensively. For example, in their paper [6], J. H. Michael and B. C. Rennie proved that if a function (defined on on a Lebesgue measurable set in the plane) is continuous in one variable and measurable in the other, then it is Lebesgue measurable in the plane. In [1], R. O. Davies proved that if $f : \mathbb{R}^2 \to \mathbb{R}$ is separately approximately continuous then it is Lebesgue measurable. J. Dravecký [2] recalled and generalized a sufficient condition for the measurability of a real-valued function on a product of two measure metric spaces. In [4], Z. Grande generalized some results from Davies [1] by weakening the condition on the sections f_x. A decade ago, G. Kwiecińska [5] obtained similar results in the context of multifunctions (also called set-valued functions) of two variables.

It is a well known result that a real-value function f of two real variables which is continuous in each variable separately (i.e. the "sections" $g_x(y) = f(x, y)$ and $g_y(x) = f(x, y)$ are continuous functions) need not be continuous at $(x, y) \in \mathbb{R}^2$. It is also known that if f is continuous at x for each y and measurable at y for each x, then f is Lebesgue measurable. The question to find a direct proof of the main result of this paper was posed on the remark in [1].

In this paper we give a solution to this posed question.

We begin by recalling the definition of measurable function.

Definition 1. Let $(X, \mathscr{A})$ and $(Y, \mathscr{A}_0)$ be two measurable spaces. A function $f : X \longrightarrow Y$ is called measurable with respect to $\mathscr{A}$ and $\mathscr{A}_0$ if $f^{-1}(E) \in \mathscr{A}$ whenever $E \in \mathscr{A}_0$.

We state now a well-known result about conditions which are equivalent to measurability of a function $f : X \longrightarrow \mathbb{R}^*$.

Theorem 1. *Let $(X, \mathscr{A})$ be a measurable space. Let $f : X \longrightarrow \mathbb{R}^*$ ($\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$). Then the following conditions are equivalent:*

(I) *f is measurable.*

(II) $f^{-1}([a, \infty]) = \{f(x) \geq a\} \in \mathscr{A}$, *for all* $a \in \mathbb{R}$.

(III) $f^{-1}([-\infty, a]) = \{f(x) < a\} \in \mathscr{A}$, *for all* $a \in \mathbb{R}$.

(IV) $f^{-1}([-\infty, a]) = \{f(x) \leq a\} \in \mathscr{A}$, *for all* $a \in \mathbb{R}$.

The above is a standard result, and the interested reader might consult [3] and [7] for a proof.

2 Main Result

Our objective is to give a new proof of the measurability of a function $f(x, y)$ which is continuous in each variable separately. To the best of our knowledge, its proof is performed by induction (see [3]). So, we will provide a new proof of the Theorem below.

Theorem 2. *Let $f(x, y)$ be a function defined on $\mathbb{R}^2$ that is continuous in each variable separately. Then f is Lebesgue measurable.*

For the sake of clarity in exposition, our proof of Theorem 2 relies in three lemmas. From now on, we will be dealing with a function $f : \mathbb{R}^2 \to \mathbb{R}$, and the symbols Ω, Ω_k,

A_n and $B(k,t)$ stand for the following sets

$$\begin{aligned}\Omega :=& f^{-1}((-\infty, a]),\\ \Omega_k :=& f^{-1}\left(\left(-\infty, a+\frac{1}{k}\right)\right),\\ A_n :=& \left\{(x,y) \in \mathbb{R}^2 : \text{there exists } (x,Y) \in \Omega \text{ with } |Y-y| < \frac{1}{n}\right\},\end{aligned}$$

$B(k,t) := \left\{(x,y) \in \mathbb{R}^2 : \text{there exists } (x,Y) \in \Omega_k \text{ with } |Y-y| < \frac{1}{n} - \frac{1}{n+t}\right\}$, where $t > 0$ is a fixed positive real number.

Let us begin with our first lemma.

Lemma 1. $\Omega = \bigcap\limits_{n=1}^{\infty} A_n$.

Proof. Let $(x,y) \in \bigcap\limits_{n=1}^{\infty} A_n$ then $(x,y) \in A_n$ for all $n \in \mathbb{N}$. Hence there exists $(x, Y_n) \in \Omega$ with $|Y_n - y| < \frac{1}{n}$. And so we have $Y_n \to y$ as $n \to \infty$. Since $f(x,y)$ is continuous at y we have

$$f(x, Y_n) \to f(x,y).$$

Note that $f(x,Y_n) \in (-\infty, a]$. Since $(-\infty, a]$ is a closed set, we get $f(x,y) \in (-\infty, a]$; and thus, $(x,y) \in f^{-1}((-\infty,a])$, which means that $(x,y) \in \Omega$. Therefore,

$$\bigcap_{n=1}^{\infty} A_n \subseteq \Omega. \tag{1}$$

For the converse inclusion, we take $Y = y$. Clearly, $\Omega \subseteq A_n$ for each $n \in \mathbb{N}$, so

$$\Omega \subseteq \bigcap_{n=1}^{\infty} A_n. \tag{2}$$

From (1) and (2) we obtain $\Omega = \bigcap\limits_{n=1}^{\infty} A_n$. $\square$

Lemma 2. *$B(k,t)$ is an open set.*

Proof. For $(x,y) \in B(k,t)$, we will show that there exists $\varepsilon > 0$ such that

$$B_\epsilon(x,y) \subseteq B(k,t).$$

In order to prove this, let $(x,y) \in B(k,t)$, then there exists $(x,Y) \in \Omega_k$ and

$$|Y - y| < \frac{1}{n} - \frac{1}{n+t}.$$

Thus, $(x,Y) \in f^{-1}\left((-\infty, a+1/k)\right)$.

