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Cohomology of OS Algebras for Quadratic Arrangements

Kelly Jeanne Pearson
Mathematics Department, Murray State University, USA

Abstract. The Orlik–Solomon algebra is a graded algebra defined by the partially ordered set of subspace intersections of the hyperplanes in an arrangement. Define the cohomology of an Orlik–Solomon algebra as that of the complex formed by its homogeneous components with the differential defined via multiplication by an element of degree one. We study the dimension of the Orlik–Solomon algebra when the arrangement is quadratic and the element defining the multiplication is concentrated under a rank two element in the lattice of the arrangement.

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Resumen. El álgebra de Orlik–Solomon es una álgebra graduada definida por el conjunto ordenado de los espacios intersecciones de los hiperplanos en un arreglo. Se define la cohomología de una de estas álgebras como el complejo formado por sus componentes homogéneas con la diferencial definida por vía de la multiplicación por un elemento de grado uno. Se estudia la dimensión del álgebra cuando el arreglo es cuadrático y el elemento que define la multiplicación se concentra bajo un elmento de rango dos en el retículo del arreglo.

1. Introduction

The theory of hyperplane arrangements is an area of mathematics with applications in algebra, combinatorics, topology, analysis (hypergeometric functions), and physics (KZ-equations); for example, see [5], [6], [13], [10], [4]. The allure of hyperplane arrangements lies both in the straightforward definitions needed to begin studying the topic but, more importantly, in the ability to pose interesting, yet understandable, problems and examples. We therefore begin our discussion with two motivating examples.

Example 1.1. It is not a difficult task to determine that removing n distinct points from the real line leaves $n+1$ regions. However, by raising the dimension just one, determining the number of regions which remain in the plane after removing n lines is dependent on the lines themselves and not merely n. For instance, removing the collection of lines in $\mathbb{R}^2$ given by $\{x = 0, y = 0, x + y = 0\}$ leaves 6 regions. But the collection $\{x = 0, y = 0, x + y = 1\}$ leaves 7 regions when removed from the plane. This question, of course, can be raised to any dimension: given a collection of codimension one affine spaces in $\mathbb{R}^\ell$, how many regions are left when this collection is removed from $\mathbb{R}^\ell$?

In Example 1.1, we considered a finite collection of affine subspaces of codimension one in $\mathbb{R}^\ell$. More generally, we can take F to be be any field and define the same notion.

Definition 1.2. Let F be a field. A hyperplane is an affine subspace of codimension one in F^ℓ. A hyperplane arrangement in F^ℓ is a finite collection of hyperplanes in F^ℓ, written $\mathcal{A} = \{H_1, \ldots, H_n\}$.

Example 1.3. We now switch our attention to an arrangement of hyperplanes in $\mathbb{C}^\ell$. In Example 1.1, we considered the space obtained by removing the hyperplanes from $\mathbb{R}^\ell$. Similarly, we define the complement space $M := \mathbb{C}^\ell \setminus \cup_{i=1}^n H$. Momentarily, let $\ell = 1$ and we see the hyperplanes of $\mathbb{C}$ are points in the complex plane (the hyperplanes have complex codimension one); hence, M is path connected. In general, for any hyperplane arrangement in $\mathbb{C}^\ell$ with $\ell \geq 1$, we have M is a path connected space. So, the question of the number of connected components of M is a trivial question. However, one can consider the cohomology algebra with coefficients in a commutative ring $\mathcal{K}$, denoted $H^*(M, \mathcal{K})$, and ask the question: can $H^*(M, \mathcal{K})$ be represented by generators and relations related to the collection of hyperplanes?

Allowing Example 1.1 to guide and motivate us, it is apparent the intersections of the hyperplanes play an important role as to the number of components of the complement space; in fact, the pattern of intersections of the hyperplanes is the determining factor. It is also apparent in Example 1.3 that the pattern of intersections of the hyperplanes is pivotal to understanding $H^*(M, \mathcal{K})$. Encoding the pattern of intersections of the hyperplanes in a combinatorial object is the purpose of the following definition, given first by Zaslavsky in [15].

Definition 1.4. Let $\mathcal{A}$ be an arrangement of hyperplanes in $V = F^\ell$. We define the partially ordered set $L(\mathcal{A})$ with objects given by $\cap_{H \in \mathcal{B}} H$ for $\mathcal{B} \subseteq \mathcal{A}$ and $\cap_{H \in \mathcal{B}} H \neq \emptyset$; order the objects of $L(\mathcal{A})$ opposite to inclusion. Notice $\emptyset \subseteq \mathcal{A}$ gives $V \in L(\mathcal{A})$ with $V \leq X$ for all $X \in L(\mathcal{A})$. For $X \in L(\mathcal{A})$, we define $\operatorname{rank}(X) := \operatorname{codim} X$. We define $\operatorname{rank}(\mathcal{A}) := \max_{X \in L(\mathcal{A})} \operatorname{rank}(X)$.

The problem of expressing $H^*(M, \mathcal{K})$ in terms of generators and relations was first studied by Arnold [1] in the case $\mathcal{A}$ was the braid arrangement and $\mathcal{K} = \mathbb{C}$; that is, $\mathcal{A}$ was the collection of hyperplanes $\{x_i - x_j : 1 \leq i < j \leq \ell\}$. This problem was later studied by Brieskorn [3] for an arbitrary arrangement. Orlik and Solomon [11] have found a purely algebraic characterization of $H^*(M, \mathcal{K})$.

These results can be briefly summarized as follows. An algebra $A(\mathcal{A})$ (referred to as the Orlik–Solomon algebra) over $\mathcal{K}$ is constructed in terms of generators and relations using only $L(\mathcal{A})$. This is a graded algebra with $A(\mathcal{A}) \cong H^*(M, \mathcal{K})$. Hence, in Example 1.3, $H^*(M, \mathcal{K})$ can be determined by $L(\mathcal{A})$.

The Orlik–Solomon algebra $A(\mathcal{A})$ can also be used to answer the question posed in Example 1.1. Zaslavsky has proven in [15] for a hyperplane arrangement in $\mathbb{R}^\ell$, the number of regions of the complement space is the sum of the dimensions of the homogeneous components of $A(\mathcal{A})$; that is, $\sum_{i=1}^{\ell} \dim A_i(\mathcal{A})$.

The answers to the questions posed in Example 1.3 and Example 1.1 are important results in that topological invariants of the complement space were expressed in term of combinatorics. Indeed, a central question in the theory of hyperplane arrangements is the problem of expressing topological invariants of the complement space in terms of combinatorics. In this manner, it is a natural question then to consider a generalization of $H^*(M, \mathcal{K})$ to cohomology with local coefficients.

For $a \in A_1(\mathcal{A})$, one can define a local coefficient system $\mathcal{L}(a)$. It turns out that $H^*(M, \mathcal{L}(a))$ relates closely to the cohomology of the Orlik–Solomon algebra. The connection between $H^*(M, \mathcal{L}(a))$ and the cohomology of the Orlik–Solomon algebra has been studied in many papers, for instance [9].

The cohomology of the Orlik–Solomon algebra is defined below. For a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$, we let $\{a_i : \ H_i \in \mathcal{A}\}$ denote a basis for $A_1(\mathcal{A})$. This basis is discussed in §2.

Definition 1.5. We construct a cochain complex on the graded linear space $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \displaystyle\sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential

$$d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$$

forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik–Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

Recently, there have been many results concerning $\dim H^1(A(\mathcal{A}), a)$; for instance, [5], [14]. In the case char $\mathcal{K} \neq 2$, it has been shown in [9] that $\dim H^1(A(\mathcal{A}), a)$ can be determined by a particular set of elements from $L(\mathcal{A})$. In particular, $\dim H^1(A(\mathcal{A}), a)$ is determined be

$$\mathcal{X}(a) = \{X \in L(2, \mathcal{A}) : \sum_{X \subset H_i} \lambda_i a_i \neq 0, \sum_{X \subset H_i} \lambda_i = 0, |X| > 2\}$$

However, little is known about the higher dimensions $H^p(A(\mathcal{A}), a)$ for $p > 1$ [14], and this is the central concern of this paper.

In §2, the Orlik–Solomon algebra is defined. The definition of $A(\mathcal{A})$ is presented here as can be found in [12].

Definition 1.6. Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement in $V = F^\ell$ for some field F. We fix an order on $\mathcal{A}$; that is, for hyperplanes H_i and H_j in $\mathcal{A}$, we have $H_i < H_j$ if and only if $i < j$.

Let $\mathcal{K}$ be a commutative ring. Let E_1 be the linear space over $\mathcal{K}$ on n generators. Let $E(\mathcal{A}) := \Lambda(E_1)$ be the exterior algebra on E_1. We have $E(\mathcal{A}) = \bigoplus_{p\geq 0} E_p$ is a graded algebra over $\mathcal{K}$. The standard $\mathcal{K}$-basis for E_p is given by

$$\{e_{i_1} \cdots e_{i_p} :\ 1 \leq i_1 < \ldots < i_p \leq p\}.$$

Any ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S := e_{i_1} \cdots e_{i_p}$ in $E(\mathcal{A})$.

Definition 1.7. We define the map $\partial : E(\mathcal{A}) \to E(\mathcal{A})$ via the usual differential. That is,

$$\begin{aligned} \partial(1) :=& 0 \\ \partial(e_i) :=& 1, \end{aligned}$$

and for $p \geq 2$,

$$\partial(e_{i_1} \cdots e_{i_p}) := \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p}\ .$$

Definition 1.8. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be a subset of $\mathcal{A}$. We say S is dependent if $\bigcap S \neq \emptyset$ and $\operatorname{rank}(\bigcap S) < |S|$.

