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A mathematical interpretation of the point splitting procedure in quantum field theory

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Abstract. In Quantum Field Theory, expressions which prove to be useful to describe observables of the theory might diverge or have an ambiguous meaning. In this paper, we present and compare known regularization procedures which provide a good interpretation of some of these expressions. On the basis of previous works on the subject ([8], [10], [13], [20]), we relate the so called "point splitting"procedure familiar to physicists to heat kernel and $\zeta-$function regularization used both in mathematics and physics.

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Resumen. En teoría cuántica de campos, algunas expresiones que son útiles para describir observables de la teoría pueden diverger o tener un significado ambiguo. En este artículo, presentamos y comparamos algunos procedimientos conocidos de regularización que ofrecen una buena interpretación de algunas de estas expresiones. Basándonos en previos trabajos sobre el tema ([8], [10], [13], [20]), relacionamos el conocido procedimiento de "separación puntual" familiar para los físicos, con la regularización por el núcleo del operador del calor y la función ζ, ambas usadas en matemáticas y física.

1. Introduction

This article, which is a summary of the author's Master's thesis in Mathematics [11], offers a survey of known regularization methods in mathematics and physics with the aim of clarifying some relations between them. In particular, we review how heat kernel regularization relates to $\zeta-$regularization, each leading to a priori different notions of regularized determinants and how the point splitting procedure can be interpreted as heat kernel regularization in disguise. Although most of the results presented here are known or belong to folklore knowledge (e.g. Theorem 5), we feel that a compact presentation somewhat clarifies various approaches used in both mathematics and physics to make sense of divergent expressions. The starting point for this work is the lecture notes [13], in which regularization procedures for divergent integrals of symbols of pseudo-differential operators (PDO's) are investigated. Here, we study a related problem, namely, how to define the integral on the diagonal of the distribution kernel of a PDO. Whereas such a kernel is smooth outside the diagonal, it can have singularities on the diagonal. We implement regularization methods (see Theorem 1) to extend it to a function (which is not even continuous) on the whole space (see Subsection 2.3).

After some preliminaries on PDO's and symbols, in Section 3 we investigate $\zeta-$regularization and the $\zeta-$function associated with an elliptic operator. The Mellin transform relates this regularization procedure to the heat kernel procedure (see Subsection 2.4); this yields a relation between the $\zeta-$determinant and the renormalized heat kernel determinant of an admissible operator (see Theorem 5). Underlying these two types of regularization procedures are corresponding regularization procedures for traces, the heat kernel regularization procedure and the $\zeta-$function regularization procedure. Regularized traces obtained via the two methods differ by a term involving the Wodzicki residue (see formula (18)). One usually expects regularization procedures to give rise to anomalies; in Section 5 we show how conformal anomalies can arise for regularized determinants. Finally, in Sections 6 and 7, we recall how $\zeta-$determinants come up as partition functions in quantum field theory and how the point splitting procedure in quantum field theory can be seen as a heat kernel procedure in disguise and therefore be related to $\zeta-$function regularization.

2. Pseudo-differential Operators on $\mathbb{R}^n$

The following material is taken from [4]. Given $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ two points in $\mathbb{R}^n$, the Euclidean scalar product and norm will be denoted by $x{\cdot}y = x_1y_1+\cdots+x_ny_n$ and $|x| = (x{\cdot}x)^{1/2}$ resp. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we set the following notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n,\ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ \alpha! = \alpha_1! \cdots \alpha_n!$, and finally, $d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $D_x^\alpha = (-i)^{|\alpha|} d_x^\alpha$, as a suitable notation for multiple partial differentiation. We denote by $\mathcal{S}$ the Schwartz class on $\mathbb{R}^n$.

Remark 1. In the following, we let $dx, dy, d\xi, etc.$, denote the Lebesgue measure on $\mathbb{R}^n$ with an additional normalizing factor of $(2\pi)^{-n/2}$.

If $f \in \mathcal{S}$, the *Fourier transform* of f is defined by $\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx$, for $\xi \in \mathbb{R}^n$. For $s \in \mathbb{R}$ we denote the corresponding Sobolev space by $H^s(\mathbb{R}^n)$.

If $a \in \mathbb{N}$, a *linear partial differential operator of order* a is a polynomial expression $P = P(x,D) = \sum_{|\alpha|\le a} a_\alpha(x) D_x^\alpha$ where the $a_\alpha(x) \in C^\infty(\mathbb{R}^n, \mathbb{C})$. The symbol $\sigma(P) = \sigma$ is defined by: $\sigma(P)(x,\xi) = \sigma(x,\xi) = \sum_{|\alpha|\le a} a_\alpha(x)\xi^\alpha$ which is a polynomial of order a in the dual variable $\xi \in \mathbb{R}^n$. If $f \in \mathcal{S}$, we can use the Fourier inversion formula to express

$$Pf(x) = \int e^{ix\cdot\xi}\sigma(x,\xi)\widehat{f}(\xi)d\xi = \iint e^{i(x-y)\cdot\xi}\sigma(x,\xi)f(y)dyd\xi.$$

Remark 2. Since the second integral does not converge absolutely, we cannot interchange the order of integration in y and ξ.

We use this formalism to define the action of pseudo-differential operators (PDO's) for a wider class of symbols $\sigma(x,\xi)$ than polynomials.

Definition 1. Let $U \subseteq \mathbb{R}^n$ be an open set and let $a \in \mathbb{R}$. We say that $\sigma(x,\xi)$ is a *symbol* of order a, and we write $\sigma \in S^a(U)$, if:

(a) $\sigma(x,\xi)$ is smooth in $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and its support in x is contained in a compact set $K \subset U$.

(b) For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$ there are constants $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{a-|\beta|} \quad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

If $U = \mathbb{R}^n$ we write $S^a(U) = S^a$.

For a symbol $\sigma \in S^a(U)$, the associated operator $P(x,D)$ is defined by:

$$P(x,D)(f)(x) = \int e^{ix\cdot\xi}\sigma(x,\xi)\widehat{f}(\xi)d\xi = \iint e^{i(x-y)\cdot\xi}\sigma(x,\xi)f(y)dyd\xi$$

as a linear operator mapping $\mathcal{S}$ to $\mathcal{S}$, and P is called a *pseudo-differential operator of order* a.

We call a symbol σ *smoothing* if $\sigma \in S^{-\infty} := \bigcap_{j=1}^\infty S^{-j}$. An equivalence relation on the class of symbols is introduced by defining $\sigma \sim \sigma'$, if $\sigma - \sigma' \in S^{-\infty}$.
If $\sigma \in S^{-\infty}$ then for the corresponding operator we have $P : H^s \to H^t$ for all s and t, consequently $P : H^s \to C^\infty$ for all s so that P is infinitely smoothing in this case. Thus, moding out by infinitely smoothing operators, given symbols $\sigma_j \in S^{a_j}$ where $a_j \to -\infty$, we write $\sigma \sim \sum_{j=1}^\infty \sigma_j$ if for every $N \in \mathbb{N}$ there is an integer K_N such that $\sigma - \sum_{j=1}^{K_N} \sigma_j \in S^{-N}$ and in this case we say that $\sum_{j=1}^\infty \sigma_j$ is an asymptotic expansion for σ.

Let $\Psi_a(U)$ denote the space of pseudo-differential operators of order a on $C_0^\infty(U)$. More generally, let $\sigma(x,\xi)$ be a matrix valued symbol, where we suppose that the components of σ all belong to $S^a(U)$. The corresponding operator P is given by a matrix of pseudo-differential operators and is a map acting on the vector valued functions with compact support in U. We shall continue denoting the collection of all such operators by $\Psi_a(U)$. The set $\bigcup_{a\in\mathbb{R}} \Psi_a(U)$ of pseudo-differential operators of all orders generates an algebra.

2.1. **Polyhomogeneous Symbols on $\mathbb{R}^n$.** Let us describe the class of pseudo-differential operators with which we are going to work, following [13].

