Lecturas Matemáticas
Volumen **18** *(1997), páginas 131-149*

Nonstandard construction of Brownian motion and G-martingales on Lie groups

Myriam Muñoz de Özak

Universidad Nacional de Colombia, Bogotá

Abstract. A nonstandard construction of Brownian motion is given and a lifting theorem for semi-martingales on Lie groups is proved. Nonstandard representations of stochastic exponential and logarithm are introduced in order to render easy the definition of G-martingales, which on Lie groups correspond to Γ-martingales on manifolds.

Key words and phrases. Manifolds, Lie algebras and Lie groups, Brownian motion, martingales, internal martingales, S-continuity, stochastic processes, stochastic differential equations, liftings of stochastic processes and of martingales.

1991 Mathematics Subject Classification. Primary 60J65. Secondary 60B15.

0. Preliminaries

For a good introduction to nonstandard analysis see [1]. The main notions and results needed in our work are the following:

We assume the existence of a set ${}^*\mathbb{R} \supseteq \mathbb{R}$, called the set of the nonstandard real numbers, and of a mapping $* : V(\mathbb{R}) \to V({}^*\mathbb{R})$, where $V_1(S) = S$,

The author acknowledges partial support from CINDEC (Universidad Nacional de Colombia), COLCIENCIAS (Colombia) and D.A.A.D. (Germany). .

$V_{n+1}(S) = V_n(S) \cup \mathfrak{P}(V_n(S))$, $\mathfrak{P}(V_n(S))$ being the set of subsets of $V_n(S)$, and $V(S) = \cup_{n \in \mathbb{N}} V_n(S)$, with three basic properties. To state these properties we introduce the following notions.

An **elementary statement** is a statement Φ built up from " $=$ ", " $\in$ ", the predicate and functional variables, the logical conectives "*and*","*or*", "*not*" and "*implies*", and the bounded quantifiers $(\forall u \in v)$, $(\exists u \in v)$.

An **internal object** A is an element of $V({}^*\mathbb{R})$ such that $A = {}^*S$, $S \in V(\mathbb{R})$. A set in $V({}^*\mathbb{R})$ which is not internal is called external.

(1) **Extension Principle**. The set ${}^*\mathbb{R}$ is a proper extension of $\mathbb{R}$ and $* : V(\mathbb{R}) \to V({}^*\mathbb{R})$ is an embedding such that ${}^*r = r$ for all $r \in \mathbb{R}$.

(2) **The Saturation Property**: Let $\{R_n : n \in \mathbb{N}\}$ be a sequence of internal objects and $\{S_m : m \in \mathbb{N}\}$ be a sequence of internal sets. If for each $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that for all $n \geq N_m$, $R_n \in S_m$, then $\{R_n : n \in \mathbb{N}\}$ can be extended to an internal sequence $\{R_\eta : \eta \in {}^*\mathbb{N}\}$ such that $R_\eta \in \cap_m S_m$ for every $\eta \in {}^*\mathbb{N} - \mathbb{N}$.

(2') **General Saturation Principle**: Let κ be an infinite cardinal. A nonstandard extension is called κ-saturated if for every family $\{X_i\}_{i \in I}$, with $card(I) < \kappa$, and the finite intersection property, the intersection $\cap_{i \in I} X_i$ is nonempty; i.e., this intersection contains some internal object.

(3) **Transfer Principle**: Let $\Phi(X_1, \cdots, X_m, x_1, \cdots, x_n)$ be an elementary statement in $V(\mathbb{R})$. Then, for any $A_1, \cdots, A_m \subseteq \mathbb{R}$ and $r_1, \cdots, r_n \in \mathbb{R}$,

$$\Phi(A_1, \cdots, A_m, r_1, \cdots, r_n)$$

is true in $V(\mathbb{R})$ if and only if

$$\Phi({}^*A_1, \cdots, {}^*A_m, {}^*r_1, \cdots, {}^*r_n)$$

is true in $V({}^*\mathbb{R})$.

System $({}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq)$ extends $(\mathbb{R}, +, \cdot, \leq)$ as an ordered field. In general, we will omit the $*$ for the operations and the order relation.

In ${}^*\mathbb{R}$ we can distinguish three kinds of numbers:

(a) $x \in {}^*\mathbb{R}$ is infinitesimal, if $|x| < r$ for each $r \in \mathbb{R}^+$.

(b) $x \in {}^*\mathbb{R}$ is a finite number, if there is a real number $r \in \mathbb{R}^+$ such that $|x| < r$.

(c) $x \in {}^*\mathbb{R}$ is an infinite number, if $|x| > r$ for each $r \in \mathbb{R}^+$.

To each finite number $x \in {}^*\mathbb{R}$ we can associate a unique real number $r := st(x) := {}^o x$ such that $x = r + \epsilon$, where ϵ is infinitesimal. We say that x is infinitely closed to y, and write $x \approx y$ if and only if $x - y$ is an infinitesimal.

In general, we use capital letters H, F, X, etc. for internal functions and processes, while h, f, x, etc. are used for standard ones. For stopping times ([11]) we will always use capital letters, and specify whether standard or non-standard is meant.

For a given set A, $^*A$ stands for the elementary extension of A, and $ns(^*A)$ denotes the nearstandard points in $^*A$. If s is an element in $ns(^*A)$, the standard part of s is denoted by $st(s)$ or ^os. For a given function f, $^*f$ stands for the elementary extension of f.

We say that the set T is S-dense if $\{^o\underline{t} : \underline{t} \in T, {}^o\underline{t} < \infty\} = [0,\infty)$ and $ns(T) := \{\underline{t} \in T : {}^o\underline{t} < \infty\}$. With T we denote an internal S-dense subset of $^*[0,\infty)$. The elements of T, or more generally, of $^*[0,\infty)$, are denoted by $\underline{s}$, $\underline{t}$, $\underline{u}$, etc... . The real numbers in $[0,\infty)$ are denoted by s, t, u, etc... We will work with different sets T, so we will always specify the definition of such T.

With $\mathbb{N}$ we denote the set of nonzero natural numbers $\{1,2,3,\cdots\}$, and $\mathbb{N}_o = \mathbb{N} \cup \{0\}$. Elements of $\mathbb{N}_o$ are denoted with n, m, l, etc..., while elements in $^*\mathbb{N} - \mathbb{N}$ will be denoted by η, N, etc... .

If $(\Omega, \mathfrak{A}, \mu)$ is an internal measure space, the corresponding Loeb space is $\underline{\Omega} = (\Omega, L(\mathfrak{A}), L(\mu))$, and $L(\mu)$ will be the unique measure extending $^o\mu$ to the σ-algebra $\sigma(\mathfrak{A})$ generated by $\mathfrak{A}$. $L(\mathfrak{A})$ will stand for the $L(\mu)$ completion of $\sigma(\mathfrak{A})$.