Definition 1.9. We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by

$$\{\partial(e_S) : S \text{ is dependent }\} \cup \{e_S :\ \cap S = \emptyset\}.$$

Definition 1.10. The Orlik–Solomon algebra, $A(\mathcal{A})$, is defined as

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A}).$$

Let $\pi : E(\mathcal{A}) \to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

Here is an outline of the paper. In §2, a linear basis for $A(\mathcal{A})$ is defined. In §3, we show this basis can be obtained as normal forms to a Gröbner basis for $I(\mathcal{A})$. We give conditions for when $I(\mathcal{A})$ has a quadratic Gröbner basis; this is dependent not only on $\mathcal{A}$ but on the order of the hyperplanes in $\mathcal{A}$. In this case, we say $\mathcal{A}$ is quadratic with respect to the order. In §4, we deal with a famous class of arrangements called supersolvable arrangements (see [2], [7], [13]). We define supersolvable arrangements here for convenience.

A hyperplane arrangement $\mathcal{A}$ is central if $\cap_{H\in\mathcal{A}} H \neq \emptyset$. Assume $\mathcal{A}$ is central. A pair $(X, Y) \in L(\mathcal{A}) \times L(\mathcal{A})$ is called a modular pair if for all $Z \in L(\mathcal{A})$ with $Z \leq Y$

$$Z \vee (X \wedge Y) = (Z \vee X) \wedge Y.$$

An element $X \in L(\mathcal{A})$ is called modular if (X, Y) is a modular pair for all $Y \in L(\mathcal{A})$. We call $\mathcal{A}$ supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = \bigcap_{H\in\mathcal{A}} H.$$

If $\mathcal{A}$ is supersolvable, we say the order on the hyperplanes respects the supersolvable structure if for a maximal modular chain

$$V = X_0 < X_1 < \cdots < X_\ell = \bigcap_{H\in\mathcal{A}} H$$

in $L(\mathcal{A})$ we have

1. X_1 is the smallest hyperplane, i.e. $X_1 = H_1$
2. For $i > 1$, we have $X_i = \cap_{j=1}^{n_i} H_j$ and if a hyperplane $H < X_i$ then $H \in \{H_1, \ldots, H_{n_i}\}$.

For $\mathcal{A}$ supersolvable, if the order respects the supersolvable structure then the respective Gröbner basis is quadratic. We use this characterization throughout §4 and §5. The following is an assumption maintained thoughout §4 and §5.

Condition A. Let $\mathcal{A}$ be a hyperplane arrangement with $\bigcap_{i=1}^{n} H_i \neq \emptyset$, and assume $\mathcal{A}$ is supersolvable. Fix $X \in L(\mathcal{A})$ with $\operatorname{rank}(X) = 2$ and X a member of a maximal modular chain in $L(\mathcal{A})$. Fix an order on the hyperplanes so that the order respects the supersolvable structure.

We consider $a \in A_1(\mathcal{A})$ so $a = \sum_{H_i < X} \lambda_i a_i$. Again, we assume $a \neq 0$ and $\sum_{i=1}^{n} \lambda_i = 0$. We call such an a concentrated under X.

We show $\dim H^k(A(\mathcal{A}), a)$ is determined combinatorially by a main result of the paper.

Theorem 4.13. *Let $\mathcal{A}$ and $X \in L(\mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Then we can compute the Hilbert series for $H^*(A(\mathcal{A}), a)$ in terms of the Hilbert series for $A(\mathcal{A})$ as follows:*

$$H(H^*(A(\mathcal{A}), a), t) = \frac{t(n_X - 2)}{1 + t(n_X - 1)} H(A(\mathcal{A}), t).$$

In §5, we study the kernel, $Z(a) = \oplus Z_i(a)$, of the chain complex $(A(\mathcal{A}), a)$ as an ideal of $A(\mathcal{A})$. We do this with the idea in mind that if $Z_k(a) = A_k(\mathcal{A}) \cdot Z_1(a)$, then $\mathcal{X}(a)$ together with $\dim A_k(\mathcal{A})$ will determine $\dim Z_k(a)$. We show in the case $\mathcal{A}$ and $X \in L(\mathcal{A})$ satisfy Condition A with a concentrated under X, this result holds, except for the top dimension. This is given in the following result.

Theorem 5.10. *Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Suppose $\ell \geq 3$. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. We have $Z_k(a)$ is generated by $Z_1(a)$ for $k < \ell$.*

2. The Orlik–Solomon Algebra and the Broken Circuit Basis

In this section, we define the Orlik–Solomon algebra and a linear basis for this algebra, referred to as the broken circuit basis; see Chapter 3 in [12]. The Orlik–Solomon algebra is a factor algebra of the exterior algebra by an ideal $I(\mathcal{A})$. In §2, we show the relationship between the broken circuit basis and a Gröbner basis for $I(\mathcal{A})$.

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement in $V = F^\ell$ for some field F. For each $H_i \in \mathcal{A}$ fix an affine functional α_i with $\operatorname{Ker}\alpha_i = H_i$. We fix an order on $\mathcal{A}$; that is, for hyperplanes H_i and H_j in $\mathcal{A}$, we have $H_i < H_j$ if and only if $i < j$.

Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ as defined in §1, and let $A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A})$ be The Orlik–Solomon algebra as defined in §1. Let $\pi : E(\mathcal{A}) \to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

We demonstrate that $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module by defining the broken circuit basis for $A(\mathcal{A})$. By Theorem 2.2 2 to follow, this is indeed a basis for $A(\mathcal{A})$.

Definition 2.1. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be an ordered subset of $\mathcal{A}$ with $i_1 < \cdots < i_p$. We say a_S is basic in $A_p(\mathcal{A})$ if

1. S is independent, and
2. For any $1 \leq k \leq p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ with $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ dependent.

The set of $\{a_S\}$ with S as above form the broken circuit basis for $A(\mathcal{A})$, whose name is justified by the following theorem.

Theorem 2.2. *As a $\mathcal{K}$-module, $A(\mathcal{A})$ is a free, graded module. The broken circuit basis forms a basis for $A(\mathcal{A})$.*

Proof. This is proven in Theorem 3.55 in [12]. ☑

The following example demonstrates the use of the broken circuit basis for computing $\dim A_p(\mathcal{A})$.

Example 2.3. Let dim $V = \ell$, and let $\mathcal{A}$ be the braid arrangement in V given by

$$Q(\mathcal{A}) = \prod_{1 \le i < j \le \ell} (x_i - x_j).$$

Let H_{ij} correspond to the hyperplane given by $x_i - x_j = 0$. Order the hyperplanes lexicographically; that is, $H_{ij} < H_{mn}$ if either $i < m$ or $i = m$ and $j < n$. We will write $a_{H_{ij}} = a_{ij}$ in $A_1(\mathcal{A})$.

In order to compute dim $A_p(\mathcal{A})$, we need to describe the elements of the broken circuit basis in $A_p(\mathcal{A})$. Let $a := a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_p j_p}$ be an element of the broken circuit basis in $A_p(\mathcal{A})$. By definition of the hyperplanes, we have $i_k < j_k$.

Suppose $j_1 = j_2$. Without loss of generality, we may assume $i_1 < i_2$. Then $\{H_{i_1 j_1}, H_{i_2 j_2}, H_{i_1 i_2}\}$ is dependent with $H_{i_1 i_2}$ being minimal in the set; this contradicts the assumption a is in the broken circuit basis. In a similar fashion, we have $j_1 < j_2 < \cdots < j_p$. Moreover, if $i_1 = i_2$, then $\{H_{i_1 j_1}, H_{i_2 j_2}, H_{j_1, j_2}\}$ is dependent; but the minimal element of this set is $H_{i_1 j_1}$. Therefore, a is still an element of the broken circuit basis. Hence, there are no restrictions on i_k other than $j_k > i_k$.

It is now just a matter of counting the possibilities we have for $\{i_1 j_1, ..., i_p j_p\}$ with the restrictions $j_1 < j_2 < \cdots < j_p$ and $i_k < j_k$ for $k = 1, ..., p$.

Fix $j_1, ..., j_p$. There are $\ell - j_k$ choices for i_k for each $k = 1, ..., p$. Thus,

$$\begin{aligned} \dim\ A_p(\mathcal{A}) &= \sum_{i_p = 1 + i_{p-1}}^{\ell - 1} \cdots \sum_{i_2 = 1 + i_1}^{\ell - p + 1} \sum_{i_1 = 1}^{\ell - p} \Big(\prod_{k=1}^{p} (\ell - j_k) \Big) \\ &= \sum_{1 \le j_1 < j_2 < \cdots < j_p \le \ell - 1} j_1 j_2 \cdots j_p. \end{aligned}$$

As usual, if $p = 0$, then this sum is taken to be 1.

The dimensions of $A_1(\mathcal{A})$ and $A_2(\mathcal{A})$ can be easily simplified. Obviously, we have $\dim A_1(\mathcal{A}) = \binom{\ell}{2}$. For the dimension of $A_2(\mathcal{A})$, consider minimally dependent sets of three hyperplanes. Any such set must be of the form $\{H_{ij}, H_{ik}, H_{jk} : \quad i < j < k\}$. There are $\binom{\ell}{3}$ of these sets. Hence, $\dim\ A_2(\mathcal{A}) = \dim\ E_2 - \binom{\ell}{3}$. Using the fact $n = \binom{\ell}{2}$, we arrive at

$$\dim\ A_2(\mathcal{A}) = \frac{\ell(\ell-1)(\ell-2)(3\ell-1)}{24}.$$

3. A Gröbner Basis for $I(A)$

In this section, we establish the relationship between the broken circuit basis and a Gröbner basis for the ideal $I(\mathcal{A})$.