Definition 2. Let $U \subseteq \mathbb{R}^n$ an open set. For $a \in \mathbb{R}$, a symbol $\sigma \in S^a(U)$ is called a *polyhomogeneous classical symbol of order* a if for any $N \in \mathbb{N}$, there is an integer K_N and there are functions σ_{a_j}, ψ such that

$$\sigma(x,\xi) - \sum_{j=0}^{K_N} \psi(\xi)\sigma_{a_j}(x,\xi) \in S^{-N}(U) \qquad \forall (x,\xi) \in U \times \mathbb{R}^n,$$

where $\lim_{j\to\infty} a_j = -\infty$, $\sigma_{a_j} \in S^{a_j}(U)$ is positively homogeneous in ξ of order a_j, i.e., $\sigma_{a_j}(x,t\xi) = t^{a_j}\sigma_{a_j}(x,\xi)$ $\forall t > 0$, and ψ is a smooth cut-off function on $\mathbb{R}^n$ such that $\psi(\xi) = 0$ for all $|\xi| \leq \frac{1}{4}$, $\psi(\xi) = 1$ for all $|\xi| \geq \frac{1}{2}$. In short we write

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \psi(\xi)\sigma_{a_j}(x,\xi).$$

In practice, we often have $a_j = a - j$. We denote by $CS^a(U)$ the set of polyhomogeneous classical symbols of order a over U and by $Cl^a(U)$ the corresponding set of pseudo-differential operators. A pseudo-differential operator with polyhomogeneous classical symbol is called *classical pseudo-differential operator*.

Remark 3. The class of symbols we have just defined is a subset of a more general class of symbols [9], the class of *log-polyhomogeneous symbols*, which are symbols with an expansion as in Definition 2, such that

$$\sigma_{a_j}(x,\xi) = \sum_{l=0}^{k} \sigma_{a_j,l}(x,\xi)\log^l|\xi|,$$

with $\sigma_{a_j,l}(x,\xi)$ positively homogeneous in ξ of order a_j.

Definition 3. ([5, 23]) Let $\sigma \in CS^a(U)$ be a symbol with asymptotic expansion $\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \psi(\xi)\sigma_{a_j}(x,\xi)$. The *noncommutative residue* of σ is defined by

$$\operatorname{res}(\sigma)(x) := \int_{S^{n-1}} \sigma_{-n}(x,\omega)d\omega \tag{1}$$

where σ_{-n} is the homogeneous component of order $-n$ of σ.

In order to simplify notations, we often omit the explicit mention of the variable $x \in U$ which is then understood. Hence, the coefficients which we define in the following are to be understood as functions on U.

2.2. **Finite Parts of Integrals of Symbols.** For a symbol σ, the integral $\int_{\mathbb{R}^n} \sigma(x,\xi)d\xi$ diverges in general so that it is necessary to extract a finite part. We present two ways to do so, the cut-off procedure and the meromorphic function approach, both of which are widely used by mathematicians and physicists in various disguises (HADAMARD, RIESZ (see [13]), MORETTI [10], WALD [22]).

Proposition 1. *Let $\sigma \in CS^a(U)$. Then for fixed $N \in \mathbb{N}$ sufficiently large, σ can be written $\sigma(\xi) = \sum_{j=0}^{K_N} \psi(\xi)\sigma_{a_j}(\xi) + g_N(\xi)$, with σ_{a_j}, ψ and $g_N \in S^{-N}(U)$ as in Definition 2. We have*

$$\int_{B(0,R)} \sigma(\xi)d\xi \underset{R\to\infty}{\sim} \sum_{j=0}^{\infty} \alpha_j(\sigma)R^{a_j+n} + \beta(\sigma)\log R + \alpha(\sigma)\,,$$

with constant term given by

$$\alpha(\sigma) = \int_{\mathbb{R}^n} g_N + \sum_{j=0}^{K_N} \int_{B(0,1)} \psi(\xi)\sigma_{a_j}(\xi)d\xi - \sum_{\substack{j=0\\ a_j+n\neq 0}}^{K_N} \frac{1}{a_j+n} \int_{S^{n-1}} \sigma_{a_j}(\omega)d\omega$$

and the coefficient of the logarithmic term is given by: $\beta(\sigma) = \mathrm{res}(\sigma)$.

Extracting a finite part from the asymptotic expansion of $\int_{B(0,R)} \sigma(\xi)d\xi$ we set for $\sigma \in CS^a(\mathbb{R}^n)$:

$$\mathrm{fp}^{\text{cut-off}}\int_{\mathbb{R}^n} \sigma(\xi)d\xi := \lim_{R\to\infty}\Big(\int_{B(0,R)} \sigma(\xi)d\xi - \Big(\sum_{\substack{j=0\\ a_j+n\neq 0}}^{\infty} \alpha_j(\sigma)R^{a_j+n} + \beta(\sigma)\log R\Big)\Big)$$

$$= \alpha(\sigma) = \int_{\mathbb{R}^n} g_N + \sum_{j=0}^{K_N} \int_{B(0,1)} \psi(\xi)\sigma_{a_j}(\xi)d\xi - \sum_{\substack{j=0\\ a_j+n\neq 0}}^{K_N} \frac{1}{a_j+n} \int_{S^{n-1}} \sigma_{a_j}(\omega)d\omega.$$

This finite part is independent of the reparametrization of R when $\beta(\sigma) = 0$. In the case $a_j = a - j$, we have that if the order a is not an integer, then $a_j \neq -n$, $\forall j \in \mathbb{N}$, hence $\mathrm{res}(\sigma) = 0$. And if $a < -n$, then $a_j + n < 0$, $\forall j \in \mathbb{N}$ and hence $\mathrm{fp}^{\text{cut-off}} \int_{\mathbb{R}^n} \sigma(\xi)d\xi = \int_{\mathbb{R}^n} \sigma(\xi)d\xi$, since $\forall j \in \mathbb{N},\ \alpha_j(\sigma) = 0 = \beta(\sigma)$.

Proposition 2. *Let $\sigma \in CS^a(U)$. Then for fixed $N \in \mathbb{N}$ sufficiently large, σ can be written $\sigma(\xi) = \sum_{j=0}^{K_N} \psi(\xi)\sigma_{a_j}(\xi) + g_N(\xi)$, with σ_{a_j}, ψ and $g_N \in S^{-N}(U)$ as in Definition 2. The map $z \mapsto \int_{\mathbb{R}^n} |\xi|^{-z}\sigma(\xi)d\xi$ is meromorphic with a possible simple pole at $z = 0$, with residue given by $\mathrm{res}(\sigma)$, and a finite part at $z = 0$ given by $\alpha(\sigma)$.*

Extracting a finite part from the meromorphic function $z \mapsto \int_{\mathbb{R}^n} |\xi|^{-z}\sigma(\xi)d\xi$, we set for $\sigma \in CS^a(\mathbb{R}^n)$:

$$\begin{aligned}\mathrm{fp}^{\mathrm{Riesz}} \int_{\mathbb{R}^n} \sigma(\xi)d\xi &:= \lim_{z\to 0}\left(\int_{\mathbb{R}^n} |\xi|^{-z}\sigma(\xi)d\xi - \frac{1}{z}\operatorname*{Res}_{z=0}\int_{\mathbb{R}^n} |\xi|^{-z}\sigma(\xi)d\xi\right)\\ &= \int_{\mathbb{R}^n} g_N + \sum_{j=0}^{K_N}\int_{B(0,1)}\psi(\xi)\sigma_{a_j}(\xi) - \sum_{\substack{j=0\\ a_j+n\neq 0}}^{K_N}\frac{1}{a_j+n}\int_{S^{n-1}}\sigma_{a_j}(\omega)d\omega.\end{aligned}$$

Combining the two propositions yields the following theorem:

Theorem 1. *Let $\sigma \in CS^a(U)$. Then for fixed $N \in \mathbb{N}$ sufficiently large, σ can be written $\sigma(\xi) = \sum_{j=0}^{K_N}\psi(\xi)\sigma_{a_j}(\xi) + g_N(\xi)$, σ_{a_j}, ψ and $g_N \in S^{-N}(U)$ as in Definition 2.*

(i) *The coefficient of the logarithmic term in the asymptotic expansion of $\int_{B(0,R)}\sigma(\xi)d\xi$ when $R \to \infty$, is the residue at $z = 0$ of the meromorphic function $z \mapsto \int_{\mathbb{R}^n}|\xi|^{-z}\sigma(\xi)d\xi$.*