When we say that $F : A \to B$ is an internal function, we mean that the domain, range and graph of F are internal concepts.

1. Introduction

There has been recently a great interest among researchers in the field of probability for a better understanding of brownian motion on Lie groups. The first paper in the field (F. Perrin: "Etude mathématique de Mouvement Brownien de Rotation". Ann. Ecole Normale Sup. 45) is of 1928. It was followed much later by K. Ito ([14], 1950), K. Yosida ([32], 1952) and H.P. McKean ([23], 1960, [24], 1969). McKean introduces the notion of a multiplicative stochastic integral and proceeds to construct the brownian motion in the Lie group by means of the exponential map applied to the brownian motion defined in the Lie algebra of the group. He uses Ado's Theorem ([28]) to identify locally the Lie group as a subgroup of $GL(m.\mathbb{R})$, and defines the brownian motion as the limit

$$\begin{aligned} \zeta_n(t) &= 1 \qquad\qquad (t=0),\\ \zeta_n(t) &= \zeta_n(l2^{-n})\exp[a(t)-a(l2^{-n})] \quad (t \geq 0, l=[2^n t]),\\ \zeta_\infty(t) &= \lim_{n\to\infty} \zeta_n(t), \end{aligned}$$

where $a = \sqrt{e}b + ft$, $e, f \in C^1(\mathbb{R})$ and b is a brownian motion in the Lie algebra of the group.

Later on M. Ibero [12], M. Emery [7] and R.L. Karandikar [17, 18, 19] looked at the multiplicative integral respectively from the points of view of diffusions, discontinuous semimartingales and stochastic calculus, resorting to integration by parts. They also restrict themselves to matrix Lie groups.

Finally, M. Hakim-Dowek and D. Lépingle [9] resort to the geometry of the Lie group, based on the formalism of Stochastic Differential Geometry as developed by L. Schwartz [28, 29], P. Malliavin [22], J.M. Bismut [3], P.A. Meyer [25, 26], N. Ikeda and S. Watanabe [13] and K.D. Elworthy [6] for manifolds, applied to the particular case of the Lie groups. They do not resort to the multiplicative integral. Instead, they define the brownian motion as a solution to a certain stochastic differential equation. The geometric point of view allows them, by handling in an appropriate way the Stratonovich integral, to avoid the use of tangent vectors or order-2 differential forms, the route followed by Meyer and Schwartz. They introduce the stochastic exponential of a semi-martingale in the Lie algebra of the group as a solution of the stochastic differential equation

$$dX_t = X_t dM_t,$$

where M_t is a semi-martingale in the Lie algebra of the group. To this purpose, given a martingale M on the Lie algebra, they define a sequence of stochastic processes on the Lie group by

$$X_o^n = X_o,$$
$$X_t^n = X_{t_k}^n \exp\left(\frac{t - t_k}{R2^n}(M_{t_{k+1}} - M_{t_k})\right),$$

where $0 \leq \frac{kR}{2^n} = t_k \leq t \leq t_{k+1} = \frac{(k+1)R}{2^n} \leq R$. Then they observe that on $[0, R]$ this sequence is uniformly convergent in probability to a solution of the differential equation. They call this solution the stochastic exponential of M. In the special case where the semi-martingale is the brownian motion on the Lie algebra, the stochastic exponential is the brownian motion on the Lie group. They denote by $\mathfrak{E}(M)$ the stochastic exponential of M, define the stochastic logarithm $\mathfrak{L}(X)$ of a semi-martingale X on the Lie group, and by means of the approximation by sums of terms $\log(X_{t_k}^{-1} X_{t_{k+1}})$, and they conclude that $\mathfrak{E}(M)$ and $\mathfrak{L}(X)$ are inverse operators. Finally they define a G-martingale on the Lie group as the stochastic exponential of a local martingale in the Lie algebra of the group. This corresponds to the notion of Γ-martingale of R.W.R Darling [5] and P.A. Meyer [25] associated to a connection Γ on the Lie group.

In this paper we intend, using [9] and nonstandard analysis techniques, to define brownian motion and one and two parameter G-martingales on Lie groups. In this respect, Loeb's results [21] on conversion from nonstandard to standard measure spaces, which allowed Anderson [2] to give nonstandard representations for brownian motion and Ito's integration in a very simple form, as well as the development by Hoover and Perkins [11], and independently by Lindstrϕm [20], of one parameter integration with respect to a semi-martingale, have allowed us to give a nonstandard definition of brownian motion and G-martingales on Lie groups.

2. Brownian motion on Lie groups

Let G be a Lie group of dimension d. We assume polysaturation (i.e. *card* $(V(G))$ - saturation (see [1])) for the embedding $^* : V(G) \to V(^*G)$ in order to define concepts like monad and nearstandardness in topological spaces.

If E is a topological space, $s \in E$ and O_s is the family of open sets containing s, the monad $\mu(s)$ of s is

$$\mu(s) := \cap\{^*O : O \in O_s\}.$$

We say that $s \in {}^*E$ is nearstandard, if there is $t \in E$ such that $s \in \mu(t)$.

Remark 1.

(i) $A \subseteq G$ is an open set if and only if $\mu(s) \subseteq {}^*A$ for all $s \in A$.
(ii) $A \subseteq G$ is closed if and only if for all $s \in G$ and $t \in {}^*A$, if $t \in \mu(s)$ then $s \in A$.
(iii) $A \subseteq G$ is compact if and only if for all $s \in {}^*A$ there is a $t \in A$ such that $s \in \mu(t)$. This means that all the points of A are nearstandard.
(iv) G is Hausdorff if and only if for $s, t \in G$, $s \neq t$, $\mu(s) \cap \mu(t) = \emptyset$.

A topological manifold G of dimension d is a topological space with the following properties:

(i) G is Hausdorff.
(ii) G is locally euclidean of dimension d, i.e., for each point $p \in G$ there is a neighborhood U of p which is homeomorphic to an open set U' in $\mathbb{R}^d$.
(iii) G has a countable basis of open sets.

By means of the transfer principle we can give an internal definition of a topological manifold.

A differentiable or C^∞ structure on a topological manifold M is a family $\mathfrak{U} = \{U_\alpha, \phi_\alpha\}$ of coordinate neighborhoods such that:

(i) The U_α cover M, i.e. $\cup U_\alpha = M$

(ii) For any α, β, the coordinate neighborhoods (U_α, ϕ_α) and (U_β, ϕ_β) are C^∞ compatible.