We establish some definitions and notations regarding Gröbner bases. These are standard notations and results which can be found in [8]. We include them here for clarity.

Let V be a module over a commutative ring $\mathcal{K}$. Let $B \subset V$ be a $\mathcal{K}$-basis. Suppose B is ordered with $<$; this means the order is linear and that $(B, <)$ is well ordered.

Definition 3.1. Let $v \in V$. Since B is a $\mathcal{K}$-basis, we can write $v = \sum_{b_i \in B} \alpha_i b_i$ for $\alpha_i \in \mathcal{K}$ and $b_i \in B$. Since B is ordered and there are only finitely many nonzero terms in the summation, there is a maximal element $b_i \in B$ with $\alpha_i \neq 0$; say this element is b_1. We define $\mathrm{Tip}(v) := b_1$.

Definition 3.2. Let $W \subseteq V$. We define $\mathrm{Tip}\, W := \{\mathrm{Tip}(w) : \ w \in W\}$. Define the non-tips of W to be $NT(W) := B \setminus \mathrm{Tip}\, W$.

Theorem 3.3. *Let V be a module over $\mathcal{K}$ with an ordered basis $(B, <)$. Let $W \subseteq V$ be a submodule of V with the condition:*

* *for any $w \in W$, there exists $w' \in W$ such that $\mathrm{Tip}(w) = \mathrm{Tip}(w')$ and $w' = \mathrm{Tip}(w') + \sum \gamma_i b_i$, for $\gamma_i \in \mathcal{K}$ and $b_i \in B \setminus \{\mathrm{Tip}(w')\}$.*

Then $V = W \oplus \langle NT(W) \rangle$.

Proof. We begin by showing $W \cap \langle NT(W) \rangle = 0$. Let $v \in W \cap \langle NT(W) \rangle$. We have $\mathrm{Tip}(v) \in \mathrm{Tip}\, W$ since $v \in W$. But $v \in \langle NT(W) \rangle$ implies $\mathrm{Tip}(v) \in NT(W)$. Hence, $v = 0$ as required.

Suppose $W + \langle NT(W)\rangle \neq V$. Choose $v \in V \setminus (W + \langle NT(W)\rangle)$ with $\mathrm{Tip}(v)$ minimal; that is, $\mathrm{Tip}(v) \leq \mathrm{Tip}(w)$ for any $w \in V \setminus (W + \langle NT(W)\rangle)$. Let $0 \neq \alpha \in \mathcal{K}$ so that $v = \alpha\,\mathrm{Tip}(v) + \sum \alpha_i b_i$ for $\alpha_i \in \mathcal{K}$ and $b_i \in B \setminus \{\mathrm{Tip}(v)\}$.

Suppose $\mathrm{Tip}(v) \in NT(W)$. We construct an element with a smaller tip by considering $v - \alpha\,\mathrm{Tip}(v)$. Then $\mathrm{Tip}(v - \alpha\,\mathrm{Tip}(v)) < \mathrm{Tip}(v)$; hence, $v - \alpha\,\mathrm{Tip}(v) \in W + \langle NT(W)\rangle$. This implies $v - \alpha\,\mathrm{Tip}(v) = w + n$ for $w \in W$ and $n \in \langle NT(W)\rangle$. We solve the equation for v to see that

$$v = w + (n + \alpha\,\mathrm{Tip}(v)) \in W + \langle NT(W)\rangle.$$

This is a contradiction to the choice of v.

Suppose $\mathrm{Tip}(v) \in \mathrm{Tip}\,W$. Then there exists $w \in W$ so that $\mathrm{Tip}(v) = \mathrm{Tip}(w)$. By the condition (*) on W, we may assume $w = \mathrm{Tip}(w) + \sum \gamma_i b_i$ for $\gamma_i \in \mathcal{K}$ and $b_i \in B \setminus \{\mathrm{Tip}(w)\}$. Then $\mathrm{Tip}(v - \alpha w) < \mathrm{Tip}(v)$; hence, by the choice of v, $v - \alpha w \in W + \langle NT(W)\rangle$. This implies $v - \alpha w = w' + n$ for $w' \in W$ and $n \in \langle NT(W)\rangle$. By solving for v, we have $v = (w' + \alpha w) + n \in W + \langle NT(W)\rangle$, a contradiction. ☑

Corollary 3.4. *Let V be a vector space over a field $\mathcal{K}$ with an ordered basis $(B, <)$. If $W \subseteq V$ is a subspace of V, then $V = W \oplus \langle NT(W)\rangle$.*

Proof. It will suffice to show W satisfies condition (*) as given in Theorem 3.3. Let $w \in W$. Then we have that $w = \gamma\,\mathrm{Tip}(w) + \sum \gamma_i b_i$ for $0 \neq \gamma, \gamma_i \in \mathcal{K}$ and that $b_i \in B \setminus \{\mathrm{Tip}(w)\}$. Since W is a subspace of V and $\mathcal{K}$ is a field, we have $\gamma^{-1} w \in W$, and we take $w' := \gamma^{-1} w$. ☑

Definition 3.5. Given a module V over $\mathcal{K}$ with an ordered basis $(B, <)$ and a submodule $W \subseteq V$, we define $\mathcal{G} \subset W$ to be a Gröbner basis of W if $\mathrm{Tip}\,\mathcal{G} = \mathrm{Tip}\,W$.

We now define Gröbner bases in algebras. Again, these are standard and can be found in [8] for the case R is commutative.

Let R be a $\mathcal{K}$-algebra and let B be a $\mathcal{K}$-basis of R. Suppose $(B, <)$ is well ordered; that is, the order is linear and any subset $C \subseteq B$ has a minimal element $c \in C$.

Example 3.6. Consider the exterior algebra on n generators, $E(\mathcal{A})$, with the standard basis $B = \{e_{i_1} \cdots e_{i_p} : \ 1 \leq i_1 < \cdots < i_p \leq p\}$. We can give B the degree lexicographic (DegLex) order. That is,

- if $p < q$, then $e_{i_1} \cdots e_{i_p} < e_{j_1} \cdots e_{j_q}$,
- if $k_0 = \min\{k : \ i_k \neq j_k\}$ with $i_{k_0} < j_{k_0}$, then $e_{i_1} \cdots e_{i_p} < e_{j_1} \cdots e_{j_p}$.

Then B is a $\mathcal{K}$-basis of $E(\mathcal{A})$ and with respect to DegLex, $(B, <)$ is well ordered.

Definition 3.7. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B, <)$ be well ordered. We say B is monomial if for any $b, b' \in B$ we have $\mathrm{Tip}(b'b), \mathrm{Tip}(b'b) \in B$ unless they are zero.

Example 3.8. Consider $E(\mathcal{A})$ with the well ordered basis $(B, <)$ given in Example 3.6. Then B is monomial.

Definition 3.9. Let R be a $\mathcal{K}$-algebra and let B be a $\mathcal{K}$-basis of R. Let $(B, <)$ be well ordered, and let B be monomial. We say the order $(B, <)$ is monomial if the following are satisfied:

1. Let $b_1, b_2, c \in B$ with $b_1 > b_2$. If $cb_i \neq 0$ for $i = 1, 2$, then $\mathrm{Tip}(cb_1) > \mathrm{Tip}(cb_2)$ and $\mathrm{Tip}(b_1c) > \mathrm{Tip}(b_2c)$.
2. If $1 \in B$, then $1 < b$ for all $1 \neq b \in B$. If $1 \notin B$, then for all $b, b' \in B$ we have $\mathrm{Tip}(bb') > b, b'$ and $\mathrm{Tip}(b'b) > b, b'$ unless zero appears.

Example 3.10. Consider the exterior algebra $E(\mathcal{A})$ with the standard basis B ordered with the DegLex order as in Example 3.6. Then $(B, <)$ is monomial.

Definition 3.11. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B, <)$ be well ordered and monomial. Let $\mathcal{G} \subseteq R$. Let $(\mathrm{Tip}\, \mathcal{G}) \subseteq B$ be defined by the smallest set containing $\mathrm{Tip}\, \mathcal{G}$ so that the following holds:

(1) for any $g \in (\mathrm{Tip}\ \mathcal{G})$ and any $b \in B$, we have either $\mathrm{Tip}(bg)$, $\mathrm{Tip}(gb) \in (\mathrm{Tip}\ \mathcal{G})$ or $bg = 0$.

Definition 3.12. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis for R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$. Let $\mathcal{G} \subseteq I$. We say $\mathcal{G}$ is a Gröbner basis for I if $(\mathrm{Tip}\ \mathcal{G}) = \mathrm{Tip}\, I$.

Definition 3.13. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis for R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$. Define the non-tips of I to be $NT(I) := B \setminus (\mathrm{Tip}\, I)$.

Theorem 3.14. *Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$. If $\mathcal{K}$ is a field, then $R = I \oplus \langle NT(I)\rangle$ as $\mathcal{K}$-modules. Moreover, $NT(I)$ is a $\mathcal{K}$-basis for R/I.*

Proof. The statement $R = I \oplus \langle NT(I)\rangle$ as $\mathcal{K}$-modules follows from Corollary 3.4. Let $\pi: \ R \to \langle NT(I)\rangle$ be the canonical projection. It follows that $NT(I)$ is a $\mathcal{K}$-basis for R/I. ☑

Definition 3.15. Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis of R. Let $(B,<)$ be well ordered and monomial. Let $\mathcal{G} \subseteq R$. We say $lc(\mathcal{G}) = 1$ if for any $g \in \mathcal{G}$ with $g = \gamma\,\mathrm{Tip}(g) + \sum \gamma_i b_i$ for $0 \neq \gamma, \gamma_i \in \mathcal{K}$ and $b_i \in B \setminus \{\mathrm{Tip}(g)\}$, we have $\gamma = 1$.

