Lecturas Matemáticas
Volumen **26** *(2005), páginas 5–11*

On the boundeness of the associated sequence of Mann iteration for several operator classes with applications

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Abstract. We prove that the associated sequence of Mann iteration is decreasing and hence bounded provided that the operator satisfy minimal assumptions. In particular we obtain for a nonexpansive operator that the associated sequence of Ishikawa iteration is decreasing for a nonexpansive operator. Applications to the convergence of Mann iteration are given.

Key words and phrases. Fixed point, associated Mann sequence, associated Ishikawa sequence.

2000 Mathematics Subject Classification. Primary: 40A05, 47H10.

Resumen. Se demuestra que la sucesión asociada a una iteración de Mann es decreciente y, por lo tanto, acotada si el operador satisface unas hipótesis mínimas. En particular, para un operador no expansivo se obtiene que la sucesión asociada a la iteración de Ishikawa es decreciente para un operador no epansivo. Se dan algunas aplicaciones a la convergencia de la iteración de Mann.

1. Preliminaries

Let X be a real inner product space, let $B \subset X$ be a nonempty, convex set. Let $T : B \to B$ be a map. Let $x_1 \in B$, be an arbitrary fixed point. We consider the iteration

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n. \tag{1}$$

The sequence $(\alpha_n)_{n\geq 1}$ satisfies:

$$(\alpha_n)_{n\geq 1} \subset (0,1),\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0. \tag{2}$$

A prototype for $(\alpha_n)_{n\geq 1}$ is $(1/n)_{n\geq 1}$. Iteration (1) is known as Mann iteration, see [2]. Supposing that T has a unique fixed point x^*, we can associated for (1), a nonnegative sequence $(\|x_n - x^*\|)_n$. We will name this sequence *the associated Mann sequence to iteration* (1).

Ishikawa iteration is given by, see [1]:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}. \end{aligned} \tag{3}$$

The sequences $(\alpha_n)_n, (\beta_n)_n$ are in $(0,1)$. Supposing again that T has a unique fixed point x^*, we can associated for Ishikawa iteration, a nonnegative sequence $(\|x_n - x^*\|)_n$. We will name this sequence *the associated Ishikawa sequence.* When T is a nonexpansive map, the associated Ishikawa (respectively Mann) sequence is decreasing.

Proposition 1. *Let X be a linear and normed space, let $B \subset X$ be a nonempty, convex set. Let $T : B \to B$ be a nonexpansive map (i.e.$\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in B$), then the associated Ishikawa sequence is decreasing.*

Proof. Let $x^* = Tx^*$. For all $x, y \in B$ we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - Tx^*)\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|Ty_n - Tx^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|y_n - x^*\| \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|(1-\beta_n)(x_n - x^*) + \beta_n(Ty_n - Tx^*)\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n ((1-\beta_n)\|x_n - x^*\| + \beta_n \|Tx_n - Tx^*\|) \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n ((1-\beta_n)\|x_n - x^*\| + \beta_n \|x_n - x^*\|) \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned}$$ ☑

Setting $\beta_n = 0$, for all $n \in N$, we get a similar result for Mann iteration.

Proposition 2. *Let X be a linear and normed space, let $B \subset X$ be a nonempty, convex set. Let $T : B \to B$ be a nonexpansive map (i.e.$\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in B$), then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing.*

Let us denote by I the identity map.

We need the following definition:

Definition 3. The map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in B. \tag{4}$$

In a real inner product space we have equivalently the following definition: T is strongly pseudocontractive map if there exists $k \in (0, 1)$such that

$$\langle Tx - Ty, x - y \rangle \leq k \|x - y\|^2, \text{ for all } x, y \in B. \tag{5}$$

Remark 4. When $k = 1$ in (5) then we deal with an pseudocontractive map. A map S is (strongly) accretive if and only if $I - S$ is (strongly) pseudocontractive.

The aim of this note is to prove that this sequence converges decreasing to zero when T is a strongly pseudocontractive. Also similar results hold for an accretive map and a strongly accretive map. This will be a good tool for the convergence of Mann iteration without boundeness assumption on the operator.

The following result is proved in [1].