(iii) Any coordinate neighborhood (V, ψ) compatible with every $(U_\alpha, \phi_\alpha) \in \mathfrak{U}$ is itself in $\mathfrak{U}$.

A C^∞ manifold M is a topological manifold together with a C^∞ structure. If (U_α, ϕ_α) is a coordinate neighborhood, by the transfer principle, ${}^*U_\alpha$ is a neighborhood in ${}^*M$, ${}^*\phi_\alpha \in {}^*C^\infty$, and as we can choose a countable system $(U_n, \phi_n)_{n\in\mathbb{N}}$ of coordinate neighborhoods in M such that $\cup_n U_n = M$, we also can extend the system $({}^*U_n, {}^*\phi_n)_{n\in\mathbb{N}}$ to an internal system $({}^*U_n, {}^*\phi_n)_{n\in{}^*\mathbb{N}}$ of coordinate neighborhoods in ${}^*M$ such that $\cup_{n\in{}^*\mathbb{N}}{}^*U_n = {}^*M$. It also follows that $d^*\phi_\alpha \approx {}^*(d\phi_\alpha)$ and that ${}^*(\phi \circ \psi^{-1}) = {}^*\phi \circ {}^*\psi^{-1}$.

Let G be a Lie group, which we assume connected and locally compact. Then G is a C^∞ manifold with the group operation $(a, b) \in G \times G \to ab^{-1} \in G$ being a C^∞ mapping of $G \times G$ into G. Thus, by the transfer principle, also the operation $(a, b) \in {}^*G \times {}^*G \to ab^{-1} \in {}^*G$ is a ${}^*C^\infty$ mapping.

We will denote with e the identity element of the group operation. Then, e is also the identity element of the group operation of ${}^*G$. We recall some basic definitions and notations from standard Lie group theory (see, for example, Helgason [18]).

Let $L_g : G \to G$ be the left translation defined by $L_g(s) = gs$. Let $dL_g : T_p(G) \to T_{L_g(p)}(G)$ be given by

$$dL_g(X_p)f = X_p(f \circ L_g).$$

If X is a vector field in G, we say that X is left invariant if $dL_g(X) = X$ for all $g \in G$, or, more precisely, if $dL_g(X_p) = X_{gp}$ for all $g \in G$. The Lie algebra $\mathfrak{g}$ of the Lie group G is the vector space of all left invariant vector fields on G. A left invariant vector field $H \in \mathfrak{g}$ is uniquely determined by its value at e, because

$$H_g = H_{ge} = dL_g(H_e), \quad \text{for all } \ g \in G.$$

Every tangent vector H_e to G at e determines uniquely a left invariant vector field H by $H_g = dL_g(H_e)$. In fact,

$$dL_g(H_p) = dL_g(dL_p(H_e)) = d(L_g \circ L_p)(H_e) = H_{gp}.$$

Thus, by means of the map $H \to H_e$, we can identify the Lie algebra $\mathfrak{g}$ with the tangent space of G at e:

$$\mathfrak{g} \simeq T_e(G)$$

The elements of $\mathfrak{g}$ are left invariant, first order differential operators on G, and if $f : G \to \mathbb{R}$ is of class C^∞ and $g \in G$ then, for $H \in \mathfrak{g}$,

$$(Hf)(g) = H(f \circ L_g)(e).$$

Suppose $\dim G = d$. Then $\dim \mathfrak{g} = d$. Let (V, ϕ) be a coordinate neighborhood of G at a point p. Define

$$D_j f = \frac{\partial (f \circ \phi^{-1})}{\partial x_j} \circ \phi,$$

for $f : G \to \mathbb{R}$, $f \in C^1$. If $H \in \mathfrak{g}$ and ϕ_j, $\quad j = 1, 2, \cdots, d$, are the components of ϕ, we have

$$(Hf)(\cdot) = D_j f(\cdot) H\phi^j(\cdot).$$

By Ado's Theorem we can identify $\mathfrak{g}$ with a subalgebra of $GL(m, \mathbb{R})$ for some m. Then, if $A \in \mathfrak{g}$, we can define the exponential function as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

The exponential maps a copy of $\mathfrak{g}$ onto a Lie subgroup of the linear group $GL(m, \mathbb{R})$ and this subgroup is locally isomorphic to G. If $\dim \mathfrak{g} = d$ and $\{H_1, \cdots, H_d\}$ is a basis of $\mathfrak{g}$, we can also take $\{H_1, \cdots, H_d\}$ as a basis of ${}^*\mathfrak{g}$ (the elementary extension of $\mathfrak{g}$).

1. Definition. Let $(\Omega, \mathfrak{B}, \overline{P})$ be an internal hyperfinite probability space and let
$(\Omega, L(\mathfrak{B}), L(\overline{P}))$ be the corresponding Loeb space. We say that $M : [0, \infty) \times \Omega \to \mathfrak{g}$ is an $\mathfrak{F}_t$-semi-martingale on $\mathfrak{g}$ if $M_t = M_t^i H_i$ (using the Einstein's convention for sums), where $H_i \in \mathfrak{g}$ for $i = 1, 2, \cdots, d$ and M^i is a real valued semi-martingale for $i = 1, 2, \cdots, d$.

If the M^i are càd- làg for each i, it follows from Theorem 7.6 in [11] that there exist an internal filtration $\{\mathfrak{B}_{\underline{t}}\}$ and for each i a semi-martingale lifting $\overline{M}^i$ of M^i such that $\overline{M}_{\underline{t}} = \overline{M}^i_{\underline{t}} H_i$ is a semi-martingale lifting of M_t.

2. Definition. Let C denote the space of C^∞ real valued functions with compact support on a Lie group G. We say that $X_t : [0, \infty) \times \Omega \to G$ is a semi-martingale on G if for every $f \in C$, $f(X_t)$ is a semi-martingale on $\mathbb{R}$ (on every closed interval).

Almost all concepts we have introduced so far for Lie groups and Lie algebras can be extended by the transfer principle to their elementary extensions. We want to make explicit some definitions though.

If $g_1, g_2 \in {}^*G$, we say that they are infinitely closed, and denote it by $g_1 \approx g_2$, if $g_1^{-1}g_2 \in \mu(e)$.