Theorem 3.16. *Let R be a $\mathcal{K}$-algebra, and let B be a $\mathcal{K}$-basis for R. Let $(B,<)$ be well ordered and monomial. Let $I \triangleleft R$ with $I = (\mathcal{G})$ as an ideal in R. Suppose $lc(\mathcal{G}) = 1$. Then $\mathcal{G}$ is a Gröbner basis of I if and only if $R = I \oplus \langle NT(\mathcal{G})\rangle$ as $\mathcal{K}$-modules.*

Proof. Suppose $\mathcal{G}$ is a Gröbner basis of I. Then $\mathrm{Tip}\, I = (\mathrm{Tip}\, \mathcal{G})$ by Definition 3.12. Hence, $NT(\mathcal{G}) = NT(I)$. Since $lc(\mathcal{G}) = 1$, $R = I \oplus \langle NT(\mathcal{G})\rangle$ follows from Theorem 3.3.

Suppose $R = I \oplus \langle NT(\mathcal{G})\rangle$. We need to show $\mathrm{Tip}\, I = (\mathrm{Tip}\, \mathcal{G})$.

Let $g \in \mathrm{Tip}\, \mathcal{G}$ and $b \in B$ so that $\mathrm{Tip}(bg) \neq 0$. Since $g \in \mathrm{Tip}\, \mathcal{G}$, there exists $h \in \mathcal{G}$ so that $\mathrm{Tip}(h) = g$. Since $h \in \mathcal{G}$ and I is generated by $\mathcal{G}$, we have $h \in I$. Hence, $bh \in I$ and $\mathrm{Tip}(bh) \in \mathrm{Tip}\, I$. Since the order is monomial, $\mathrm{Tip}(bh) = \mathrm{Tip}(bg)$ or $bg = 0$. Therefore, $\mathrm{Tip}(bg) \in \mathrm{Tip}\, I$.

Let $g \in \mathrm{Tip}\, I$. Then there exists $h \in I$ so that $\mathrm{Tip}(h) = g$. Since B is a linear basis for R over $\mathcal{K}$, we have $h = \sum \alpha_i b_i \,\mathrm{Tip}(g_i) + \sum \beta_i n_i$ for $\alpha_i, \beta_i \in \mathcal{K}$, $b_i \in B$, $g_i \in \mathcal{G}$, and $n_i \in NT(\mathcal{G})$. Since $R = I \oplus \langle NT(\mathcal{G})\rangle$ and

$h \in I$, we must have $\beta_i = 0$ for all β_i. Hence $g = \mathrm{Tip}(h) \in (\mathrm{Tip}\,\mathcal{G})$ as required. ☑

We now apply this theory to the Orlik–Solomon algebra $A(\mathcal{A})$. Recall that for any set of ordered hyperplanes $S = \{H_{i_1}, \ldots, H_{i_p}\}$, we have $e_S = e_{i_1} \cdots e_{i_p} \in E(\mathcal{A})$.

Theorem 3.17. *Let $A(\mathcal{A})$ be the Orlik–Solomon algebra. Let B be the standard basis for $E(\mathcal{A})$ with the DegLex order. Let*

$$\mathcal{G} = \{\partial(e_S) : \; S \text{ is dependent}\} \cup \{e_S : \; \cap_{H \in S} H = \emptyset\}.$$

$NT(\mathcal{G})$ is a linear basis for $A(\mathcal{A})$.

Proof. By definition, $\mathcal{G}$ generates $I(\mathcal{A})$ as an ideal in $E(\mathcal{A})$. Also, $lc(\mathcal{G}) = 1$.

We show $\mathcal{G}$ is a Gröbner basis of $I(\mathcal{A})$.

Let $\mathrm{Tip}(bg) \in (\mathrm{Tip}\,\mathcal{G})$ for $b \in B$ and $g = \mathrm{Tip}(h)$ for $h \in \mathcal{G}$. Since $\mathcal{G}$ generates $I(\mathcal{A})$, $h \in I(\mathcal{A})$. Since $I(\mathcal{A})$ is an ideal, $bh \in I(\mathcal{A})$, so $\mathrm{Tip}(bh) \in \mathrm{Tip}\, I(\mathcal{A})$. But $\mathrm{Tip}(bh) = \mathrm{Tip}(bg)$.

Let $g \in \mathrm{Tip}\, I(\mathcal{A})$. Then $g = e_S$ for $S = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \mathcal{A}$. We consider different cases for S.

If $\cap_{H \in S} H = \emptyset$, then $e_S \in \mathrm{Tip}\,\mathcal{G}$.

Suppose $\cap_{H \in S} H \neq \emptyset$ for the remainder of the proof.

If S is dependent, then let $H := \min S$. Then $e_{S \setminus \{H\}} \in \mathrm{Tip}\,\mathcal{G}$. We then have $g = \mathrm{Tip}(e_H e_{S \setminus \{H\}}) \in (\mathrm{Tip}\,\mathcal{G})$.

Suppose S is independent. If there exists H_0 with $H_0 < \min S$ and $\{H_0\} \cup S$ is dependent, then by definition of $\mathcal{G}$ we have $g = e_S \in \mathrm{Tip}\,\mathcal{G}$.

Suppose S is independent, and suppose there does not exist $H_0 < \min S$ so that $\{H_0\} \cup S$ is dependent. Then $e_S \in NT(\mathcal{G})$.

We may apply Theorem 3.16 to conclude $\mathcal{G}$ is a Gröbner basis for I and $\langle NT(\mathcal{G}) \rangle$ is a $\mathcal{K}$-basis for $A(\mathcal{A})$. ☑

We now consider the case that $\mathcal{A}$ is central and give a characterization of when $\mathrm{Tip}\,\mathcal{G}$ is generated by elements of degree two; that is, any element $g \in \mathrm{Tip}\,\mathcal{G}$ may be written as $\mathrm{Tip}(e_S e_T)$ for $|T| = 2$

Definition 3.18. A Gröbner basis $\mathcal{G}$ is quadratic if for any $g \in \operatorname{Tip} \mathcal{G}$, there exists $h \in \mathcal{G}$ so that $\deg(h) = 2$ and $g = \operatorname{Tip}(bh)$ or $g = \operatorname{Tip}(hb)$ for some $b \in B$.

Definition 3.19. Let $\mathcal{A}$ be a central hyperplane arrangement. Order the hyperplanes via $<$. Let

$$\begin{aligned} BC := \{S \subseteq \mathcal{A} : \quad & \text{there is } H < \min S \text{ so that } \{H\} \cup S \\ & \text{is minimally dependent}\}. \end{aligned}$$

We say $\mathcal{A}$ is quadratic with respect to $<$ to mean for $S \in BC$, there exists $T \in BC$ with $T \subseteq S$ and $|T| = 2$.

Proposition 3.20. *Let $\mathcal{A}$ be a central hyperplane arrangement. If $\mathcal{A}$ is quadratic under an order $<$ of the hyperplanes, then* $\operatorname{Tip} I(\mathcal{A})$ *is generated by elements of degree two, i.e. $\mathcal{G}$ is a quadratic Gröbner basis.*

Proof. Let $S \subseteq \mathcal{A}$ be dependent. Let $R \subset S$ be minimally dependent. Fix $H_0 := \min R$; let $\tilde{R} := R \setminus \{H_0\}$. Then $\tilde{R} \in BC$. Since $\mathcal{A}$ is quadratic, there exists $T \in BC$ with $T \subseteq \tilde{R}$ and $|T| = 2$. Then $e_T \in \operatorname{Tip} \mathcal{G}$ with degree two. Moreover, $e_{S \setminus \min S} = \operatorname{Tip}(e_{S \setminus (T \cup \min S)} \cdot e_T)$ as required. ☑

A central hyperplane arrangement $\mathcal{A}$ is called supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H.$$

Definition 3.21. Let $\mathcal{A}$ be a central hyperplane arrangement with order $<$ on the hyperplanes. If $\mathcal{A}$ is supersolvable, we say the order on the hyperplanes respects the supersolvable structure if for a maximal modular chain

$$V = X_0 < X_1 < \cdots < X_\ell = \cap_{H \in \mathcal{A}} H$$

in $L(\mathcal{A})$ we have

1. X_1 is the smallest hyperplane, i.e. $X_1 = H_1$
2. For $i > 1$, we have $X_i = \cap_{j=1}^{n_i} H_j$ and if a hyperplane $H < X_i$ then $H \in \{H_1, \ldots, H_{n_i}\}$.

Theorem 3.22. (Björner and Ziegler [2]) *Let $\mathcal{A}$ be a central hyperplane arrangement. $\mathcal{A}$ is supersolvable if and only if $\mathcal{A}$ is quadratic under an order that respects the supersolvable structure.*

Proof. This is Theorem 2.8 in [2]. ☑

Example 3.23. This example illustrates the importance of the choice of order on the hyperplanes. Let the arrangement $\mathcal{A}$ be given by the functionals $\{x, x-y, x+y, y, x-z, x+z, y+z, y-z, z\}$; order the hyperplanes as they are written. Then $\mathcal{A}$ is supersolvable as a maximal chain of modular elements is given by

$$V < \{x = 0\} < \{x = y = 0\} < \{0\}.$$

We can check to see that $\mathcal{A}$ is quadratic with this order by noticing the element $H_1 \cap H_2 \cap H_3 \cap H_4 \in L(\mathcal{A})$ is modular and part of a maximal modular chain in $L(\mathcal{A})$.