Lemma 5. [1]. *Let X be a real inner product space , the following relation is true for all $x, y \in X$, and for all $\lambda \in (0, 1)$:*

$$\|(1 - \lambda)x + \lambda y\|^2 = (1 - \lambda) \|x\|^2 + \lambda \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2. \tag{6}$$

2. Some useful inequalities

We need the following inequality

Lemma 6. *If X is a real inner product space, then the following relation is true*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle, \quad \text{for all } \ x, y \in X. \tag{7}$$

Proof. For all $x, y \in X$ we have

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2 \langle y, x \rangle \leq \|x\|^2 + 2 \langle y, y \rangle + 2 \langle y, x \rangle \\ &\leq \|x\|^2 + 2 \langle y, x + y \rangle. \end{aligned}$$ ☑

Also we need the convergence of a sequence supplied by an inequality.

Lemma 7. [3]. *Let* $(a_n)_n$ *be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n, \tag{8}$$

where $\lambda_n \in (0,1)$, *for all* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, *and* $\sigma_n = o(\lambda_n)$. *Then* $\lim\limits_{n\to\infty} a_n = 0$.

3. Main results

We are able now to give the following result:

Theorem 8. *Let* X *be a real inner product space and let* $B \subset X$ *be a nonempty convex set, and* $T : B \to B$ *a strongly pseudocontractive operator with constant* k. *Then the associated Mann sequence* $(\|x_n - x^*\|)_n$ *is decreasing.*

Proof. Strongly pseudocontractivity assures the uniqueness of the fixed point. Thus the sequence $(\|x_n - x^*\|)_n$ is well defined. Using (6), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - Tx^*)\|^2 \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|Tx_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 . \end{aligned}$$

If there exists the number k_0 such that $\|x_{k_0} - x^*\| = 0$, then it is easy to see that $\|x_n - x^*\| = 0$, for all $n \geq k_0$. Hence, we have the conclusion.

If $\|x_n - x^*\| \neq 0$, for all $n \in N$, we will show that

$$\frac{\|x_{n+1} - x^*\|^2}{\|x_n - x^*\|^2} = (1-\alpha_n) + \alpha_n \frac{\|Tx_n - x^*\|^2}{\|x_n - x^*\|^2} - \alpha_n(1-\alpha_n)\frac{\|Tx_n - x_n\|^2}{\|x_n - x^*\|^2} \leq 1.$$

The last inequality is equivalent to

$$-1 + \frac{\|Tx_n - x^*\|^2}{\|x_n - x^*\|^2} - (1-\alpha_n)\frac{\|Tx_n - x_n\|^2}{\|x_n - x^*\|^2} \leq 0 \Leftrightarrow$$

$$\frac{\|Tx_n - x^*\|^2}{\|x_n - x^*\|^2} \leq 1 + (1-\alpha_n)\frac{\|Tx_n - x_n\|^2}{\|x_n - x^*\|^2}. \tag{9}$$

The proof is complete if we are able to prove this last inequality. From strongly pseudocontractivity we have

$$\frac{\|Tx_n - x^*\|^2}{\|x_n - x^*\|^2} \leq 1 + k\frac{\|Tx_n - x_n\|^2}{\|x_n - x^*\|^2}.$$

Because $\lim_{n\to\infty} \alpha_n = 0$, there exists n_0 such that for all $n \geq n_0$ we have $\alpha_n \leq 1-k$. Hence to see that

$$1 + k\frac{\|Tx_n - x_n\|^2}{\|x_n - x^*\|^2} \leq 1 + (1-\alpha_n)\frac{\|Tx_n - x_n\|^2}{\|x_n - x^*\|^2}.$$

Therefore (9) is true. ☑

The following remark help us to extend Theorem 8 for a strongly accretive map. Remark 4 assures that the map T is strongly pseudocontractive map with $k \in (0,1)$ if and only if $(I-T)$ is strongly accretive map. Let us to consider the following operator equation

$$Sx = z,$$

where S is a strongly accretive map and z is a given point in X. Consider the map $Tx = z + (I-S)x$, for all $x \in D(S)$. A fixed point for T will be a solution for the equation $Sx = z$. The above (see also Remark 4) assures that if S is a strongly accretive map then T is strongly pseudocontractive and conversely.

Corollary 9. *Let X be a real inner product space and $S : X \to X$ a strongly accretive operator with constant k. Then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing, where $(x_n)_n$ is given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(z + (I-S)\right)x_n,$$

with $(\alpha_n)_n$ satisfying (2),where x^* is a solution for $Sx = z$.

Let us to consider the following operator equation

$$x + Sx = z,$$

where S is a strongly accretive map and z is a given point. Consider the map $Tx = z - Sx$, for all $x \in D(S)$. A fixed point for T will be a solution for the equation $x + Sx = z$.

Remark 10. If S is strongly accretive map then $T = z - S$ is strongly pseudocontractive map.