Remark 2. The group elements $g_1, g_2 \in {}^*G$ are infinitely closed if and only if for all $f \in C$, ${}^*f(g_1^{-1}g_2) \approx {}^*f(e)$. In fact, if $g_1 \approx g_2$, then, by the uniform continuity of f, ${}^*f(g_1^{-1}g_2) \approx {}^*f(e)$, i.e., ${}^*f(g_1) \approx {}^*f(g_2)$. On the other hand, if for any $f \in C$, ${}^*f(g_1^{-1}g_2) \approx {}^*f(e)$, let (V, ϕ) be a coordinate neighborhood of e and U be an open set containing e such that $\overline{U} \subseteq V$ and $\overline{U}$ is compact. Also let ϕ^i, $i = 1, 2, \cdots, d$, be functions in C which coincide with the components of ϕ on $\overline{U}$. Then

$$ {}^*\phi^i(g_1^{-1}g_2) \approx {}^*\phi^i(e), \qquad i = 1, \cdots, d, $$

so that ${}^*\phi(g_1^{-1}g_2) \approx {}^*\phi(e)$, and by the continuity of ϕ^{-1} we have $g_1^{-1}g_2 \approx e$.

3. Definition. Let $X : [0, \infty) \times \Omega \to G$ be a continuous $\mathfrak{F}_t$-semi-martingale. A semi-martingale lifting of $\overline{X}$ of X is an internal S-continuous $\mathfrak{B}_t$-semi-martingale such that, for all $f \in C$, $st({}^*f(\overline{X}_{\underline{t}})) = f(X_{{}^\circ\underline{t}})$ a.s. (i.e., $\overline{P}$- almost everywhere)

We say that the stochastic differential equation

$$ dX_t = X_t dM_t \tag{1} $$

has a solution in G, if there exists a semi-martingale X on G such that for all $f \in C$ we have

$$ \begin{aligned} f(X_t) &= f(X_o) + \int_0^t (H_i f)(X_s) \partial M_s^i \\ &= f(X_o) + \int_0^t (H_i f)(X_s) dM_s^i + \frac{1}{2} \int_0^t (H_i H_j f)(X_s) d[M^i, M^j]_s. \end{aligned} $$

The H_i's are, as before, a basis of $\mathfrak{g}$. The first stochastic integral corresponds to the Stratonovich integral. The second, to Ito's integral.

4. Definition.

(a) A brownian motion on a Lie group is a semi-martingale which satisfies differential equation (1), where $M_t = b_t^i H_i + tH_o$, $\sum_{i=1}^d b_t^i H_i$ is a brownian motion on $\mathfrak{g}$ and $H_o \in \mathfrak{g}$. Its differential generator is the operator $\frac{1}{2} \sum_i H_i^2 + H_0$.

(b) A semimartingale X on G which satisfies (1) is called a stochastic exponential and is denoted by $\mathfrak{E}(M)$.

Let $\overline{M}_{\underline{t}} = \chi^i_{\underline{t}} H_i + \underline{t} H_o$, where $H_o = \sum_{i=1}^{d} a_i H_i \in \mathfrak{g}$, $\{H_i : i = 1, 2, \cdots, d\}$ is a basis of ${}^*\mathfrak{g}$ and the $\chi^i_{\underline{t}}$ are *-independent Anderson's infinitesimal random walks defined on ${}^*[0, R]$ (see [2]). Then $\overline{M}_{\underline{t}}$ is an S-continuous internal $\mathfrak{B}_{\underline{t}}$-semi-martingale.

Let $\eta \in {}^*\mathbb{N} - \mathbb{N}$ and let

$$T := T_\eta = \left\{0, \frac{R}{2^\eta}, \frac{2R}{2^\eta}, \cdots, \frac{(2\eta - 1)R}{2^\eta}, R\right\}$$

be a hyperfinite time line.

Define an internal stochastic process $B : T \times \Omega \to {}^*G$ as follows:

$$\begin{aligned} B_o &= e, \\ B_{\underline{t}} &= \prod_{j=1}^{k+1} {}^*\exp(\overline{M}_{\underline{t}_j} - \overline{M}_{\underline{t}_{j-1}}), \end{aligned} \tag{2}$$

where

$$\underline{t} = \underline{t}_{k+1}, \quad \underline{t}_k = \frac{kR}{2^\eta}, \quad \underline{t} \in T.$$

Then

5. Theorem. *The standard part $st(B_{\underline{t}})$ is a brownian motion with values in G.*

Proof. First we extend the function $B_{\underline{t}}$ continuously to $\underline{t} \in {}^*[0, R]$ by defining $\overline{B}_{\underline{t}}$ as follows:

$$\begin{aligned} \overline{B}_o &= e, \\ \overline{B}_{\underline{t}} &= \prod_{j=1}^{k} {}^*\exp(\overline{M}_{\underline{t}_j} - \overline{M}_{\underline{t}_{j-1}})^* \exp\left((\frac{\underline{t} - \underline{t}_k}{R2^{-\eta}})(\overline{M}_{\underline{t}_{k+1}} - \overline{M}_{\underline{t}_k}) \right), \end{aligned}$$

where

$$\underline{t} \in [\underline{t}_k, \underline{t}_{k+1}], \quad \underline{t}_k = \frac{kR}{2^\eta}, \quad \underline{t} \in T.$$

We have:

(1) $\overline{B}_{\underline{t}}$ is an S-continuous function (see [1]) for $\underline{t} \in {}^*[0, R]$ a.s. If $A \in {}^*\mathfrak{g}$, we can, by Ado's Theorem, identify A with a matrix. So, for fixed A, the function $\exp(\underline{t}A)$ is an S-continuous function. If $\underline{s}, \underline{t} \in [\underline{t}_k, \underline{t}_{k+1}]$ we have $\underline{s} \approx \underline{t}$. Then

$${}^*\exp\left((\frac{\underline{t} - \underline{t}_k}{R2^{-\eta}})(\overline{M}_{\underline{t}_{k+1}} - \overline{M}_{\underline{t}_k}) \right) \approx {}^*\exp\left((\frac{\underline{s} - \underline{t}_k}{R2^{-\eta}})(\overline{M}_{\underline{t}_{k+1}} - \overline{M}_{\underline{t}_k}) \right)$$

and so $\overline{B}_{\underline{t}} \approx \overline{B}_{\underline{s}}$. This implies that $B_{\underline{t}}$ is S- continuous, meaning that $st(B_{\underline{t}})$ is continuous.

(2) Now we show that $st(B_{\underline{t}})$ satisfies the stochastic differential equation (1).

First we make a few observations about some nonstandard concepts in manifolds. Sometimes we will write f and ϕ instead of $^*f$ and $^*\phi$ respectively, if there is no risk of confusion.