However, if $\mathcal{A}$ is given by $\{x-y, x-z, y-z, x, x+y, y, x+z, y+z, z\}$ with the hyperplanes ordered as they are written, then $\mathcal{A}$ is not quadratic under this order because $S = \{H_1, H_2, H_4, H_8\}$ is minimally dependent so $\{H_2, H_4, H_8\} \in BC$. However, $\{H_2, H_4\}, \{H_2, H_8\}, \{H_4, H_8\} \notin BC$. Notice the element $H_1 \cap H_2 \cap H_3 \in L(\mathcal{A})$ is not modular.

4. The Dimension of $H^k(A(\mathcal{A}), a)$ for $\mathcal{A}$ Quadratic

We construct a cochain complex on the homogeneous components of $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik–Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

In this section, we work under special conditions and compute the Hilbert series for $H^*(A(\mathcal{A}), a)$ in terms of the Hilbert series for $A(\mathcal{A})$. We maintain the following assumption throughout the remainder of the paper.

Condition A. Let $\mathcal{A}$ be a central hyperplane arrangement, and assume $\mathcal{A}$ is supersolvable. Fix $X \in L(\mathcal{A})$ with $\operatorname{rank}(X) = 2$ and X a member of a maximal modular chain in $L(\mathcal{A})$. Fix an order on the hyperplanes so that the order respects the supersolvable structure. Then we have $\mathcal{A}_X = \{H_1, \dots, H_{n_X}\}$.

Recall from §3 that $\mathcal{A}$ satisfying Condition A implies $\mathcal{A}$ is quadratic under this order.

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a central hyperplane arrangement in V. The lattice, $L(\mathcal{A})$, of subspace intersections formed by the hyperplanes and ordered opposite to inclusion is ranked (via codimension) and atomic. This allows us to discuss the rank of each element from the lattice and to associate to it the hyperplanes which contain it. The following notational conventions are maintained throughout the remainder of the paper.

Notational Conventions

1. For $X \in L(\mathcal{A})$, we write $i \in X$ to mean X is contained in the hyperplane H_i.
2. For $X \in L(\mathcal{A})$, we write $X = \{i_1, ..., i_p\}$ to mean
   (i) X is the intersection of the hyperplanes $\{H_{i_1}, ..., H_{i_p}\}$,
   (ii) if $X \subseteq H$ then $H \in \{H_{i_1}, ..., H_{i_p}\}$.
3. If $\operatorname{rank}(X) = p$, then we write $X \in L(p, \mathcal{A})$.

Theorem 4.1. *Let $\mathcal{A}$ be a central hyperplane arrangement. Let $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. If $\sum_{i=1}^{n} \lambda_i \neq 0$, then $H^*(A(\mathcal{A}), a) = 0$.*

Proof. This is given in Proposition 2.1 in [14]. ☑

By Theorem 4.1, we may now assume $\sum_{i=1}^{n} \lambda_i = 0$.

Definition 4.2. Let B_k be the set of indices $\vec{j} \subseteq \{1, \dots, n\}$ so that $a_{\vec{j}}$ is basic (with respect to the broken circuit basis) in $A_k(\mathcal{A})$.

Definition 4.3. Let M_k be the matrix of the map $d_k : A_k \xrightarrow{a\cdot} A_{k+1}$ in the broken circuit basis.

Definition 4.4. Let $X \in L(2,\mathcal{A})$. Let a be a nonzero element of $A_1(\mathcal{A})$; we write $a = \sum_{i=1}^{n} \lambda_i a_i$. Assume $\lambda_i = 0$ for $i \notin X$ and $\sum_{i=1}^{n} \lambda_i = 0$. In this case, we say a is concentrated under X.

In the setting of Definition 4.3 and Definition 4.4, M_k is a $|B_{k+1}| \times |B_k|$ matrix. We compute the rank of M_k by considering the span of the column space of M_k. Let $X = \{1, ..., n_X\} \in L(2,\mathcal{A})$. We need to consider the types of basic elements of A_k. Let $\vec{\jmath} = \{j_1, ..., j_p\}$ be a subset of $\vec{n}$. For $\mathcal{A}$ and X satisfying Condition A, we have the following types of elements from B_k.

1. $S = (\alpha, \vec{\jmath})$ for $\vec{\jmath} \in B_{k-1}$ and $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\alpha \in \{1, ..., n_X\}$.
2. $S = (1, \vec{\jmath})$ for $j_1 \in \{2, ..., n_X\}$ and $\vec{\jmath} \in B_{k-1}$.
3. $S = \vec{\jmath}$ for $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\vec{\jmath} \in B_k$.

Lemma 4.5. *Let $\mathcal{A}$ and $X \in L(2,\mathcal{A})$ be as in Condition A. Let $1 < k < \ell$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Fix $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\vec{\jmath} \in B_{k-1}$. Then the set of columns of M_k labeled by $1\vec{\jmath}, 2\vec{\jmath}, ..., n_X\vec{\jmath}$ are the same. If $k = 1$, then the columns of M_k labeled by $1, 2, \ldots, n_X$ are the same.*

Proof. Fix $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\vec{\jmath} \in B_{k-1}$. Notice $(\alpha, \vec{\jmath}) \in B_k$ for any $\alpha \in \{1, ..., n_X\}$. For $\alpha \in \{1, ..., n_X\}$, we have

$$a \cdot a_{\alpha\vec{\jmath}} = \sum_{i<\alpha} \lambda_i a_{i\alpha\vec{\jmath}} - \sum_{i=\alpha+1}^{n_X} \lambda_i a_{\alpha i\vec{\jmath}}.$$

If $\alpha = 1$, then we have $a \cdot a_{\alpha\vec{\jmath}} = -\sum_{i=2}^{n_X} \lambda_i a_{1i\vec{\jmath}}$. If $\alpha > 1$, then we have

$$a \cdot a_{\alpha\vec{\jmath}} = \lambda_1 a_{1\alpha\vec{\jmath}} + \sum_{1<i<\alpha} \lambda_i a_{i\alpha\vec{\jmath}} - \sum_{i=\alpha+1}^{n_X} \lambda_i a_{\alpha i\vec{\jmath}}.$$

However,

$$\begin{aligned} a_{i\alpha\vec{\jmath}} =& a_{1\alpha\vec{\jmath}} - a_{1i\vec{\jmath}}, \\ a_{\alpha ij} =& a_{1ij} - a_{1\alpha j}, \text{ and} \\ \sum_{i=1}^{n} \lambda_i =& 0 \end{aligned}$$

implies $a \cdot a_{\alpha\vec{\jmath}} = -\sum_{2\leq\alpha\leq n_X} \lambda_i a_{1i\vec{\jmath}}$. Therefore, the $\alpha\vec{\jmath}$ columns are the same for any $1 \leq \alpha \leq n_X$ as required. Since $\mathcal{A}$ is quadratic under this order, $a_{1i\vec{\jmath}} \neq 0$. That is, if $\{H_1, H_i, H_{\vec{\jmath}}\}$ is dependent, then $\{H_i, H_{\vec{\jmath}}\}$ is minimally dependent since $\vec{\jmath} \in B_{k-1}$. Hence, $\{H_i, H_{j_k}\}$ is minimally dependent for some j_k. But this implies $H_{j_k} \in X$, a contradiction.

In the case $k = 1$, the same proof works. ☑

In light of the above theorem, we define

$$|\,\vec{\jmath} \in B_0 :\ \vec{\jmath} \subseteq \{n_X + 1, ..., n\}| := 1$$

for ease in computations.

Lemma 4.6. *Let $\mathcal{A}$ be a central hyperplane arrangement with* $\operatorname{rank}(\mathcal{A}) = \ell$. *Let $0 < k < \ell$. Let $X = \{1, ..., n_X\}$ be in $L(2, \mathcal{A})$. Let $0 \neq a \in A_1$ be concentrated under X. Fix $\vec{\jmath} \in B_{k-1}$ with $j_1 \in \{2, \ldots, n_X\}$. The column of M_k labeled by $1\vec{\jmath}$ is the zero column.*

Proof. This is immediate since any three elements under X are dependent; in particular, we have

$$a \cdot a_{1\vec{\jmath}} = \sum_{i=1}^{n_X} \lambda_i a_i a_{1\vec{\jmath}} = 0. \square$$

Lemma 4.7. *Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Let $0 < k < \ell$. The set of columns given by $\vec{\jmath}$ for $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\vec{\jmath} \in B_k$ are linearly independent.*

Proof. This follows because $a_{i\vec{\jmath}}$ is basic in $A_{k+1}(\mathcal{A})$ for $i \in \{1, ..., n_X\}$ since $\mathcal{A}$ is quadratic under this order. Indeed, if $a_{i\vec{\jmath}}$ is not basic, then we have two cases. Let $S = \{H_{j_1}, \ldots, H_{j_k}\}$. If $\{H_i\} \cup T$ is dependent for any $T \subseteq S$, then $a_{\vec{\jmath}}$ is not basic, a contradiction. If there exists $H < H_i$ so that $\{H, H_i\} \cup S$ is dependent, then this set is minimally dependent since a_S is basic. Since $\mathcal{A}$ is quadratic, this implies $H_{j_k} < X$ for some k, a contradiction. ☑

Theorem 4.8. *Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 < k < \ell$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. We have*

$$\operatorname{rank} d_k = \left|\left\{\vec{\jmath} \in B_{k-1} : \ \vec{\jmath} \subseteq \{n_X + 1, ..., n\}\right\}\right| + \\ \left|\left\{\vec{\jmath} \in B_k : \ \vec{\jmath} \subseteq \{n_X + 1, ..., n\}\right\}\right|.$$