(ii) *The finite part $\mathrm{fp}^{\mathrm{cut\text{-}off}}\int_{\mathbb{R}^n}\sigma(\xi)d\xi$ of $\int_{B(0,R)}\sigma(\xi)d\xi$ when $R \to \infty$ and the finite part $\mathrm{fp}^{\mathrm{Riesz}}\int_{\mathbb{R}^n}\sigma(\xi)d\xi$ of the map $z \mapsto \int_{\mathbb{R}^n}|\xi|^{-z}\sigma(\xi)d\xi$ at $z = 0$ coincide so that we can set*

$$\mathrm{fp}\int_{\mathbb{R}^n}\sigma(\xi)d\xi := \mathrm{fp}^{\mathrm{Riesz}}\int_{\mathbb{R}^n}\sigma(\xi)d\xi = \mathrm{fp}^{\mathrm{cut\text{-}off}}\int_{\mathbb{R}^n}\sigma(\xi)d\xi.$$

Let $a_j = a - j$, then for any integer N larger than n,

(iii) *If $a \in -\mathbb{N}$ and $a \geq -n$, then $\mathrm{res}(\sigma) = \int_{S^{n-1}}\sigma_{-n}(\omega)d\omega$ and*

$$\mathrm{fp}\int_{\mathbb{R}^n}\sigma(\xi)d\xi = \int_{\mathbb{R}^n} g_N + \sum_{j=0}^{K_N}\int_{B(0,1)}\psi(\xi)\sigma_{a_j}(\xi)d\xi - \sum_{\substack{j=0\\ a_j+n\neq 0}}^{K_N}\frac{1}{a_j+n}\int_{S^{n-1}}\sigma_{a_j}(\omega)d\omega.$$

(iv) *If $a \notin -\mathbb{N}$ or if $a < -n$, then $\mathrm{res}(\sigma) = 0$ and*

$$\mathrm{fp}\int_{\mathbb{R}^n}\sigma(\xi)d\xi = \int_{\mathbb{R}^n} g_N + \sum_{j=0}^{K_N}\Big(\int_{B(0,1)}\psi(\xi)\sigma_{a_j}(\xi)d\xi - \frac{1}{a_j+n}\int_{S^{n-1}}\sigma_{a_j}(\omega)d\omega\Big).$$

2.3. **Kernel of a PDO.** If $\Omega \subseteq \mathbb{R}^n$ is an open set, we denote by $\mathcal{D}(\Omega)$ the set of smooth functions with compact support on Ω, endowed with the topology of the uniform convergence of the functions and their derivatives on compact subsets of Ω, and by $\mathcal{D}'(\Omega)$ its dual, i.e. the space of distributions on Ω (see [19]). If A is a PDO on Ω, with symbol $\sigma \in S^a(\Omega)$, for any function $f \in \mathcal{S}(\mathbb{R}^n)$

we have

$$f(x) = \int e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi = \iint e^{i(x-y)\cdot\xi}\sigma(x,\xi)f(y)dyd\xi$$
$$= \int K_A(x,y)f(y)dy$$

where the *kernel* of the operator A is defined by

$$K_A(x,y) := \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)d\xi\,.$$

A smoothing operator has a smooth kernel but the operators we are interested in typically have a distribution kernel. If $a < -n$ the integral

$$\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)d\xi$$

converges for every $x,\ y$. If $a \geq -n$ the integral

$$\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)d\xi$$

can be interpreted as a distribution in $\mathcal{D}'(\Omega\times\Omega)$ in the following way: If $\varphi \in \mathcal{D}(\Omega\times\Omega)$, we set ([3])

$$K(\varphi) = \iiint e^{i(x-y)\cdot\xi}\sigma(x,\xi)\varphi(x,y)dx\,dy\,d\xi.$$

The kernel of a PDO is an infinitely differentiable function outside the diagonal. The kernel associated with the product of two pseudo-differential operators P and Q is given by the convolution of the corresponding kernels:

$$K_{PQ}(x,y) = K_P \star K_Q(x,y) = \int K_P(x,z)K_Q(z,y)dz\,.$$

Generalizing Definition 1 we can consider symbols of complex order. In this case condition (b) reads: For all $(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n$ there are constants $C_{\alpha,\beta}$ such that $|D_x^\alpha D_\xi^\beta\sigma(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{\Re(a)-|\beta|}\quad\forall(x,\xi)\in K\times\mathbb{R}^n$.

If $U\subseteq\mathbb{R}^n$, we call a family of symbols $\sigma(z,x,\cdot)\in CS^{\alpha(z)}(U)$ *holomorphic* if

$$\sigma(z,x,\xi)\sim\textstyle\sum_{j=0}^\infty\psi(\xi)\sigma_{\alpha(z)-j}(z,x,\xi)\ \text{ with }\ x\in U,\ \xi\in\mathbb{R}^n$$

is such that $z\mapsto\alpha(z)$ is holomorphic, $z\mapsto\sigma_{\alpha(z)-j}(z,\cdot,\cdot)$ is holomorphic as a map with values in $C^\infty(U\times\mathbb{R}^n)$, and for all $N >> 0$, the kernel associated to the symbol $\sigma(z,\cdot,\cdot)-\sum_{j=0}^N\psi\sigma_{\alpha(z)-j}(z,\cdot,\cdot)$ is holomorphic with values in $C^{J_N}(U\times U)$ for some J_N such that $\lim_{N\to\infty}J_N=\infty$.

Remark 4. When σ is classical, the derivatives $\sigma^{(k)}(z,\cdot,\cdot)$ are expected to be log-polyhomogeneous, see [14].

The following theorem generalizes the last results with

$$\sigma(z,x,\xi) = \sigma(x,\xi)\psi(\xi)|\xi|^{-z}$$

of order $a - z$ (with $\psi(\xi)$ a cut-off function as in Definition 2), to holomorphic families of symbols $z \mapsto \sigma(z,x,\xi)$ of order $\alpha(z)$, following [5] and [8].

Theorem 2. *Let $U \subseteq \mathbb{R}^n$ and $z \mapsto \sigma(z,x,\cdot) \in CS^{\alpha(z)}(U)$ be a holomorphic family of symbols on a domain $G \subseteq \mathbb{C}$. Then the map $z \mapsto \operatorname{fp}\int_{\mathbb{R}^n} \sigma(z,x,\xi)d\xi$ is meromorphic with possible simple poles at points $z_j \in G \cap \alpha^{-1}(\mathbb{Z} \cap [-n,+\infty))$ such that $\alpha'(z_j) \neq 0$ and the residue at such a pole is given by*

$$\operatorname*{Res}_{z=z_j} \operatorname{fp}\int_{\mathbb{R}^n} \sigma(z,x,\xi)d\xi = -\frac{1}{\alpha'(z_j)} \operatorname{res}(\sigma(z_j,x,\cdot)).$$

The function $z \mapsto \operatorname{fp}\int_{\mathbb{R}^n} \sigma(z,x,\xi)d\xi$ is holomorphic if $z \notin G \cap \alpha^{-1}(\mathbb{Z} \cap [-n,+\infty))$.

In particular we have that if $\Re(\alpha(z)) < -n$, then the map

$$z \mapsto \operatorname{fp}\int_{\mathbb{R}^n} \sigma(z,x,\xi)d\xi$$

is holomorphic.

Let us now see how far the kernel can be extended to the whole space. We distinguish two cases, the case of operators of integer order and those of non integer order.

(1) Suppose $\sigma \in S^a$ and $a \notin \mathbb{Z}$.
If $a < -n$, $K(x,y)$ is a continuous function on the whole space, and on the diagonal $K(x,x) = \int \sigma(x,\xi)d\xi$ is a convergent integral. If $a \geq -n$, it is defined by the function

$$\widehat{K}(x,y) := \begin{cases} K(x,y) & \text{if } x \neq y, \\ \operatorname{fp}\int_{\mathbb{R}^n} \sigma(x,\xi)d\xi & \text{if } x = y, \end{cases}$$

which extends to the whole space $\mathbb{R}^n \times \mathbb{R}^n$.