If $x(t)$ is a curve of class $^*C^1$ in $^*G$, $a \leq t \leq b$, with $x(t_o) = p$, the tangent vector to the curve $x(t)$ at p is the operator $X : {}^*C_p \rightarrow {}^*\mathbb{R}$ defined by

$$(Xf)_p = \frac{\Delta f(x(t))}{\Delta t}|_{t=t_o} = \frac{f(x(t_o + \delta t)) - f(x(t_o))}{\delta t}$$

with $\delta t \approx 0$. Such X is linear, and for $f, g \in C_p$,

$$X(fg)_p \approx g(p)(Xf)_p + f(p)(Xg)_p,$$

where C_p is the set of functions which are defined and are C^∞ on some neighborhood of p. We call X the internal tangent vector to G at p : $X \in {}^*T_p(G)$. We see that ^oX coincides with the standard notion of a tangent vector.

Given a local chart (U, ϕ) at p with coordinates $x_1, x_2, \cdots, x_d$, define

$$\begin{aligned}(^*\frac{\partial}{\partial x_i})(f)(p)& \\ &= \frac{(f \circ \phi^{-1})(\phi^1(p), \cdots, \phi^i(p) + \delta t, \phi^{i+1}(p), \cdots, \phi^d(p))}{\delta t} \\ &- \frac{f \circ \phi^{-1}(\phi^1(p), \cdots, \phi^i(p), \cdots, \phi^d(p))}{\delta t},\end{aligned}$$

so that $(^*\frac{\partial}{\partial x_i}) \approx (\frac{\partial}{\partial x_i})$. Then $(^*\frac{\partial}{\partial x_i})$, $i = 1, 2, \cdots, d$, is a basis of $^*T_p(G)$. Let $f : {}^*M \rightarrow {}^*M'$ be a $^*C^\infty$ function, where M, M' are manifolds. The differential at p of f is the linear map $f_* : {}^*T_p(M) \rightarrow {}^*T_{f(p)}(M')$ such that the tangent vector $X \in {}^*T_p(M)$ to the curve $x(t)$ at $p = x(t_o)$, $f_*(X)$ is the tangent vector to the curve $f(x(t))$ at $f(p) = f(x(t_o))$. Since $\delta t \approx 0$,

$$f_*(X)(h) = \frac{h(f(x(t_o + \delta t))) - h(f(x(t_o)))}{\delta t} = X(h \circ f).$$

For the left translation $L_g : {}^*G \rightarrow {}^*G$, if we consider the curve $x(t) = \exp((t - t_o)X)$, we have that $x(t_o) = e$ and $\dot{x}(t_o) = X$ $(\dot{x}(t) = dx(\frac{d}{dt})_t$), so that

$$dLg(X)(f) = (L_g)_*(X)(f) = X(f \circ L_g)_e = \frac{f(g\exp((\delta t)X)) - f(g)}{\delta t}|_{t=t_0}.$$

Call ϕ^i the extensions of the coordinate functions x_i of ϕ to a function in ${}^*C^\infty$. Let $\overline{M}_{\underline{t}} = \sum_{i=0}^d \chi^i_{\underline{t}} H_i + \underline{t} H_o$, where $H_o = \sum_{i=1}^d a_i H_i$. Then $\overline{M}^l_{\underline{t}} = \chi^l_{\underline{t}} + a_l \underline{t}$, $B_{\underline{t}_k} = B_{\underline{t}_{k-1}} \exp(\overline{M}_{\underline{t}_k} - \overline{M}_{\underline{t}_{k-1}})$. For $f \in C$, and writting f and ϕ instead of ${}^*f$, ${}^*\phi$ respectively, we have that

$$\begin{aligned} f(B_{\underline{t}_k}) - f(e) &= \sum_{j=1}^k f(B_{\underline{t}_j}) - f(B_{\underline{t}_{j-1}}) \\ &= \sum_{j=1}^k \Bigg[\sum_{i=1}^d ({}^*\frac{\partial}{\partial x_i})(f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}}))\Delta\phi^i(B_{\underline{t}_{j-1}}) \\ &\quad + (f \circ \phi^{-1})(\phi(B_{\underline{t}_j})) - (f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}})) \\ &\quad - \sum_{i=1}^d ({}^*\frac{\partial}{\partial x_i})(f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}}))\Delta\phi^i(B_{\underline{t}_{j-1}}) \Bigg]. \end{aligned}$$

From the Taylor formula for $f \circ \phi^{-1}$ it follows that

$$\begin{aligned} &\Bigg| (f\circ\phi^{-1})(\phi(B_{\underline{t}_j})) - (f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}})) \\ &\quad - \sum_{i=1}^d {}^*(\frac{\partial}{\partial x_i})(f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}}))\Delta\phi^i(B_{\underline{t}_{j-1}}) \\ &\quad - \frac{1}{2} \sum_{i,m=1}^d ({}^*\frac{\partial^2}{\partial x_m \partial x_i})(f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}}))\Delta\phi^i(B_{\underline{t}_{j-1}})\Delta\phi^m(B_{\underline{t}_{j-1}}) \Bigg| \\ \leq\ & C\|\Delta\phi(B_{\underline{t}_{j-1}})\|^3 \approx 0. \end{aligned}$$

Then, by the S-continuity of ${}^*\phi$ we can choose $\eta \in {}^*\mathbb{N} - \mathbb{N}$ such that

$$(f\circ\phi^{-1})(\phi(B_{\underline{t}_j})) - (f\circ\phi^{-1})(\phi(B_{\underline{t}_{j-1}})) - \sum_{i=1}^d ({}^*\frac{\partial}{\partial x_i})(f\circ\phi^{-1})(\phi(B_{\underline{t}_{j-1}}))\Delta\phi^i(B_{\underline{t}_{j-1}})$$

can be replace by

$$\frac{1}{2} \sum_{i,m=1}^d ({}^*\frac{\partial^2}{\partial x_m \partial x_i})(f \circ \phi^{-1})(\phi(B_{\underline{t}_{j-1}}))\Delta\phi^i(B_{\underline{t}_{j-1}})\Delta\phi^m(B_{\underline{t}_{j-1}}),$$

up to equivalence $\approx$. But

$$\begin{aligned}
\Delta\phi^i(B_{\underline{t}_{j-1}}) &= \frac{\phi^i(B_{t_j}) - \phi^i(B_{\underline{t}_{j-1}})}{\delta t}\delta t \\
&= \frac{\phi^i\left(B_{\underline{t}_{j-1}} \exp\left((\delta t \cdot 1/\delta t)(\overline{M}_{\underline{t}_j} - \overline{M}_{\underline{t}_{j-1}})\right)\right) - \phi^i(B_{\underline{t}_{j-1}})}{\delta t}\delta t \\
&= dL_{B_{\underline{t}_{j-1}}}\left(1/\delta t(\overline{M}_{\underline{t}_j} - \overline{M}_{\underline{t}_{j-1}})\right)(\phi^i)\delta t \\
&= \sum_{l=1}^{d} 1/\delta t\left(\overline{M}^l_{\underline{t}_j} - \overline{M}^l_{\underline{t}_{j-1}}\right) H_l(\phi^i(B_{\underline{t}_{j-1}}))\delta t \\
&= \sum_{l=1}^{d}\left(\overline{M}^l_{\underline{t}_j} - \overline{M}^l_{\underline{t}_{j-1}}\right) H_l(\phi^i(B_{\underline{t}_{j-1}})),
\end{aligned}$$