Proof. Lemmata 4.5, 4.6, and 4.7 imply the $\operatorname{rank} d_k$ is the number of $1\vec{\jmath}$ for $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\vec{\jmath} \in B_{k-1}$ and the number of $\vec{\jmath}$ for $\vec{\jmath} \subseteq \{n_X + 1, ..., n\}$ and $\vec{\jmath} \in B_k$. Notice in the case that $k = 0$, we have $\operatorname{rank} d_0 = 1$ since $a \neq 0$. ☑

Theorem 4.9. *Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 < k < \ell$. Let $0 \neq a \in A_1$ be concentrated under X. We have $\dim Z_k(a) = (n_X - 1) \operatorname{rank} d_{k-1}$.*

Proof. We use Theorem 4.8 and calculate:

$$\begin{aligned}
\dim Z_k(a) =& \dim A_k - \operatorname{rank} d_k \\
=& |\{\vec{\jmath} \in B_k\}| - \left|\left\{\vec{\jmath} \in B_{k-1} : \ \vec{\jmath} \subseteq \{n_X + 1, ..., n\}\right\}\right| \\
& - \left|\left\{\vec{\jmath} \in B_k : \ \vec{\jmath} \subseteq \{n_X + 1, ..., n\}\right\}\right| \\
=& \left|\left\{\vec{\jmath} \in B_k : \ j_1 \in \{1, ..., n_X\}\right\}\right| \\
& - \left|\left\{\vec{\jmath} \in B_{k-1} : \ \vec{\jmath} \subseteq \{n_X + 1, ..., n\}\right\}\right|.
\end{aligned}$$

Consider the first term above. Since $\mathcal{A}$ is quadratic, for any $\alpha \in X$ and $\vec{\jmath} \in B_{k-2}$, we have $1\alpha\vec{\jmath} \in B_k$. Hence,

$$\left|\left\{\vec{\jmath} \in B_k : \ j_1 \in \{1, ..., n_X\}\right\}\right| = |\{\alpha\vec{\jmath} \in B_k : \ \alpha \in X, \vec{\jmath} \in B_{k-1}, j_1 > n_X\}| \\ + |\{1\alpha\vec{\jmath} \in B_k : \alpha \in X, \vec{\jmath} \in B_{k-2}, j_1 > n_X\}|.$$

Returning to our calculations, we now have

$$\begin{aligned}\dim Z_k(a) =& |\{\alpha\vec{\jmath} \in B_k : \ \alpha \in X, \vec{\jmath} \in B_{k-1}, j_1 > n_X\}| + \\ & |\{1\alpha\vec{\jmath} \in B_k : \alpha \in X, \vec{\jmath} \in B_{k-2}, j_1 > n_X\}| - \\ & |\{\vec{\jmath} \in B_{k-1} : j_1 > n_X\}|.\end{aligned}$$

Consider the first and third terms. Since $\mathcal{A}$ is quadratic, for any $\vec{\jmath} \in B_{k-1}$ with $j_1 > n_X$, we have $\alpha\vec{\jmath} \in B_k$ for any $\alpha \in X$. Hence, the sum of the first and third terms can be expressed as $(n_X - 1)|\{\vec{\jmath} \in B_{k-1} : \ j_1 > n_X\}|$. The middle term as written above is $|\{1\alpha\vec{\jmath} \in B_k : \ \alpha \in X, \vec{\jmath} \in B_{k-2}, u_1 > n_X\}|$, and gives $n_X - 1$ choices for α. Hence, the middle term can be simplified to $(n_X - 1)|\{\vec{\jmath} \in B_{k-2} : \ j_1 > n_X\}|$. Continuing with our calculations, we have

$$\begin{aligned}\dim Z_k(a) =& (n_X - 1) \cdot |\{\vec{\jmath} \in B_{k-1} : j_1 > n_X\}| + \\ & (n_X - 1) \cdot |\{\vec{\jmath} \in B_{k-2} : j_1 > n_X\}| \\ =& (n_X - 1) \ \operatorname{rank} d_{k-1}. \ \boxtimes\end{aligned}$$

Theorem 4.10. *Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $k < \ell$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Then*

$$\dim H^k(A(\mathcal{A}), a) = (n_X - 2) \operatorname{rank} d_{k-1}.$$

Proof. We use Theorems 4.9 and 4.9 to compute:

$$\begin{aligned}\dim H^k(A(\mathcal{A}), a) &= \dim Z_k(a) - \operatorname{rank} d_{k-1} \\ &= (n_X - 1) \operatorname{rank} d_{k-1} - \operatorname{rank} d_{k-1} \\ &= (n_X - 2) \operatorname{rank} d_{k-1}. \ \boxtimes\end{aligned}$$

Theorem 4.11. *Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. For $0 < k < \ell$, we have*

$$\dim H^k(A(\mathcal{A}), a) = (n_X - 2) \sum_{i=1}^{k} (-1)^{i-1} (n_X - 1)^{i-1} \dim A_{k-i},$$

and for $k = \ell$, we have

$$\dim H^\ell(A(\mathcal{A}), a) = \dim A_\ell + \sum_{i=1}^{\ell} (-1)^{i} (n_X - 1)^{i-1} \dim A_{\ell-i}.$$

Proof. We consider the first statement. For $k = 1$, the statement clearly holds true as $\dim H^1(A(\mathcal{A}), a) = n_X - 2$. Fix $1 < k < \ell - 1$ and suppose the statement is true for $k-1$. By Theorem 4.10, Theorem 4.9, and the induction hypothesis, we have

$$\begin{aligned}
\dim H^k(A(\mathcal{A}), a) =& (n_X - 2) \operatorname{rank} d_{k-1} \\
=& (n_X - 2)[\dim A_{k-1} - \dim Z_{k-1}(a)] \\
=& (n_X - 2) \dim A_{k-1} - (n_X - 2) \dim Z_{k-1}(a) \\
=& (n_X - 2) \dim A_{k-1} - (n_X - 1) \dim Z_{k-1}(a) \\
& \qquad\qquad + \dim Z_{k-1}(a) \\
=& (n_X - 2) \dim A_{k-1} - (n_X - 1) \dim Z_{k-1}(a) \\
& \qquad\qquad + (n_X - 1) \operatorname{rank} d_{k-2} \\
=& (n_X - 2) \dim A_{k-1} - (n_X - 1) \dim H^{k-1}(A(\mathcal{A}), a) \\
=& (n_X - 2) \dim A_{k-1} \\
& - ((n_X - 1)(n_X - 2) \sum_{i=1}^{k-1} (-1)^{i-1} (n_X - 1)^{i-1} \dim A_{k-1-i} \\
=& (n_X - 2) \sum_{i=1}^{k} (-1)^{i-1} (n_X - 1)^{i-1} \dim A_{k-i}.
\end{aligned}$$

We now consider the second statement. We first prove for $1 \leq k < \ell$,

$$\dim Z_k(a) = \sum_{i=1}^{k} (-1)^{i-1} (n_X - 1)^{i} \dim A_{k-i}. \qquad (*)$$

For $k = 1$, (*) holds since $\dim Z_1(a) = n_X - 1$. Fix $1 < k < \ell$ and suppose (*) holds for $k-1$. Then

$$\begin{aligned}
\dim Z_k(a) =& (n_X - 1)\operatorname{rank} d_{k-1} \\
=& (n_X - 1)(\dim A_{k-1} - \dim Z_{k-1}(a)) \\
=& (n_X - 1)\dim A_{k-1} \\
& \qquad - (n_X - 1)\sum_{i=1}^{k-1}(-1)^{i-1}(n_X - 1)^i \dim A_{k-1-i} \\
=& \sum_{i=1}^{k}(-1)^{i-1}(n_X - 1)^i \dim A_{k-i}.
\end{aligned}$$

Hence, (*) is true for all $1 \leq k < \ell - 1$ and we use it to prove the second statement of the theorem.

Indeed, we have the following which proves the theorem:

$$\begin{aligned}
\dim H^\ell(A(\mathcal{A}), a) &= \dim A_\ell - \operatorname{rank} d_{\ell-1} \\
&= \dim A_\ell - \dim A_{\ell-1} + \dim Z_{\ell-1}(a) \\
&= \dim A_\ell - \dim A_{\ell-1} \\
& \qquad + \sum_{i=1}^{\ell-1}(-1)^{i-1}(n_X - 1)^i \dim A_{\ell-1-i} \\
&= \dim A_\ell + \sum_{i=1}^{\ell}(-1)^{i}(n_X - 1)^{i-1} \dim A_{\ell-i}. \quad \square
\end{aligned}$$

Definition 4.12. The Hilbert series of a graded algebra A over $\mathcal{K}$ is defined to be

$$H(A,t) := \sum_{i=1}^{\infty}(\dim_{\mathcal{K}} A_i)t^i.$$

Theorem 4.13. *Let $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ be as in Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. We can compute the Hilbert* *series for $H^*(A(\mathcal{A}),a)$ in terms of the Hilbert series for $A(\mathcal{A})$ as follows:*

$$H(H^*(A(\mathcal{A}),a),t) = \frac{t(n_X-2)}{1+t(n_X-1)} H(A(\mathcal{A}),t).$$

Proof. In the proof of Theorem 4.11, we have for $1 \le k < \ell$

$$\dim H^k(A(\mathcal{A}),a) = (n_X-2)\dim A_{k-1} - (n_X-1)\dim H^{k-1}(A(\mathcal{A}),a).$$

So, the series holds for $k < \ell$.