(2) Suppose that $\sigma \in S^a$ and $a \in \mathbb{Z}$. If $z \mapsto \sigma(z,x,\xi) \in CS^{\alpha(z)}(U)$ is a holomorphic family of symbols on a domain $G \subseteq \mathbb{C}$ such that $\sigma(0,x,\xi) = \sigma(x,\xi)$, then we have the corresponding family of kernels $K_z(x,y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(z,x,\xi)d\xi$. We define

$$\widehat{K_z}(x,y) := \begin{cases} K_z(x,y) & \text{if } x \neq y, \\ \operatorname{fp}\int_{\mathbb{R}^n} \sigma(z,x,\xi)d\xi & \text{if } x = y. \end{cases}$$

Let $\triangle = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$, then
  (a) $\widehat{K_z}$ is an infinitely differentiable function in $\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle = \triangle^c$.
  (b) The map $z \mapsto \widehat{K_z}(x,y)\Big|_{\triangle^c}$ is holomorphic on the complex plane.

(c) For $x \in \mathbb{R}^n$, the map $z \mapsto \widehat{K_z}(x,x)$ is meromorphic with simple poles at the points $z \in G \cap \alpha^{-1}(\mathbb{Z} \cap [-n,+\infty))$ with $\alpha'(z) \neq 0$.
(d) If $\Re(\alpha(z)) < -n$, then $\widehat{K_z}$ is an infinitely differentiable function on $\mathbb{R}^n \times \mathbb{R}^n$, since in this case it is well defined on the diagonal.
(e) If $\Re(\alpha(z)) < -n$, then the map $z \mapsto \widehat{K_z}(x,y)$ is holomorphic.

2.4. **The Mellin Transform.** Given $\lambda > 0$, the map $\phi_\lambda : r \mapsto e^{-\lambda r}$ lies in $\mathcal{S}((0,+\infty))$ and the function $z \mapsto \int_0^\infty t^{z-1}e^{-\lambda t}dt$ is holomorphic on the half plane $\Re(z) > 0$ and extends to a meromorphic function on the complex plane with simple poles in $-\mathbb{N}$, with residues given by $\frac{(-\lambda)^k}{k!}$. If we consider the change of variable $u : t \mapsto \lambda t$ in $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$, we have

$$\lambda^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}e^{-\lambda t}dt\,. \tag{2}$$

This formula is valid for $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, and represents a meromorphic function on the plane. More generally, if $\phi \in \mathcal{S}((0,+\infty))$, the *Mellin transform* of ϕ is defined by (see [2]):

$$\mathcal{M}(\phi)(z) := \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\phi(t)dt\,.$$

Proposition 3. *Let $f \in C^\infty(0,+\infty)$ be a function with asymptotic expansion for small t, of the form $f(t) \sim \sum_{k\geq -n} f_k t^{\frac{k}{q}} + g\log t$, where n is the dimension of the space, and suppose that f decays exponentially at infinity, that is, for some $\lambda > 0$ and t sufficiently large, $|f(t)| \leq Ce^{-t\lambda}$ for some constant C. Let $\gamma := -\int_0^\infty (\log r)e^{-r}dr$ be the Euler constant. Then*

(1) *The Mellin transform $\mathcal{M}(f)$ is a meromorphic function with poles contained in the set $\frac{n}{q} - \frac{\mathbb{N}}{q}$.*
(2) *The Laurent series of $\mathcal{M}(f)$ around $z = 0$ is $-gz^{-1} + (f_0 - \gamma g) + O(z)$.*

Thus we can define the finite part at $z = 0$ of the Mellin transform of a function f that satisfies the assumptions of the above proposition:

$$\operatorname*{fp}_{z=0} \mathcal{M}(f)(z) = \left.\frac{d}{dz}\right|_{z=0} z\mathcal{M}(f)(z) = f_0 - \gamma g. \tag{3}$$

3. The ζ function

The results we have described for operators acting on functions on $\mathbb{R}^n$ can be generalized to operators acting on sections of a vector bundle on a manifold. Throughout this paper, M is a Hausdorff, connected, closed (i.e., compact and without boundary), oriented, C^∞ m-dimensional manifold endowed with a riemannian metric g, and E is a vector bundle on M of rank N. Write $C^\infty(M,E)$ for the smooth sections of E.

3.1. **Pseudo-differential Operators on Manifolds.** A differential operator of order k on the space of sections $C^\infty(M,E)$ is a linear map $A : C^\infty(M,E) \to C^\infty(M,E)$, such that for $s \in C^\infty(M,E)$ and for local coordinates $x_1,\dots,x_m$ of M, $A(s)(x_1,\dots,x_m) = \sum_{|\alpha|\le k} a_\alpha(x) D_x^\alpha s(x_1,\dots,x_m)$, where the $a_\alpha(x)$ are $N \times N$ square matrices whose entries are smooth functions in the variable $x = (x_1,\dots,x_m)$. Associated to the operator A is the symbol:

$$\sigma(A)(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha, \quad \xi \in T_x^*(M)\,.$$

We can extend this definition to pseudo-differential operators. In this case the sum which represents the symbol of the operator is an infinite sum that need not converge. A pseudo-differential operator $A : C^\infty(M,E) \to C^\infty(M,E)$ is *classical* of order a if its symbol is polyhomogeneous, i.e., if the following conditions are satisfied:

(1) In any local coordinates over which E is trivial, the symbol of A has an expansion $\sigma(A)(x,\xi) \sim \sum_{j=0}^\infty \sigma_j(x,\xi)$, where $\sigma_j(x,\xi)$ is an $N\times N$-matrix-valued function, which is positively homogeneous in ξ of degree $a-j$.
(2) There exist local coordinates for which the term $\sigma_0(x,\xi)$ is not identically zero.

We call $\sigma_0(A)$ the *principal (or leading) symbol of* A; it corresponds to the positively homogeneous component of σ of highest degree. It is defined by

$$\sigma_0(A)(x,\xi) := \lim_{t\to+\infty} \sigma(x,t\xi)t^{-a}\,.$$

If $\sigma_0(A)(x,\xi)$ is invertible for $\xi \neq 0$, the operator A is *elliptic*. For $s \in \mathbb{R}$, $H^s(M,E)$ denotes the Sobolev space on $C^\infty(M,E)$, $Cl(M,E)$ denotes the algebra of classical PDO's which act on $C^\infty(M,E)$, and for any $\alpha \in \mathbb{R}$, let $Cl^\alpha(M,E)$ be the subset of operators in $Cl(M,E)$ of order α.

Example 1. If $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $G = (|\det(g_{ij})|)^{\frac{1}{2}}$, and we define g^{jk} by $\sum_{j=1}^m g_{ij}g^{jk} = \delta_{ik}$, the *Laplace-Beltrami operator* corresponding to the metric g is defined by

$$\Delta_g = G^{-1}\sum_{i=1}^m\sum_{k=1}^m \frac{\partial}{\partial x_k}\left\{g^{ik}G\frac{\partial}{\partial x_i}\right\}.$$

Δ_g is an elliptic PDO with principal symbol $\sigma_0(x,\xi) = -|\xi|^2$ (see [1]).

Definition 4. [12] An operator A of non negative order has *principal angle* θ, if for every $(x,\xi) \in T^*(M)\setminus i(M)$ ($i(M)$ denotes the zero section of $T^*(M)$), the map $\sigma_0(A)(x,\xi)$ has no eigenvalues on the ray $L_\theta = \{re^{i\theta} : r \ge 0\}$. A has *Agmon angle* θ, if θ is a principal angle of A, and the spectrum of A does not meet L_θ; in this case, we call L_θ a *spectral cut* of A.

If A has a principal angle θ, then A is elliptic. If A has an Agmon angle θ, then A is elliptic and invertible. The operator A is *admissible* if A is of order zero and admits a spectral cut, or if A is of positive order and has a principal angle (see [12]); in particular such an operator is elliptic.