so that

$$\begin{aligned}
&f(B_{\underline{t}_k}) - f(e) \\
&\quad\approx \sum_{j=1}^{k}\Bigg[\sum_{i=1}^{d}\sum_{l=1}^{d}(^*\frac{\partial}{\partial x_i})(f\circ\phi^{-1})(\phi(B_{\underline{t}_{j-1}}))(\overline{M}^l_{\underline{t}_j} - \overline{M}^l_{\underline{t}_{j-1}})H_l(\phi^i(B_{\underline{t}_{j-1}})) \\
&\qquad + \frac{1}{2}\sum_{i,m=1}^{d}\sum_{n,l=1}^{d}(^*\frac{\partial^2}{\partial x_m\partial x_i})(f\circ\phi^{-1})\phi(B_{\underline{t}_{j-1}}) \\
&\qquad \cdot(\overline{M}^l_{\underline{t}_j} - \overline{M}^l_{\underline{t}_{j-1}})H_l(\phi^i(B_{\underline{t}_{j-1}}))(\overline{M}^n_{\underline{t}_j} - \overline{M}^n_{\underline{t}_{j-1}})H_n(\phi^m(B_{\underline{t}_{j-1}}))\Bigg] \\
&\quad= \sum_{j=1}^{k}\Bigg[\sum_{l=1}^{d}(H_l f)(B_{\underline{t}_{j-1}})(\overline{M}^l_{\underline{t}_j} - \overline{M}^l_{\underline{t}_{j-1}}) \\
&\qquad + \frac{1}{2}\sum_{n,l=1}^{d}(H_nH_l f)(B_{\underline{t}_{j-1}})(\overline{M}^l_{\underline{t}_j} - \overline{M}^l_{\underline{t}_{j-1}})(\overline{M}^n_{\underline{t}_j} - \overline{M}^n_{\underline{t}_{j-1}})\Bigg] \qquad (3) \\
&\quad= \sum_{j=1}^{k}\Bigg[\sum_{l=1}^{d}(H_l f)(B_{\underline{t}_{j-1}})(\chi^l_{\underline{t}_j} - \chi^l_{\underline{t}_{j-1}}) + \sum_{l=1}^{d}(H_l f)(B_{\underline{t}_{j-1}})a_l(\underline{t}_j - \underline{t}_{j-1}) \\
&\qquad + \frac{1}{2}\sum_{n,l=1}^{d}(H_nH_l f)(B_{\underline{t}_{j-1}})(\chi^l_{\underline{t}_j} - \chi^l_{\underline{t}_{j-1}})(\chi^n_{\underline{t}_j} - \chi^n_{\underline{t}_{j-1}}) \\
&\qquad + \frac{1}{2}\sum_{n,l=1}^{d}(H_nH_l f)(B_{\underline{t}_{j-1}})a_l a_n(\underline{t}_j - \underline{t}_{j-1})^2\Bigg]
\end{aligned}$$

$$= \sum_{l=1}^{d} \int_0^{\underline{t}} (H_l f)(B_{\underline{s}}) d\chi^l_{\underline{s}} + \frac{1}{2} \sum_{n,l=1}^{d} \delta_{n,l} \int_0^{\underline{t}} (H_n H_l f)(B_{\underline{s}}) d\underline{s} + \int_0^{\underline{t}} (H_o f)(B_{\underline{s}}) d\underline{s}$$

$$= \sum_{l=1}^{d} \int_0^{\underline{t}} (H_l f)(B_{\underline{s}}) d\chi^l_{\underline{s}} + \int_0^{\underline{t}} \left(\frac{1}{2} \sum_{l=1}^{d} (H_l^2 f) + (H_o f) \right) (B_{\underline{s}}) d\underline{s}$$

$$= \sum_{l=1}^{d} \int_0^{\underline{t}} (H_l f)(B_{\underline{s}}) \partial \chi_{\underline{s}} + \int_0^{\underline{t}} (H_o f)(B_{\underline{s}}) d\underline{s}.$$

Now, f has compact support. So, for $b^i_t = {}^o\chi^i_{\underline{t}}$, we have

$$f({}^oB_{\underline{t}}) - f(e) = \sum_{l=1}^{d} \int_0^t (H_l f)({}^oB_s) \partial b_s + \int_o^t (H_o f)({}^oB_s) ds,$$

provided $\underline{t} \approx t$. Since we have seen that ${}^oB_t$ satisfies the stochastic differential equation (1), using the connection between the Ito and the Stratonovich Calculus ([29]), we obtain

$$\begin{aligned} df({}^oB_t) &= (H_i f)({}^oB_t) db^i_t + \left(\frac{1}{2} \sum_i H_i^2 + H_o \right) (f)({}^oB_t) dt \\ &= (H_i f)({}^oB_t) \partial B^i_t + (H_o f)(B_t) dt. \end{aligned}$$

It follows that $\frac{1}{2} \sum_i H_i^2 + H_o$ is the generator of ${}^oB_t$.
(3) ${}^oB_t \in G$ a.s. In fact, $[0, R]$ is a compact set. So, if $\underline{t} \in T$, $\underline{t}$ is nearstandard, and if $f \in C$ has compact support, ${}^*f$ is nearstandard. Let (V_n, ϕ_n) be a system of coordinate neighborhoods in G and for each n let V'_n be a subordinate neighborhood such that $\overline{V'_n} \subseteq V_n$ and $\overline{V'_n}$ is compact. Let $\phi^i_n \in C$ be such that it coincides with the i-component of ϕ_n on $\overline{V'_n}$. The sequence $({}^*V_n, {}^*V'_n, {}^*\phi_n)_{n \in \mathbb{N}}$ can be extended to an internal sequence $({}^*V_n, {}^*V'_n, {}^*\phi_n)_{n \in {}^*\mathbb{N}}$ with the same properties. For fixed $\underline{t} \in T$, $B_{\underline{t}} \in {}^*G$. Then, there is $m \in {}^*\mathbb{N}$ such that $B_{\underline{t}} \in {}^*V'_m$, and ${}^*\phi^i_m(B_{\underline{t}})$ is an S-continuous *-real valued internal nearstandard semi-martingale. So, by Proposition 2.3 in [11], $st({}^*\phi^i_m(B_{\underline{t}}))$ exists, is finite and

$$st({}^*\phi^i_m(B_{\underline{t}})) = {}^o({}^*\phi^i_m(B_{\underline{t}})) = \phi^i_m({}^oB_{\underline{t}}).$$

Applying ϕ_m^{-1} we see that $B_{\underline{t}} \in {}^*V'_m$, is nearstandard, and there exists $N_t \in G$ such that ${}^oB_{\underline{t}} = N_t$. ☑

From the proof of equation (3) in Theorem 5 it follows that the nonstandard definition of the stochastic exponential can be given by means of equation (2).