We now check for $k = \ell$. For $k < \ell$, we have

$$\begin{aligned}\dim Z_k(a) =& (n_X-1)\operatorname{rank} d_{k-1} \\ =& (n_X-1)(\dim A_{k-1} - \dim Z_{k-1}(a)).\end{aligned}$$

Hence, we may use the series $\sum_{i=0}^{\infty} \frac{t(n_X-1)}{1+t(n_X-1)} H(A(\mathcal{A}),a)$ to compute $\dim Z_k(a)$ for $k < \ell$. Since $\dim H^\ell(A(\mathcal{A}),a) = \dim A_\ell - \dim A_{\ell-1} + \dim Z_{\ell-1}$, we find $\dim H^\ell(A(\mathcal{A}),a)$ by taking the coefficient of t^ℓ in the series $(1+t)H(A(\mathcal{A}),t) + \frac{t(n_X-1)}{1+t(n_X-1)} H(A(\mathcal{A}),a)$. By obtaining a common denominator and adding, we have $\dim H^\ell(A(\mathcal{A}),a)$ is given by the coefficient of t^ℓ in the series

$$\frac{t(n_X-2)}{1+t(n_X-1)} H(A(\mathcal{A}),t)$$

as required. ☑

5. The Ideal $Z(a) = \oplus Z_k(a)$ for $\mathcal{A}$ quadratic

We now consider $Z(a) = \oplus Z_k(a)$ as an ideal of $A(\mathcal{A})$. We endeavor to show that if $\mathcal{A}$ and $X \in L(2,\mathcal{A})$ are as in Condition A with a concentrated under X, then we have $Z_k(a)$ is generated by $Z_1(a)$ (that is, $Z_k(a) = A_{k-1}(\mathcal{A}) \cdot Z_1(a)$) except in the top dimension ℓ.

We recall the following description of $Z_1(a)$ from Libgober and Yuzvinsky [9].

Theorem 5.1. *Let $\mathcal{A}$ be a central hyperplane arrangement. Let $x \in A_1(\mathcal{A})$ with $x = \sum_{i=1}^{n} x_i a_i$. Then $x \in Z_1(a)$ if and only if the following conditions hold:*

1. *For every $Y \in L(2)$ with $|Y| > 2$ and $a(Y) \neq 0$ but $\sum_{i \in Y} \lambda_i = 0$, we have $\sum_{i \in Y} x_i = 0$.*
2. *For every other $Y \in L(2)$ and every pair $i < j$ from Y, we have $\lambda_i x_j - \lambda_j x_i = 0$.*

We use this description to prove the following lemma.

Lemma 5.2. Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X = \{1, ..., n_X\}$ be in $L(2, \mathcal{A})$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. If $z, w \in Z_1(a)$ and both nonzero, then $z \in Z_1(w)$ and $\dim(z \cdot A_1(\mathcal{A})) = \dim(w \cdot A_1(\mathcal{A}))$.

Proof. Let $z, w \in Z_1(a)$. It will suffice to show $z \in Z_1(w)$. We show conditions (1) and (2) above hold for any $Y \in L(2, \mathcal{A})$. Let $Y \in L(2)$ with $Y = \{i_1, ..., i_k\}$. We consider the following three cases.

Case 1. Suppose $Y = X$. If $|X| > 2$, then since $z, w \in Z_1(a)$, $a(X) \neq 0$, and $\sum_{i \in X} \lambda_i = 0$, condition (1) gives $\sum_{i \in X} z_i = \sum_{i \in X} w_i = 0$ as required. If $|X| = 2$, then condition (2) together with $a(X) \neq 0$ gives $z_1 = -z_2$ and $w_1 = -w_2$; hence, $z_1 w_2 - z_2 w_1 = 0$ as required.

Case 2. Suppose $i_1 > n_X$. In this case, we have $a(Y) = 0$. It will suffice to show $z(Y)$ and $w(Y)$ are both zero. Since $a \neq 0$, we may assume without loss of generality that $\lambda_1 \neq 0$. Consider the element $W_j \in L(2)$ which contains $\{H_1, H_{i_j}\}$. Then $a(W_j) \neq 0$ and $\sum_{i \in W_j} \lambda_i = \lambda_1 \neq 0$. By condition (2), we have $z_{i_j} = w_{i_j} = 0$ for all $1 \leq j \leq k$.

Case 3. Suppose $i_1 \in X$. Then $\sum_{i \in Y} \lambda_i = \lambda_{i_1}$. If $\lambda_{i_1} \neq 0$, then by condition (2), $z_{i_j}, w_{i_j} = 0$ for all $j > 1$. Hence, $z_{i_j} w_{i_m} - z_{i_m} w_{i_j} = 0$ for any $H_{i_m}, H_{i_j} \in Y$. If $\lambda_{i_1} = 0$, then we follow the same approach as Case 2 to obtain $z(Y)$ and $w(Y)$ are linearly dependent. In particular, assume $\lambda_1 \neq 0$. Then consider W_j as defined previously, noting $W_1 = X$. We have $z_{i_j} = w_{i_j} = 0$ for all $2 \leq j \leq k$. Hence, $z(Y)$ and $w(Y)$ are linearly dependent. The lemma now follows. ☑

Lemma 5.3. *Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X \in L(2,\mathcal{A})$ with $X = \{1, ..., n_X\}$. Let $0 \neq a \in A_1$ be concentrated under X. Assume $\lambda_1 \neq 0$. Then $Z_1(a)$ has a basis given by $\{a_1 - a_k\}$ for $2 \leq k \leq n_X$.*

Proof. By straightforward computation and the assumption $\sum_{i=1}^{n_X} \lambda_i = 0$, we have that $a_1 - a_k \in Z_1(a)$ for $2 \leq k \leq n_X$. Indeed, we compute

$$\begin{aligned} a \cdot (a_1 - a_k) =& (\sum_{i=1}^{n_X} \lambda_i a_i)(a_1 - a_k) \\ =& -\sum_{i=2}^{n_X} \lambda_i a_{1i} - \sum_{i<k} \lambda_i a_{ik} + \sum_{k<i<n_X} \lambda_i a_{ki}. \end{aligned}$$

Since $a_{ik} = a_{1k} - a_{1i}$ and $a_{ki} = a_{1i} - a_{1k}$, we substitute and have

$$\begin{aligned} a \cdot (a_1 - a_k) =& -\sum_{i=1}^{n_X} \lambda_i a_{1k} \\ =& 0. \end{aligned}$$

Obviously, $\{a_1 - a_k : \ 2 \leq k \leq n_X\}$ is a set of linearly independent elements from $A_1(\mathcal{A})$. Let $z \in Z_1(a)$. By the proof of Lemma 5.2, we have $z_i = 0$ for any $i > n_X$. Moreover, $\sum_{i=1}^{n_X} z_i = 0$ implies z is a linear combination of $\{a_1 - a_k : \ 2 \leq k \leq n_X\}$. ☑

Theorem 5.4. *Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X \in L(2,\mathcal{A})$ so that $X = \{1, ..., n_X\}$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated in $X \in L(2,\mathcal{A})$. We have the following description of $Z_1(a)$:*

$$Z_1(a) = \left\{ \sum_{i=1}^{n} x_i a_i : \ x_j = 0 \text{ for } j \notin X, \sum_{i=1}^{n} x_i = 0 \right\}$$

Proof. This follows immediately from Lemma 5.3. ☑

Lemma 5.5. *Let $\mathcal{A}$ be a central hyperplane arrangement. Let $X \in L(2,\mathcal{A})$ with $X = \{1, ..., n_X\}$. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under*

X. Let z_i, z_k be basic elements of $Z_1(a)$ as given in Lemma 5.3. We have $A_1(\mathcal{A})z_i \cap A_1(\mathcal{A})z_k = 0$.

Proof. Suppose $z_i = a_1 - a_i$ and $z_k = a_1 - a_k$. Let $\gamma \in A_1(\mathcal{A})$. Then by computation

$$z_i\gamma = \left(\sum_{j=1}^{n_X} \gamma_j\right)a_{1i} + \sum_{j>n_X} \gamma_j a_{1j} - \sum_{j>n_X} \gamma_j a_{ij}.$$

So, for $z_i\gamma = z_k\sigma$ with $\gamma, \sigma \in A_1(\mathcal{A})$, we have

$$\left(\sum_{j=1}^{n_X} \gamma_j\right)a_{1i} + \sum_{j>n_X} \gamma_j a_{1j} - \sum_{j>n_X} \gamma_j a_{ij} = \left(\sum_{j=1}^{n_X} \sigma_j\right)a_{1k} + \sum_{j>n_X} \sigma_j a_{1j} - \sum_{j>n_X} \sigma_j a_{kj}.$$

Since $i \neq k$, $\sum_{j=1}^{n_X} \gamma_j = \sum_{j=1}^{n_X} \sigma_j = 0$. Since $i \neq k$ and $n_X < j \leq n$, a_{kj} and a_{ij} are distinct basic elements of $A_2(\mathcal{A})$; this forces $\sigma_j = \gamma_j = 0$ for $n_X < j \leq n$. By Theorem 5.4, this implies $\gamma, \sigma \in Z_1(a)$ as required. ☑

Theorem 5.6. *Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. We have $Z_2(a)$ is generated by $Z_1(a)$, i.e. $Z_2(a) = A_1(\mathcal{A}) \cdot Z_1(a)$.*

Proof. We follow the argument given in Theorem 4.9 and compute

$$\dim Z_2(a) = (n_X - 1)(n - n_X) + n_X - 1.$$

By using Lemma 5.2 and Lemma 5.5, we compute $\dim A_1(\mathcal{A}) \cdot Z_1(a)$ to be

$$(n_X - 1)(n - n_X + 1).$$

Since these two quantities are equal and we have the containment $A_1(\mathcal{A}) \cdot Z_1(a) \subseteq Z_2(a)$, the result now follows. ☑