Let $A \in Cl(M,E)$ be a self-adjoint pseudo-differential operator of order $a \geq 0$ which admits an Agmon angle θ. Under these conditions, we can define the complex power A^z_θ ($z \in \mathbb{C}$), and the logarithm $\log_\theta A$ (see [12] and [20]). For $\Re(z) < 0$, A^z_θ is a bounded operator on any space $H^s(M,E)$ of sections of E of the Sobolev class H^s, defined by the integral:

$$A^z_\theta = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z (A - \lambda I)^{-1} d\lambda\,,$$

where

$$\begin{aligned}
\Gamma_\theta &= \Gamma_{1,\theta} \cup \Gamma_{2,\theta} \cup \Gamma_{3,\theta},\\
\Gamma_{1,\theta} &= \{\lambda = re^{i\theta}, r \geq R\},\\
\Gamma_{2,\theta} &= \{\lambda = Re^{i\phi}, \theta \geq \phi \geq \theta - 2\pi\},\\
\Gamma_{3,\theta} &= \{\lambda = re^{i(\theta-2\pi)}, r \geq R\},
\end{aligned}$$

R is a small positive number such that Γ_θ does not meet the spectrum of A, $\lambda^z = e^{z\log\lambda}$ where $\log\lambda = \ln|\lambda| + i\theta$ on $\Gamma_{1,\theta}$ and $\log\lambda = \ln|\lambda| + i(\theta - 2\pi)$ on $\Gamma_{3,\theta}$. (see [12]).

For any $z \in \mathbb{C}$, A^z_θ is a classical, elliptic pseudo-differential operator of order az. Let $A \in Cl^a(M,E)$ be an admissible pseudo-differential operator of nonnegative order with spectral cut L_θ. For arbitrary $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, the map $z \mapsto A^z_\theta$ defines a holomorphic function from $\{z \in \mathbb{C}, \Re(z) < k\}$ to the space of bounded linear maps from $H^s(M,E)$ to $H^{s-ak}(M,E)$ and we can set $\log_\theta A := \left[\frac{\partial}{\partial z}A^z_\theta\right]_{z=0}$. $\log_\theta A$ defines a PDO which is a bounded operator from $H^s(M,E)$ to $H^{s-\epsilon}(M,E)$ for any $\epsilon > 0$ and any $s \in \mathbb{R}$. When $a = 0$, $\log_\theta A$ is polyhomogeneous of nonpositive order. When $a > 0$, this operator is not polyhomogeneous: In local coordinates (x,ξ) on T^*M, the symbol of $\log_\theta A$ reads (see [12]):

$$\sigma(\log_\theta A) = a\log|\xi| + \text{ a polyhomogeneous symbol of order 0.}$$

From now on, when we take logarithms and powers of an admissible operator of positive order, we shall consider that it is with respect to a fixed Agmon angle of the operator.

Under certain conditions, a PDO has a well defined trace or determinant. A PDO A is trace-class if and only if the order of A is less than $-m$, where m is the dimension of M.

Theorem 3. [17] *Let M be a compact oriented manifold, endowed with a Riemannian metric g and let A be a bounded operator on $L^2(M, d\mu_g)$ defined* *by* $Au(x) = \int_M k(x,y)u(y)d\mu_g(y)$ *and suppose that* $x \mapsto k(x,x) \in L^1(M)$. *Then A is of trace-class, and* $\mathrm{tr}(A) = \int_M k(x,x)d\mu_g(x)$.

This can be extended to operators acting on sections of a vector bundle E. The corresponding statement is:

Theorem 4. [17] *Let A be an operator of order less than* $-m$ *on* $C^\infty(M,E)$, *with kernel k. Then A is of trace-class and* $\mathrm{tr}(A) = \int_M \mathrm{tr}_x\, k(x,x)d\mu_g(x)$, *where the trace of the right hand side is the canonical trace on endomorphisms of the finite dimensional vector space* E_x *(the fiber of E over* $x \in M$*).*

3.2. **The ζ determinant.** If Q is a hermitian matrix whose eigenvalues λ_1, ..., λ_N are positive numbers, for $z \in \mathbb{C}$ we introduce the function (see [1]) $\zeta_{Q,N}(z) := \sum_{n=1}^N \lambda_n^{-z}$. Then the determinant of Q is recovered by the formula

$$\det(Q) = \prod_{i=1}^N \lambda_i = e^{-\zeta'_{Q,N}(0)} \tag{4}$$

where $\zeta'_{Q,N}(0) := \frac{d}{dz}\zeta_{Q,N}(z)|_{z=0}$. The infinite-dimensional generalization of this scheme consists in considering Q as a nonnegative operator which admits an Agmon angle, acting on a Hilbert space, with spectrum given by the eigenvalues $\{\lambda_n\}_{n=1}^\infty$. For $z \in \mathbb{C}$ *the ζ function of Q* is defined by ([6] and [15]):

$$\zeta_Q(z) := {\sum_n}' \frac{1}{\lambda_n^z}, \tag{5}$$

where $\sum'$ means that we are summing only over the nonzero eigenvalues of Q. We shall restrict ourselves to the case where Q is an admissible, positive pseudo-differential operator, of positive order q, on the space of sections $C^\infty(M,E)$. In this case Q has discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$, each λ_n in the spectrum is an eigenvalue whose space of generalized eigenfunctions is finite-dimensional, and $\lim_{n\to\infty} \lambda_n = +\infty$ (see [4] and [21]). The formal determinant of the operator, given by the product of all its eigenvalues, diverges. Nevertheless, in the case of an admissible operator of positive order q acting on the smooth sections of an m-dimensional manifold M, we have the asymptotic formula $\lambda_n \sim Cn^{\frac{q}{m}}$ for some constant C (see [1] and [20]). With this, the sum (5) is absolutely convergent for $\Re(z) > \frac{m}{q}$ and therefore $\zeta_Q(z)$ is holomorphic in this region of the plane. In other words, when Q is positive, Q^{-z} has a finite trace for $\Re(z)$ sufficiently large. In [5] and [20] it is proved that $\zeta_Q(z)$ admits a meromorphic continuation to the whole complex plane, which is regular at $z = 0$.

Equation (2) implies that for $\Re(\lambda) > 0$:

$$\lambda^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-t\lambda} dt = \mathcal{M}(e^{-t\lambda})(z),$$

where $\mathcal{M}$ is the Mellin transform defined previously. This can be extended to matrices; if Q is an $n \times n$ hermitian matrix, with positive eigenvalues λ_i, $1 \leq i \leq n$, and $f_Q(t) = \mathrm{tr}(e^{-tQ}) = \sum_{i=1}^n e^{-t\lambda_i}$, then for $z \in \mathbb{C}$,

$$\zeta_Q(z) = \mathrm{tr}(Q^{-z}) = \sum_{i=1}^n \lambda_i^{-z} = \sum_{i=1}^n \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-t\lambda_i} dt = \mathcal{M}(f_Q)(z). \quad (6)$$

It further extends to an admissible pseudo-differential operator Q of positive order q, with positive principal symbol. The operator e^{-tQ}, the solution of the heat equation $Q(u) = \frac{\partial u}{\partial t}$ for $u \in C^\infty(M, E)$, has a kernel $e^{-tQ}(x, y)$ defined on $M \times M$, that on the diagonal admits the following asymptotic expansion when $t \to 0^+$ (see [1]):

$$e^{-tQ}(x, x) = \sum_{n=0}^{N-1} t^{\frac{n-m}{q}} a_n(x) + O(t^{\frac{N-m}{q}}) \quad (7)$$

for each $N = 1, 2, \ldots$, the $a_n(x)$ being certain Q-dependent scalar functions. Let n_0 be the dimension of the null space of Q. Then

$$\mathrm{tr}'(e^{-tQ}) = \sum\nolimits_n' e^{-t\lambda_n} = \mathrm{tr}(e^{-tQ}) - n_0\,;$$

the function $f(t) = \mathrm{tr}'(e^{-tQ})$ satisfies the assumptions of Proposition 3. From (7) and from Theorem 4 we have $\mathrm{tr}'(e^{-tQ}) = \sum_{\substack{n<N \\ n\neq m}} a_n t^{\frac{n-m}{q}} + (a_m - n_0) + O(t)$.

Therefore,

$$\Gamma(z)\zeta_Q(z) := \int_0^\infty t^{z-1}\, \mathrm{tr}'(e^{-tQ}) dt \quad (8)$$

is well defined for $\Re(z) > \frac{m}{q}$.