In order to give a nonstandard definition of the stochastic logarithm we first show that for each semi-martingale on G we can find a semi-martingale lifting.

6. Definition. Let $A \subseteq [0,\infty)\times\Omega$. We say that A is an open set if $A_w = \{t : (t,w)\in A\}$ is open for each $w\in\Omega$.

A function $\varphi : [0,+\infty)\times\Omega\to\mathbb{R}$ is C^∞ if for each $w\in\Omega$ the function $\varphi(\cdot,w) : [0,+\infty)\to\mathbb{R}$ is C^∞.

7. Theorem. *Let $X : [0,\infty)\times\Omega\to G$ be a continuous $\mathfrak{F}_t$-semi-martingale. Then, there exist an internal filtration $\{\mathfrak{B}'_{\underline{t}}\}$ and a $\mathfrak{B}'_{\underline{t}}$ semi-martingale lifting $\overline{X} : T\times\Omega\to {}^*G$ of X.*

Proof. Let (V_n,ϕ_n) be a countable coordinate system of G and assume that for each n we have chosen an open set V'_n such that $\overline{V'_n}$ is a compact subset of V_n. Let ϕ^i_n be functions in C (Definition 2) that coincide with the components of ϕ_n on V'_n. We write $\overline{\phi}_n = (\phi^1_n,\cdots\phi^d_n)$. In the same way we can define for each n a function $\psi_n : \mathbb{R}^d\to G$ whose components are in C and coincide with the components of ϕ_n^{-1} on $\phi(\overline{V'_n})$. Then $\overline{\phi}_n(X_t)$ is an $\mathbb{R}^d$ valued continuous semi-martingale which coincides with $\phi_n(X_t)$ on $X^{-1}(\overline{V'_n})$. From the continuity of X_t we then have that for each $w\in\Omega$, $\{t : (t,w)\in X^{-1}(\overline{V'_n})\}$ is compact. Fom Theorem 1.2.12 in [27] there is an internal filtration $\{\mathfrak{B}_{\underline{t}}\}$, and a $\mathfrak{B}_{\underline{t}}$-$S$-continuous semi-martingale lifting $Y^n_{\underline{t}}$ of $\overline{\phi}_n(X_t)$. From the continuity of X_t, it also follows that if $(t,w)\in X^{-1}(\overline{V'_n})$ then, for $\underline{t}\approx t$, we have that

$$st(Y^n(\underline{t},w)) = \phi_n(X)(t,w) \qquad a.s.$$

Define $Z^n_{\underline{t}} = {}^*\psi_n(Y^n_{\underline{t}})$. This is an S-continuous internal semi-martingale with respect to $\mathfrak{B}_{\underline{t}}$ on ${}^*G$, and if $(t,w)\in X^{-1}(\overline{V'_n})$, we have for $\underline{t}\approx t$ that $st(Z^n(\underline{t},w)) = X(t,w)$ a.s.

We also have that if $(t,w)\in X^{-1}(\overline{V'_n})\cap X^{-1}(\overline{V'_m})$ and $\underline{t}\approx t$ then

$$st(Z^n(\underline{t},w)) = X(t,w) = st(Z^m(\underline{t},w)) \qquad a.s.$$

Now, $\{X^{-1}(V'_n)\}_{n\in\mathbb{N}}$ is an open cover of $[0,+\infty)\times\Omega$. Let Φ be a partition of unity in C subordinated to this cover. Then Φ is a countable set, say $\Phi = \{\varphi_n\}$. If $\varphi\in\Phi$, φ is of bounded variation, that is, φ is a semi-martingale.

We can extend the sequence $\{{}^*V_n, {}^*\phi_n, {}^*V'_n, {}^*\psi_n, Y^n, Z^n, {}^*\varphi_n\}_{n\in\mathbb{N}}$ to an internal sequence $\{{}^*V_n, {}^*\phi_n, {}^*V'_n, {}^*\psi_n, Y^n, Z^n, {}^*\varphi_n\}_{n\in{}^*\mathbb{N}}$ with the same properties. Let $\eta\in{}^*\mathbb{N}-\mathbb{N}$. For each $n\le\eta$, define $Z_{\underline{t}} = Z^n_{\underline{t}}$ for $(\underline{t},w)\in{}^*(X^{-1}(V'_n))\cap(T\times\Omega)$. We have that $Z_{\underline{t}}$ is S-continuous (from the definition of $Z^n_{\underline{t}}$) and that for $f\in C$,

$${}^*f(Z_{\underline{t}}) = \sum_{n=1}^{\eta} {}^*\varphi_n\cdot{}^*f(Z^n_{\underline{t}}).$$

Also, the right side of the above identity being a hyperfinite sum of the product of two internal $\mathfrak{B}_{\underline{t}}$-semi-martingales, is an internal semi-martingale. If $\underline{t} \in T$ and $n \leq \eta$, there is $t \in V'_n$ such that, for $w \in \Omega$, $(\underline{t}, w) \approx (t, w)$ and $(\underline{t}, w) \in {}^*X^{-1}(V'_n)$. Then

$$st(Z_{\underline{t}}) = st(Z^n_{\underline{t}}) = X_t,$$

and so $Z_{\underline{t}}$ is the desired lifting. ☑

Now we show that we can also define a nonstandard version of the stochastic logarithm of a semi-martingale on G as a semi-martingale on $\mathfrak{g}$. First we recall the notations in [9]. For $x \in G$, there is a neighborhood N_x in $T_x(G)$ which is mapped diffeomorphically by the exponential map onto a neighborhood U_x of x in G.

For $x = e$, let $N = N_e$ and $U = U_e$. Then, for $x \in U$, $x \to (\log^i(x))_{1 \leq i \leq d} \in \mathbb{R}^d$ if and only if $x = \exp(\log^i(x) H_i)$ ($\{H_i : i = 1, \cdots, d\}$ is a basis of $\mathfrak{g}$ and $(\log^i(x))_{1 \leq i \leq d}$ are the canonical coordinates).