Lemma 5.7. *Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $\ell \geq 3$. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. Let $\vec{\jmath} \in B_k$ for $k < \ell$. Suppose $\gamma a_{\vec{\jmath}} \in Z_k(a)$ for some $\gamma \in \mathcal{K}$. If $j_1 > n_X$, then $\gamma = 0$.*

Proof. Suppose $j_1 > n_X$. Since $\mathcal{A}$ is quadratic, $a_{\alpha\vec{\jmath}} \in B_{k+1}$ for any $\alpha \in X$. Since $\gamma a_{\vec{\jmath}} \in Z_k(a)$, we must have $\gamma = 0$. ☑

Lemma 5.8. *Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. Let $\vec{\jmath} \in B_k$ for $2 \leq k < \ell$. Suppose $a_{\vec{\jmath}} \in Z_k(a)$. If $j_1 = 1$ and $j_2 \in X$, then $a_{\vec{\jmath}} \in A_1(\mathcal{A}) \cdot Z_1(a)$.*

Proof. Without loss of generality, we may assume $\lambda_1 \neq 0$. Suppose $j_1 = 1$ and $j_2 \in X$. Then $(a_1 - a_\alpha)a_{1j_2} = 0$ for all $2 \leq \alpha \leq n_X$. Hence, $a_{1j_2} \in Z_2(a)$, and by Theorem 5.6, $Z_2(a)$ is generated by $Z_1(a)$. Thus, $a_{\vec{\jmath}} \in A_1(\mathcal{A}) \cdot Z_1(a)$. ☑

Lemma 5.9. *Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Let $0 \neq a \in A_1(\mathcal{A})$ be concentrated under X. Let $\vec{\jmath}' \in B_{k-1}$ with $\vec{\jmath}' \cap X = \emptyset$. For $k < \ell$, if $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{\jmath}'} \in Z_k(a)$ with $\gamma_\alpha \in \mathcal{K}$, then $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{\jmath}'} \in A_{k-1}(\mathcal{A}) \cdot Z_1(a)$.*

Proof. Suppose $\vec{\jmath} \cap X \neq \emptyset$ with $j_1 \in X$ and $j_2 \notin X$. Then $\vec{\jmath}' := \{j_2, ..., j_k\}$ is in B_{k-1}. Since $\mathcal{A}$ is quadratic, we have $a_{\alpha\vec{\jmath}'} \in B_k$ for any $\alpha \in X$. Assume $\lambda_1 \neq 0$. By Lemma 5.3, we may express a as $a = \sum_{\alpha=2}^{n_X} c_\alpha(a_1 - a_\alpha)$. By computing,

$$aa_{\vec{\jmath}} = \left(\sum_{\alpha=2}^{n_X} c_\alpha\right)a_{1\vec{\jmath}} - \sum_{\alpha=2}^{n_X} c_\alpha a_{\alpha\vec{\jmath}}.$$

But $\alpha\vec{\jmath}$ begins with αj_1 for $2 \leq \alpha \leq n_X$. For $j_1 = 1$, we have

$$aa_{\vec{\jmath}} = \sum_{\alpha=2}^{n_X} c_\alpha a_{1\alpha\vec{\jmath}'}.$$

If $j_1 \neq 1$, then $a_{\alpha\vec{\jmath}}$ is not basic and we have $a_{\alpha\vec{\jmath}} = a_{1\vec{\jmath}} - a_{1\alpha\vec{\jmath}'}$; but we still obtain

$$aa_{\vec{\jmath}} = \sum_{\alpha=2}^{n_X} c_\alpha a_{1\alpha\vec{\jmath}'}.$$

Fix $\vec{j}' \in B_{k-1}$ with $\vec{j}' \cap X = \emptyset$. For any $\alpha \in X$, we have $\alpha\vec{j}' \in B_k$. Let $\gamma_\alpha \in \mathcal{K}$ so that $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'} \in Z_k(a)$ as in the assumption of the lemma. We have

$$a\left(\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'}\right) = \sum_{\alpha=1}^{n_X} \gamma_\alpha \left(\sum_{i=2}^{n_X} c_i a_{1i\vec{j}'}\right) = \sum_{i=2}^{n_X} \left(\sum_{\alpha=1}^{n_X} \gamma_\alpha\right) c_i a_{1i\vec{j}'}$$

Since $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'} \in Z_k(a)$, we have $\sum_{\alpha=1}^{n_X} \gamma_\alpha = 0$. Hence, $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_\alpha \in Z_1(a)$ by Theorem 5.4, so $\sum_{\alpha=1}^{n_X} \gamma_\alpha a_{\alpha\vec{j}'}$ is generated by $Z_1(a)$. ☑

Theorem 5.10. *Suppose $\mathcal{A}$ and $X \in L(2, \mathcal{A})$ satisfy Condition A. Suppose $\ell \geq 3$. Let $a \in A_1(\mathcal{A})$ be a nonzero element concentrated under X. We have $Z_k(a)$ is generated by $Z_1(a)$ for $k < \ell$.*

Proof. Theorem 4.10 shows $Z_2(a)$ is generated by $Z_1(a)$. Let $\gamma \in Z_k(a)$ for $k \geq 3$. Then $\gamma = \sum \gamma_{\vec{j}} a_{\vec{j}}$ for $\vec{j} \in B_k$. We now decompose γ by considering different types of $\vec{j}$. There are three possibilities for $\vec{j}$.

1. Suppose $j_1 > n_X$. Then by Lemma 5.8, we have $\gamma_{\vec{j}} = 0$.
2. Suppose $j_1 = 1$ and $j_2 \in X$. Then by Lemma 5.9, we have $a_{\vec{j}}$ is generated by $Z_1(a)$.
3. Suppose $j_1 \in X$ and $j_2 \notin X$. Then $\vec{j}' = \{j_2, ..., j_k\}$ is in B_{k-1}. We have
$$\sum_{\alpha=1}^{n_X} \gamma_{\alpha\vec{j}'} a_{\alpha\vec{j}'} \in Z_k(a).$$
By Lemma 5.9, this implies $\sum_{\alpha=1}^{n_X} \gamma_{\alpha\vec{j}'} a_{\alpha\vec{j}'}$ is generated by $Z_1(a)$.
Since each summand of γ is generated by $Z_1(a)$, this implies γ is generated by $Z_1(a)$ ☑

We provide examples demonstrating the results of the previous two sections and examples where dropping hypotheses cause the results to fail.

Example 5.11. Let $\mathcal{A}$ be the arrangement given by the functionals $\{x, x-y, x+y, y, x-z, x+z, y+z, y-z, z\}$; order the hyperplanes as they are written. Then $\mathcal{A}$ is supersolvable and the order respects the supersolvable structure. Let a be concentrated under $X = \{1, 2, 3, 4\} \in L(2, \mathcal{A})$. The indices for the broken circuit basis for $A_2(\mathcal{A})$ are

$$\{12, 13, 14, 15, 16, 17, 18, 19, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39, 45, 46, 47, 48, 49\}.$$

Checking Theorem 4.13, we see

$$\begin{aligned}\frac{t(n_X-2)}{1+t(n_X-1)}H(A(\mathcal{A}), t) =& \frac{2t}{1+3t}(1+9t+23t^2+15t^3)\\ =&(2t)(t+1)(5t+1)\\ =&10t^3+12t^2+2t\end{aligned}$$

We now check the dimensions of $H^k(A(\mathcal{A}), a)$ by computing

$$\dim Z_1(a) = 3 \text{ and } \operatorname{rank} d_1 = 6,$$

$$\dim Z_2(a) = 18 \text{ and } \operatorname{rank} d_2 = 23 - 18 = 5.$$

Therefore, the dimensions of $H^k(A(\mathcal{A}), a)$ match the Hilbert series above.

Moreover, $\dim Z_2(a) = 18$ and $\dim A_1 \cdot Z_1(a) = 18$, so $Z_2(a) = A_1 \cdot Z_1(a)$.

Example 5.12. However, if $\mathcal{A}$ is the arrangement given by the functionals $\{x, x-y, x+y, y, x-z, x+z, y+z, y-z, z\}$ with the hyperplanes ordered as they are written, then the indices for the broken circuit basis for $A_2(\mathcal{A})$ are

$$\{12, 13, 14, 15, 16, 17, 18, 19, 24, 25, 26, 27, 28, 29, 34, 35, 36, 37, 38, 39, 48, 59, 67\}.$$

We also have $\mathcal{A}$ is not quadratic under this order because $S = \{H_1, H_2, H_4, H_8\}$ is minimally dependent but $|\{H_2, H_4, H_8\}| \neq 2$. Notice the element $H_1 \cap H_2 \cap H_3 \in L(\mathcal{A})$ is not modular. Even though $\mathcal{A}$ is supersolvable arrangement, we show the formulas derived earlier do not hold in this case because the order does not respect the supersolvable structure. Let a be concentrated under $\{1,2,3\} \in L(2,\mathcal{A})$. Then $\dim Z_2(a) = 17$ and $\operatorname{rank} d_1 = 7$, so $\dim Z_2(a) \neq 2 \cdot \operatorname{rank} d_1$.

Moreover, $\dim Z_2(a) = 17$ and $\dim A_1 \cdot Z_1(a) = 14$, so $Z_2(a) \neq A_1 \cdot Z_1(a)$.

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Kelly Jeanne Pearson, Departament of Mathematics
Murray State University, Murray , Murray, KY 42071, USA
e-mail: kelly.pearson@murraystate.edu