Proposition 4. *If Q is an admissible PDO of positive order q, the function $\zeta_Q(z)$, that is well defined for $\Re(z) > \frac{m}{q}$, is regular at $z = 0$ and admits a meromorphic extension to the whole complex plane. Moreover, if $R_N(z) = \int_0^1 t^{z-1} O(t) dt + \int_1^\infty t^{z-1}\, \mathrm{tr}'(e^{-tQ}) dt$,*

$$\zeta_Q(0) = a_m - n_0, \quad \textit{and} \quad \zeta_Q'(0) = \sum_{\substack{n<N \\ n\neq m}} \frac{q \cdot a_n}{n-m} + \gamma(a_m - n_0) + R_N(0). \quad (9)$$

As a consequence of this analysis and generalizing (4), it makes sense to introduce the *ζ-determinant* of Q:

$$\det_\zeta(Q) := e^{-\zeta_Q'(0)}. \quad (10)$$

3.3. **The heat kernel determinant.** Let Q be an admissible, positive, pseudo-differential operator of positive order q on the space of sections $C^\infty(M,E)$ and let $\tilde{Q}$ denote the restriction of Q to the subspace orthogonal to the null space of Q. For $\varepsilon > 0$ we consider the function $h_\varepsilon : (0,\infty) \to (0,\infty)$ given by

$$h_\varepsilon(\lambda) := \exp\left(-\int_\varepsilon^\infty \frac{e^{-t\lambda}}{t} dt\right). \tag{11}$$

By the spectral theorem $h_\varepsilon(\tilde{Q})$ makes sense as a positive bounded operator.

Proposition 5. [1] *If Q is an elliptic, positive operator on $C^\infty(M,E)$, $h_\varepsilon(\tilde{Q})$ is of the type "Identity+a trace-class operator".*

As a consequence of this proposition, it is possible to define the determinant of $h_\varepsilon(\tilde{Q})$, and it makes sense to define the ε*−regularized heat kernel determinant* of Q as ([1]):

$$\det_\varepsilon(Q) := \det(h_\varepsilon(\tilde{Q})) = \exp(\operatorname{tr}(\log h_\varepsilon(\tilde{Q}))), \tag{12}$$

being a finite number (see equation (21) in Section 6). This formula is obvious in the finite dimensional case with $\varepsilon = 0$. Letting ε tend to 0 in (12), by Proposition 3 we get infinity. It is therefore necessary to subtract from (12) the divergent part before taking such a limit. This procedure is called regularization, which motivates the terminology *regularized heat kernel determinant* of Q for the quantity

$$\det{}'(Q) := \exp\left\{\lim_{\varepsilon\to 0}[\log\det_\varepsilon(Q) - (\text{divergent terms })]\right\}. \tag{13}$$

Let us investigate the relation between the ζ-determinant and the regularized heat kernel determinant of an operator.

Theorem 5. *If Q is an admissible operator of positive order q, then*

$$\det{}'(Q) = e^{\gamma\zeta_Q(0)}\det_\zeta(Q).$$

Remark 5. This result can be found in [1] for the case of the Laplacian.

Proof. Equations (11) and (12) imply that

$$\log\det_\varepsilon(Q) = -\int_\varepsilon^\infty \frac{\operatorname{tr}'(e^{-tQ})}{t} dt \tag{14}$$

since summing over the nonzero eigenvalues of Q, $\operatorname{tr}(e^{-t\tilde{Q}}) = \operatorname{tr}'(e^{-tQ})$. Now, from equations (9) and (10), we have that

$$\log\det_\zeta(Q) = -\sum_{\substack{n<N\\ n\neq m}} \frac{q\cdot a_n}{n-m} - \gamma(a_m - n_0) - R_N(0).$$

Following the asymptotic expansion given in (7) and Proposition 4, we can split the integral over $[\varepsilon, \infty)$ into the sum of the integrals over $[\varepsilon, 1)$ and $[1, \infty]$, and if

$$w = \sum_{\substack{n<N \\ n \neq m}} \frac{q \cdot a_n}{n-m}$$

we obtain

$$\begin{aligned}
&\log \det_\zeta(Q) = -w - \gamma(a_m - n_0) \\
&- \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left(\mathrm{tr}'(e^{-tQ}) - \sum\nolimits_{\substack{n<N \\ n \neq m}} a_n t^{\frac{n-m}{q}} - (a_m - n_0) \right) \frac{dt}{t} - \int_1^\infty \frac{\mathrm{tr}'(e^{-tQ})}{t} dt \\
&= -\gamma\zeta_Q(0) + \lim_{\varepsilon \to 0} \left[\log \det_\varepsilon(Q) - \left(\sum\nolimits_{\substack{n<N \\ n<m}} \frac{q \cdot a_n}{n-m} \varepsilon^{\frac{n-m}{q}} + \zeta_Q(0) \log \varepsilon \right) \right] \\
&= -\gamma\zeta_Q(0) + \log(\det{}'(Q)). \qquad ☑
\end{aligned}$$

Remark 6. We have the equality between the two determinants when $\zeta_Q(0) = 0$, which happens, for example, when Q is a differential operator which admits an Agmon angle, and is of even order on an odd-dimensional manifold ([18], [20]).

In the case $Q = -\Delta_g$ and $\dim(M) = 2$, $a = \frac{Vol_g(M)}{4\pi}$, $b = \frac{\chi(M)}{6} - 1$, and we have

$$\begin{aligned}
\log \det_\zeta(-\Delta_g) &= a - \gamma b - \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left(\mathrm{tr}'(e^{t\Delta_g}) - \frac{a}{t} - b \right) \frac{dt}{t} - \int_1^\infty \frac{\mathrm{tr}'(e^{t\Delta_g})}{t} dt \\
&= -\gamma b + \log(\det{}'(-\Delta_g)) \,. \qquad (15)
\end{aligned}$$

From (13) and (15) we deduce that $\det'(-\Delta_g) = e^{\gamma((\chi(M)/6)-1)} \det_\zeta(-\Delta_g)$, this shows that the ζ-determinant of $-\Delta_g$ coincides with the heat kernel determinant up to a finite positive constant depending only on the topology of M.

4. **Noncommutative residue of a PDO**

The representation of the ζ-function for a pseudo-differential operator Q of order $q > 0$ on $C^\infty(M, E)$ as a Mellin transform

$$\zeta_Q(z) = \mathrm{tr}(Q^{-z}) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \, \mathrm{tr}'(e^{-tQ}) dt$$

allows to describe the asymptotic behaviour of $\mathrm{tr}(e^{-tQ})$ when t tends to zero:

$$\mathrm{tr}(e^{-tQ}) \underset{t \to 0}{\sim} \alpha_m(Q) + \sum_{\substack{j=0 \\ j \neq m}}^\infty \alpha_j(Q) t^{-z_j} + \sum_{k=1}^\infty \beta_k(Q) t^k \log t \,, \qquad (16)$$

where $z_j = \frac{m-j}{q}$, $j \in \mathbb{N} - \{m\}$, (see [23]). Logarithmic terms do not arise ($\beta_k = 0$) when Q is a differential operator. If we consider any pseudo-differential operator A on $C^\infty(M,E)$ of order a, choosing an elliptic operator Q of order q larger than a, then for any real parameter u, the ζ-function of $Q+uA$ is well defined (see [7]). In his doctoral thesis, Wodzicki defined the noncommutative residue of A by the formula $\operatorname{res}(A) = q \cdot \left.\frac{d}{du}\right|_{u=0} \left(\operatorname{Res}_{z=-1} \zeta_{Q+uA}(z)\right)$.

In reference with Definition 3 in Section 2, let $dx = dx_1 \wedge \ldots \wedge dx_m$ be the locally defined coordinate form on M and let $d\xi$ be the volume form on T^*M, or the restriction of $d\xi$ to the unit cosphere bundle $S^*M \subseteq T^*M$ or the unit cosphere S^*_xM at a fixed $x \in M$. Then

$$\operatorname{res}(\sigma(A))(x) = \int_{S^*_xM} \operatorname{tr}_x \sigma_{-m}(A)(x,\xi)d\xi,$$

is (nontrivially) a global top degree form on M whose integral is *the noncommutative residue* of A (see [7])

$$\operatorname{res}(A) = \int_M \operatorname{res}(\sigma(A))dx.$$

Following [5] and [8] we can consider holomorphic families of symbols in $Cl(M,E)$ and we have the following generalization of Theorem 2. The proof can be found in [8].