Define $g_Y(x) = f(x,Y)$. Let $(x,Y) \in f^{-1}((-\infty, a+1/k))$, then $f(x,Y) \in (-\infty, a+1/k)$, so $g_Y(x) \in (-\infty, a+1/k)$. Hence,

$$x \in g_Y^{-1}\left(\left(-\infty, a+\frac{1}{k}\right)\right),$$

and $g_Y^{-1}((-\infty, a+1/k))$ is an open set because $g_Y(x)$ is continuous. So, given $\epsilon > 0$, there exists an interval $(x-\epsilon, x+\epsilon)$ such that $(x-\epsilon, x+\epsilon) \subseteq g_Y^{-1}((-\infty, a+1/k))$.

Since $g_Y(x) \in (-\infty, a+1/k)$, we may assume that

$$|Y-y| + \epsilon < \frac{1}{n} - \frac{1}{n+t}.$$

Let $(x_0, y_0) \in B_\epsilon(x,y)$, then $x_0 \in (x-\epsilon, x+\epsilon)$; so $x_0 \in g_Y^{-1}((-\infty, a+1/k))$. Then $f(x_0, Y) \in (-\infty, a+1/k)$ and $(x_0, Y) \in f^{-1}((-\infty, a+1/k))$. Thus, $(x_0, Y) \in \Omega$.

Now,

$$\begin{aligned} |Y-y_0| \leq& |Y-y| + |y-y_0| \\ <& \epsilon + |Y-y| \\ <& \frac{1}{n} - \frac{1}{n+t}. \end{aligned}$$

Therefore, $(x_0, y_0) \in B(k,t)$, this means that $B_\epsilon(x,y) \subseteq B(k,t)$, that is, $B(k,t)$ is an open set (therefore, $B(k,t)$ is Lebesgue measurable). $\square$

Lemma 3. $A_n = \bigcup\limits_{t=1}^{\infty} \bigcap\limits_{k=1}^{\infty} B(k,t)$. *In particular, A_n is Lebesgue measurable.*

Proof. Let $(x,y) \in \bigcup\limits_{t=1}^{\infty} B(k,t)$. Then there exists t_0 such that $(x,y) \in \bigcap\limits_{k=1}^{\infty} B(k,t_0)$. Hence $(x,y) \in B(k,t_0)$ for all $k \in \mathbb{N}$. Thus, there exists $(x,Y) \in \Omega_k$ such that

$$|Y-y| < \frac{1}{n} - \frac{1}{n+t} < \frac{1}{n},$$

which implies that

$$\begin{aligned} (x,Y) \in \bigcap_{k=1}^{\infty} f^{-1}\left(\left(-\infty, a+\frac{1}{k}\right)\right) &\Rightarrow (x,Y) \in f^{-1}\left(\bigcap_{k=1}^{\infty}\left(-\infty, a+\frac{1}{k}\right)\right) \\ &\Rightarrow (x,Y) \in f^{-1}((-\infty, a]) \\ &\Rightarrow (x,Y) \in \Omega. \end{aligned}$$

So $(x,Y) \in A_n$ and

$$\bigcup_{t=1}^{\infty} \bigcap_{k=1}^{\infty} B(k,t) \subseteq A_n. \tag{3}$$

On the other hand, let $(x,y) \in A_n$. Then, there exists $(x,Y) \in \Omega$ with $|Y-y| < \dfrac{1}{n}$. Hence, for some $t_0 > 0$, $|Y-y| < \dfrac{1}{n} - \dfrac{1}{n+t_0}$. Moreover, $\Omega \subseteq \Omega_k$ for all $k \in \mathbb{N}$. So,

$(x,y) \in B(k,t_0)$ for all $k \in \mathbb{N}$, and $(x,y) \in \bigcap_{t=1}^{\infty} B(k,t_0)$. Then $(x,y) \in \bigcup_{t=1}^{\infty} \bigcap_{k=1}^{\infty} B(k,t)$, which means that

$$A_n \subseteq \bigcup_{t=1}^{\infty} \bigcap_{k=1}^{\infty} B(k,t). \tag{4}$$

From (3) and (4) we get

$$A_n = \bigcup_{t=1}^{\infty} \bigcap_{k=1}^{\infty} B(k,t).$$

Therefore, A_n is Lebesgue measurable. □

Now that we have proved the three lemmas, we are ready to prove our main result. Its proof is as follows.

Proof of Theorem 2. From Lemma 1, we have

$$\Omega = \bigcap_{n=1}^{\infty} A_n.$$

On the other hand, from Lemma 3, each A_n is measurable, which in turn implies that $\bigcap_{k=1}^{\infty} A_n$ is measurable. Finally,

$$f^{-1}((-\infty,a]) := \Omega = \bigcap_{k=1}^{\infty} A_n,$$

is measurable for all $a \in \mathbb{R}$. From Theorem 1, this implies the measurability of f. □

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