Corollary 11. *Let X be a real inner product space and $S : X \to X$ a strongly accretive operator with constant k. Then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing, where $(x_n)_n$ is given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(z - S\right)x_n,$$

with $(\alpha_n)_n$ satisfying (2).

4. Applications to the convergence of Mann iteration

We are able to give now the following application.

Theorem 12. *Let X be a real inner product space and $B \subset X$ a nonempty convex closed set. Let $T : B \to B$ be a completely continuous and strongly pseudocontractive map with fixed points. Then $(x)_n$ given by (1) strongly converges to the fixed point of T namely x^*.*

Proof. Strongly pseudocontractivity assures the uniqueness of the fixed point. Inequalities (7) and (5) lead us

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 = \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\|^2 \\
&\leq (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Tx_n - x^*, x_{n+1} - x^* \rangle \\
&\leq (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Tx_n - x^*, x_n - x^* \rangle + \\
&+ 2\alpha_n \langle Tx_n - x^*, (x_{n+1} - x^*) - (x_n - x^*) \rangle \\
&\leq (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n k \|x_n - x^*\|^2 + \\
&+ 2\alpha_n \langle Tx_n - x^*, x_{n+1} - x_n \rangle \\
&\leq \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + 2\alpha_n \|Tx_n - x^*\| \|x_{n+1} - x_n\| \\
&\leq \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + 2\alpha_n (\|Tx_n\| + \|x^*\|) \alpha_n \|Tx_n - x_n\| \\
&\leq \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + 2\alpha_n^2 (\|Tx_n\| + \|x^*\|) (\|Tx_n\| + \|x_n\|) .
\end{aligned}$$

The map T is strongly pseudocontractive, thus from Theorem 8 we know that $(\|x_n - x^*\|)_n$ is decreasing, thus it is bounded. Moreover we get that $(x_n)_n$ is bounded. Because T is completely continuous and $(x_n)_n$ is bounded, one obtain that $(\|Tx_n\|)_n$ is bounded too. Thus there exists a strict positive $M > 0$ such that

$$\|x_{n+1} - x^*\|^2 \leq \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + \alpha_n^2 M.$$

Because $\lim_{n\to\infty} \alpha_n = 0$, there exists a rank such that from there we have $(1-k) \geq \alpha_n$ and then

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + \alpha_n^2 M \\
&\leq (1 - 2(1-k)\alpha_n + (1-k)\alpha_n) \|x_n - x^*\|^2 + \alpha_n^2 M \\
&= (1 - (1-k)\alpha_n) \|x_n - x^*\|^2 + \alpha_n^2 M.
\end{aligned}$$

Let us denote by

$$a_n := \|x_n - x^*\|^2,$$
$$\lambda_n := (1-k)\alpha_n \in (0,1),$$
$$\sigma_n := \alpha_n^2 M.$$

Observe that $\sigma_n = o(\lambda_n)$ and inequality (8) is satisfied. Thus from Lemma 7, we get $\lim_{n\to\infty} a_n = 0$, that is $\lim_{n\to\infty} x_n = x^*$. ☑

Remarks 4 and 10 are good tools to extend Theorem 13 for the strongly accretive and accretive case.

Corollary 13. *Let X be a real inner product space and $B \subset X$ a nonempty convex closed set. Let $T : B \to B$ be a completely continuous and strongly accretive map with fixed points. Then $(x)_n$ given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n (z + (I-S))\, x_n,$$

with $(\alpha_n)_n$ satisfying (2), strongly converges to the solution of $Sx = z$.

Corollary 14. *Let X be a real inner product space and $B \subset X$ a nonempty convex closed set. Let $T : B \to B$ be a completely continuous and accretive map with fixed points. Then $(x_n)_n$ given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n (z - S)\, x_n,$$

with $(\alpha_n)_n$ satisfying (2), strongly converges to the solution of $x + Sx = z$.

Usually for the convergence of Mann iteration, either the set B or the sequence $(\|Tx_n\|)_n$ is bounded. From the above results one can see that in a inner product space both assumptions are redundant.

References

[1] S. Ishikawa, *Fixed Points by a New Iteration Method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.
[2] W. R. Mann, *Mean Value in Iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.
[3] X. Weng, *Fixed Point Iteration for Local Strictly Pseudocontractive Mapping*, Proc. Amer. Math. Soc. **113** (1991), 727-731.

(Recibido en junio de 2004. Aceptado para publicación en enero 2005)

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