Theorem 6. *Let f be a holomorphic function on $\mathbb{C}$ such that $f'(z) \neq 0$ for $z \in f^{-1}\{j-m : j \in \mathbb{N}\}$. Let $(B(z))_{z\in\mathbb{C}}$ be a holomorphic family of operators in $Cl(M,E)$ with $B(z)$ of order $f(z)$. The function $z \mapsto \operatorname{tr}(B(z))$ is holomorphic in the domain $\{z \in \mathbb{C} : \Re(f(z)) < -m\}$ and can be extended to a meromorphic function on the whole complex plane, with only simple poles. If they exist, the poles occur at the points $z_j = f^{-1}(j-m), j \in \mathbb{N}$. At $z = z_j$ the residue is:*

$$\operatorname{Res}_{z=z_j} \operatorname{tr}(B(z)) = -\frac{1}{f'(z_j)} \operatorname{res}(B(z_j)).$$

Applying this theorem to the holomorphic family AQ^{-z}, where $A \in Cl(M,E)$ and Q is an elliptic admissible pseudo-differential operator of positive order, we have the following theorem.

Theorem 7. *Let Q be an elliptic admissible pseudo-differential operator of order $q > 0$, which acts on $C^\infty(M,E)$. The function $\operatorname{tr}(AQ^{-z})$ defined for $\Re(z) >> 0$ admits a meromorphic extension to $\mathbb{C}$ with a simple pole at $z = 0$, and its residue is proportional to the noncommutative residue of A:*

$$\operatorname{res}(A) = q \cdot \operatorname{Res}_{z=0} \operatorname{tr}(AQ^{-z}). \tag{17}$$

The right hand side of this equation does not depend of Q and the left hand side is local by the locality of the Wodzicki residue.

Replacing Q by $Q+uA$ in (16) and differentiating $e^{-t(Q+uA)}$ with respect to u, we can see that $\operatorname{tr}(Ae^{-tQ})$ also admits an asymptotic expansion when $t \to 0$ (see [7]). This yields another formula for the noncommutative residue of A:

Theorem 8. *When $t \to 0^+$, $\mathrm{tr}(Ae^{-tQ})$ admits an asymptotic expansion*

$$\mathrm{tr}(Ae^{-tQ}) \sim \sum_j a_j t^{\frac{j-a-m}{q}} - \frac{\mathrm{res}(A)}{q} \log t + O(1).$$

In brief, in analogy with the results of Section 1.2, we have that for any pseudo-differential operator A and an appropriate elliptic operator Q of order q,

$$\begin{aligned}\mathrm{res}(A) &= q \cdot \operatorname*{Res}_{z=0} \mathrm{tr}(AQ^{-z}) \\ &= -q \cdot \text{coefficient of } \log t \text{ in the asymptotic} \\ &\qquad \text{expansion of } \mathrm{tr}(Ae^{-tQ}) \text{ as } t \to 0.\end{aligned}$$

Consequently, applying Proposition 3 to $f(t) = \mathrm{tr}(Ae^{-tQ})$ and using equation (3) we get:

Corollary 1. *For A and Q as above,*

$$\operatorname*{fp}_{z=0} \mathrm{tr}(AQ^{-z}) = \operatorname*{fp}_{z=0} \mathcal{M}(f)(z) = \operatorname*{fp}_{t=0} \mathrm{tr}(Ae^{-tQ}) - \gamma \frac{\mathrm{res}(A)}{q}. \tag{18}$$

5. Conformal Anomalies

In this section we study the variation of the determinant of Δ_g under a conformal variation of the metric: $g \mapsto e^{\alpha\varphi} g$ where $\alpha \in \mathbb{R}$, $\varphi \in C^\infty(M, \mathbb{R})$, for $\dim(M) = 2$.

Recall from equations (12) and (14) that for $\varepsilon > 0$ the regularized heat kernel determinant of $-\Delta_g$ reads:

$$\det_\varepsilon(-\Delta_g) = \exp\left(- \int_\varepsilon^\infty \frac{\mathrm{tr}(e^{t\tilde{\Delta}_g})}{t} dt \right),$$

where on the right hand side, $\tilde{\Delta}_g$ denotes the restriction of Δ_g to the orthogonal complement of its null space. The knowledge of the function $\varepsilon \mapsto \det_\varepsilon(-\Delta_g)$ depends on the knowledge of $t \mapsto \mathrm{tr}(e^{t\tilde{\Delta}_g})$, for all $t \in [\varepsilon, \infty)$ (see [1]).

To understand how the determinant depends on the parameter ε we consider a one-parameter family of Laplacians. The following proposition expresses the conformal anomaly of the determinant $\det_\varepsilon(-\Delta_g)$.

Proposition 6. [1] *Let M be a smooth compact connected manifold without boundary. Let g_0, g_1 be conformal equivalent riemannian metrics on M, such that $g_1 = e^\varphi g_0$ for some $\varphi \in C^\infty(M, \mathbb{R})$ and let Δ_{g_0}, Δ_{g_1} be the Laplace operators on M with respect to (each one of) the metrics g_0, g_1. Then we have,*

$$\det_\varepsilon(-\Delta_{g_1}) = \left[\frac{Vol_{g_1}(M)}{Vol_{g_0}(M)} \exp(-W_0(\varepsilon, \varphi)) \right] \det_\varepsilon(-\Delta_{g_0}),$$

with

$$W_0(\varepsilon,\varphi) \equiv \frac{1}{4\pi\varepsilon}(Vol_{g_1}(M) - Vol_{g_0}(M)) + \frac{1}{24\pi}\left(\frac{1}{2}\int_M \varphi(\eta)(-\Delta_{g_0}\varphi(\eta))d\mu_{g_0}(\eta) + \int_M \varphi(\eta)\mathcal{R}_{g_0}(\eta)d\mu_{g_0}(\eta) + K_0(\varepsilon,\varphi)\right),$$

$\mathcal{R}_{g_0}$ *is the scalar curvature with respect to* g_0, $K_0(\varepsilon, 0) = 0$, *and* $\lim_{\varepsilon\to 0} K_0(\varepsilon,\varphi) = 0$.

The following theorem shows that conformal anomalies coincide for ζ and heat kernel determinants, and therefore do not depend on the chosen regularization procedure.

Theorem 9. *Let* g_0 *be a fixed riemannian metric on* M. *Let* $g_\alpha \equiv e^{\alpha\varphi} g_0$, *for* $\alpha \in [0,1]$, *with* $\varphi \in C^\infty(M,\mathbb{R})$. *Set* $\Delta_\alpha \equiv \Delta_{g_\alpha}$, *a one-parameter family of Laplacians. Then*

$$\frac{\det'(-\Delta_{g_1})}{\det'(-\Delta_{g_0})} = \frac{\det_\zeta(-\Delta_{g_1})}{\det_\zeta(-\Delta_{g_0})}.$$

Proof. By Theorem 5 we have $\log\det'(-\Delta_\alpha) = \gamma\zeta_{-\Delta_\alpha}(0) + \log(\det_\zeta(-\Delta_\alpha))$. Since $\frac{d}{d\alpha}\zeta_{-\Delta_\alpha}(0) = 0$ (see [18]), we have that

$$\frac{d}{d\alpha}\log(\det'(-\Delta_\alpha)) = \frac{d}{d\alpha}\log(\det_\zeta(-\Delta_\alpha)).$$

Integration of $\frac{d}{d\alpha}\log(\det'(-\Delta_\alpha))$ with respect to α between 0 and 1 produces

$$\frac{\det'(-\Delta_{g_1})}{\det'(-\Delta_{g_0})} = \frac{\det_\zeta(-\Delta_{g_1})}{\det_\zeta(-\Delta_{g_0})}.$$ ☑

6. Partition functions and regularized determinants

For a hermitian vector bundle E on M, the space $C^\infty(M,E)$ of smooth sections of E is endowed with the inner product induced by the hermitian structure $\langle\cdot,\cdot\rangle_x$ of the fiber over $x \in M$, i.e., denoting by $d\mu_g$ the volume form on M with respect to the riemannian metric g:

$$\langle\sigma,\rho\rangle := \int_M \langle\sigma(x),\rho(x)\rangle_x d\mu_g(x), \qquad \forall\sigma,\rho \in C^\infty(M,E).$$

For a field theory with classical action $\langle\phi, A\phi\rangle = \int_M \phi A\phi\, d\mu_g$, the partition function reads

$$Z[A] = \int_{Conf(M)} e^{-\frac{1}{2}\langle\phi,A\phi\rangle}\mathcal{D}\phi, \tag{19}$$

here $\mathcal{D}\phi$ is a heuristic probability measure over the infinite dimensional space Conf(M), that is the space of configurations or smooth sections ϕ of the vector bundle on the manifold M. Such a Gaussian integral on a finite dimensional space $\mathbb{R}^n$ reads

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x,Ax\rangle} dx = (\det A)^{-1/2}, \tag{20}$$

where A is an $n \times n$ matrix. If A is a positive operator on $L^2(M,E)$ with eigenvalues $\{\lambda_j\}$, and if for all j, we have $\lambda_j = 1 + \mu_j$ with $\sum_{j=1}^{\infty} \mu_j < \infty$, the determinant of A is well defined [16]. By Proposition 5, for an elliptic, positive operator A, for any $\epsilon > 0$ the operator $h_\varepsilon(\tilde{A})$ is of the type $1 + B_\varepsilon$, with $\mathrm{tr}(B_\varepsilon) < \infty$. Then $\sum_{k=1}^{n} \log \lambda_k$ converges to $\mathrm{tr}(\log h_\varepsilon(\tilde{A})) := \lim_{n\to\infty} \sum_{k=1}^{n} \log \lambda_k$, whose exponential yields a well defined determinant. Taking a finite part when ε tends to 0 we have the formula

$$\det A = e^{\mathrm{tr}(\log A)}. \tag{21}$$

Physicists set the following Ansatz in analogy with formula (20):

$$Z[A] := (\det A)^{-\frac{1}{2}} ,$$

using an appropriate definition of the determinant of A, namely one of the definitions (10), (13) introduced in the previous sections.

From $Z[A]$ one builds the *effective action*:

$$S = S[A] := -\log Z[A].$$

In physics the effective action is expressed in terms of the *effective lagrangian*

$$S = \int_M \mathcal{L}(x) d\mu_g(x) .$$

From (21), we have the matrix identity $\mathrm{tr}(\log A) = \log(\det A)$, and hence

$$e^{-S} = (\det A)^{-\frac{1}{2}} = e^{-\frac{1}{2}\,\mathrm{tr}(\log A)} ,$$

so that the effective action can be interpreted as $S = \dfrac{1}{2}\,\mathrm{tr}(\log A)$.

Let $L(x,x')$ denote the kernel of the operator $\log A$. Under certain conditions given by Theorem 4, we have $S = \dfrac{1}{2}\displaystyle\int_M L(x,x) d\mu_g(x)$, so that the effective lagrangian is

$$\mathcal{L}(x) = \frac{1}{2} L(x,x) . \tag{22}$$

From $\mathcal{L}$ we can obtain some information of physical interest. The kernel of the operator $\log A$ diverges for $x = x'$, as in the case of the kernel of A^{-1} (see [22]). Thus equation (22) does not make sense; our aim is to give it a precise meaning. To do so, we use results of the previous section.

7. Two Regularization Procedures

7.1. **$\zeta-$regularization.** $\zeta-$regularization of a pseudo-differential operator A with symbol $\sigma \in CS^a$ is a particular instance of more general holomorphic regularization procedures.

By a holomorphic regularization of the symbol σ, $R : \sigma \mapsto \sigma(z, \cdot, \cdot)$, we mean a choice of a holomorphic family of symbols $\sigma(z, \cdot, \cdot) \in CS^{\alpha(z)}$ on a domain $G \subseteq \mathbb{C}$, such that $\sigma(0, x, \xi) = \sigma(x, \xi)$. Correspondingly, we define the family of kernels $K_z^R(x, y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(z, x, \xi)d\xi$. We consider the regularized kernel

$$\widehat{K^R}(x, y) = \operatorname*{fp}_{z=0} K_z^R(x, y) = \begin{cases} K(x, y) & \text{if } x \neq y, \\ \operatorname*{fp}_{z=0} \int_{\mathbb{R}^n} \sigma(z, x, \xi)d\xi & \text{if } x = y. \end{cases}$$

$\zeta-$regularization corresponds to choosing $\sigma(z) = \sigma(AQ^{-z})$, given by the symbol of the product of the operators A and Q^{-z}, where Q is any admissible pseudo-differential operator of positive order q and $z \in \mathbb{C}$.

This has a counterpart on the kernel level; for every z, let $\widehat{K_z}$ denote the regularized kernel corresponding to AQ^{-z}. As we previously pointed out, outside the diagonal $\widehat{K_z}$ is an infinitely differentiable function; on the diagonal $\widehat{K_z}$ has a possible simple pole at $z = 0$.

From Theorem 4, integrating $\widehat{K_z}(x, x)$ on M, we obtain $\operatorname*{fp}_{z=0} \operatorname{tr}(AQ^{-z})$,

$$\operatorname*{fp}_{z=0} \operatorname{tr}(AQ^{-z}) = \lim_{z\to 0} \left(\operatorname{tr}(AQ^{-z}) - \frac{1}{z} \operatorname*{Res}_{z=0} \operatorname{tr}(AQ^{-z}) \right).$$

From (17) we have that $\operatorname{res}(A) = q \cdot \operatorname*{Res}_{z=0} \operatorname{tr}(AQ^{-z})$ and this residue does not depend of the order of Q, so we have the formula

$$\operatorname*{fp}_{z=0} \operatorname{tr}(AQ^{-z}) = \lim_{z\to 0} \left(\operatorname{tr}(AQ^{-z}) - \frac{1}{qz} \operatorname{res}(A) \right).$$

Thus $L(x, x)$ is regularized by $\widehat{K_z}$, the regularized kernel corresponding to $\log(A)Q^{-z}$, and $S = S[A]$ is regularized by $\operatorname*{fp}_{z=0} \operatorname{tr}(\log(A)Q^{-z})$.

7.2. **Point Splitting Procedure.** Following [10], [22], the point splitting procedure to find a renormalization of a divergent quantity defined on the diagonal, boils down to replacing one of the variables x by some nearby point x' and then taking the limit $x' \to x$. A precise mathematical description of this procedure requires introducing a regularizing kernel corresponding to $e^{-\varepsilon Q}$, where ε is a small positive number and Q is a Laplace-Beltrami operator associated to the metric of the manifold and then taking the limit when ε tends to 0.

Separating the variables mathematically comes down to introducing a smoothing kernel K_ε in the following way:

i) We replace the kernel K_A of an operator A by the convolution $K_A \star K_\varepsilon$, where K_ε is the kernel of $e^{-\varepsilon\Delta_g}$, being Δ_g the Laplace-Beltrami operator associated with the metric g in the manifold M or, more generally, choosing K_ε as kernel of $e^{-\varepsilon Q}$ where Q is an elliptic operator with positive principal symbol and of positive order.

ii) We make an expansion in powers of ε. Such an expansion exists since $K_A \star K_\varepsilon$ is the kernel of $Ae^{-\varepsilon Q}$ and we know that $\mathrm{tr}(Ae^{-\varepsilon Q})$ has an asymptotic expansion in ε (Theorem 8), that is related by the Mellin transform to the expansion of $\mathrm{tr}(AQ^{-z})$, according to equation (18).

iii) We take the finite part when ε goes to 0 to obtain (see (18)):

$$\mathop{\mathrm{fp}}_{z=0} \mathrm{tr}(AQ^{-z}) = \mathop{\mathrm{fp}}_{\varepsilon=0} \mathrm{tr}(Ae^{-\varepsilon Q}) - \gamma\frac{\mathrm{res}(A)}{q}.$$

Thus $L(x,x)$ is regularized by $K_{\log(A)}\star K_\varepsilon$, the regularized kernel corresponding to $\log(A)e^{-\varepsilon Q}$, and $S = S[A]$ is regularized by $\mathop{\mathrm{fp}}_{\varepsilon=0} \mathrm{tr}(\log(A)e^{-\varepsilon Q})$.

In this way we find a relation between the $\zeta-$regularization procedure and the point splitting procedure, giving a mathematical interpretation to the last one. The two regularization methods coincide if $\mathrm{res}(A) = 0$, which happens if the homogeneous component of degree $-n$ of the symbol of A is zero ($n = \dim(M)$); that is the case, for example, if A is a differential operator, a classical pseudo-differential operator of non integer order or if the order of A is less than $-n$.

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