Let (V, ϕ) be a coordinate neighborhood of e and let $(D_j)_j$ be the differential operators associated with (V, ϕ). We have, for $x \in V$ and $f : V \to \mathbb{R}$, $f \in C^1$, that

$$H_i f(x) = H_i \phi^k D_k f(x)$$

and that

$$D_j f(x) = (D_j; \Theta^k)(x) H_k f(x)$$

where $(\Theta^k)_k$ is the dual basis of $(H_i)_i$ and $((D_j; \Theta^k)(x))_k$ is the family of coordinates of $D_j(x) \in T_x G$ with respect to the basis $(H_k(x))_k$. D_j and H_k are vector fields and

$$(D_j; \Theta^k)(x) = D_j(\log^k \circ L_{(x^{-1})})(x).$$

Now let $(U_n, \phi_n)_{n \in \mathbb{N}}$ be an open cover of G by a countable family of coordinate neighborhoods and $(h_n)_{n \in \mathbb{N}}$ be a partition of unity subordinate to U_n. For each n, let the operators $(D^n_j)_{1 \leq j \leq d}$ be as above. For a semi-martingale X on U define

$$\mathfrak{L}(X) = M^i H_i$$

where

$$M^i_t = \int_0^t h_n(X_s)(D^n_j; \Theta^i)(X_s) \partial \phi^j_n(X_s) \tag{4}$$

is the Stratonovich integral with respct to $\phi^j_n(X_s)$. The following definition shows that the nonstandard definition of the stochastic logarithm is an easy working alternative. Let $\eta \in {}^*\mathbb{N} - \mathbb{N}$ and let

$$T_\eta = \left\{0, \frac{R}{2^\eta}, \frac{2R}{2^\eta}, \cdots, \frac{(2\eta - 1)R}{2^\eta}, R\right\}$$

be a hyperfinite time line. Given a semi-martingale X on G, there exists a semi-martingale lifting $\underline{X}$ of X. Define

$$\overline{\mathfrak{L}}(\overline{X}_{\underline{t}}) = \sum_{l=0}^{k}{}^* \log \overline{X}_{\underline{t}_l}^{-1} \overline{X}_{\underline{t}_{l+1}}.$$

8. Theorem. *Let $\overline{M}$ be a semi-martingale lifting of a semi-martingale M with values on $\mathfrak{g}$. If $\overline{X}_{\underline{t}} = \overline{\mathfrak{E}}(\overline{M}_{\underline{t}})$, we have that*

$${}^o(\overline{\mathfrak{L}}(\overline{\mathfrak{E}}(\overline{M}_{\underline{t}}))) = \mathfrak{L}({}^o\overline{\mathfrak{E}}(\overline{M}_{\underline{t}})) = M_t.$$

If $\overline{X}_{\underline{t}}$ is a semi-martingale lifting of a semi-martingale X_t on G, and if $\overline{\mathfrak{L}}(\overline{X}_{\underline{t}}) = \overline{M}_{\underline{t}}$, then

$${}^o(\overline{\mathfrak{E}}(\overline{\mathfrak{L}}(\overline{X}_{\underline{t}}))) = \mathfrak{E}({}^o\overline{\mathfrak{L}}(\overline{X}_{\underline{t}})) = X_t.$$

Proof.

$$\begin{aligned}\overline{\mathfrak{L}}(\overline{X}_{\underline{t}}) &= \sum_{l=0}^{k}{}^* \log \overline{X}_{\underline{t}_l}^{-1} \overline{X}_{\underline{t}_{l+1}} \\ &= \sum_{l=0}^{k}{}^* \log {}^* \exp(\overline{M}_{\underline{t}_{l+1}} - \overline{M}_{\underline{t}_l}) \\ &= \sum_{l=0}^{k}(\overline{M}_{t_{l+1}} - \overline{M}_{\underline{t}_l}) = \overline{M}_{\underline{t}_{k+1}} - \overline{M}_{\underline{t}_o}\end{aligned}$$

for $\underline{t} = \underline{t}_{k+1}$, and ${}^o\overline{\mathfrak{L}}(\overline{X}_{\underline{t}}) = M_t$ for $\underline{t} \approx t$. The proof of the second part is similar. ☑

Hakim-Dowek and Lépingle ([9]) introduce a natural notion of local martingale on a Lie group by

9. Definition. A continuous semi-martingale X with values in G is a G-martingale if $\mathfrak{L}(X)$ is a local martingale with values in $\mathfrak{g}$.

An analogous definition holds for internal $^*G$-martingales on $^*G$.

P.A. Meyer [26] and R.W.R Darling [5] give a notion of martingale on a manifold V by taking a connection Γ on V. The authors in [9] use the fact that on a Lie group G one can choose a left invariant connection, which carries the maps $\exp(tA)$, $t \in \mathbb{R}$, to geodesics. The connection has no torsion, so that $\nabla_A A = 0$ for any left invariant vector field A, and that $\nabla_A B = 0$ for $A, B \in \mathfrak{g}$. For this type of connection, they have shown in [9] that the Γ-martingales coincide with the G-martingales.

A local martingale with values in $\mathfrak{g}$ is given by $M_t = \sum_{i=1}^d M_t^i H_i$, where $\{H_i\}$ is a basis in $\mathfrak{g}$ and $(M_t^1, \cdots, M_t^d)$ is a local martingale on $\mathbb{R}^d$.

10. Theorem.

(a) *Let X be a G-martingale on G. Then, there is an internal $^*G$-martingale $\overline{X}$ on $^*G$ such that $st(\overline{X}) = X$ a.s.*

(b) *Let $\overline{X}$ be an internal $^*G$-martingale on $^*G$. Then $st(\overline{X})$ is a G-martingale on G.*

Proof.

(a) If X is a G-martingale, $\mathfrak{L}(X)$ is a continuous local martingale on $\mathfrak{g}$ with paths in $D(\mathbb{R}^d)$. By Theorem 5.6 in [11], there is a $\mathfrak{B}_{\underline{t}}$- S-continuous local martingale lifting $\overline{M}$ of $\mathfrak{L}(X)$. It follows that $\overline{\mathfrak{E}}(\overline{M})$ is an internal semi-martingale on $^*G$. Thus, $\overline{\mathfrak{E}}(\overline{M})$ is an internal $^*G$-martingale on $^*G$.

(b) If $\overline{X}$ is an internal $^*G$-martingale, $\overline{X}$ is an internal S-continuous semi-martingale on $^*G$ such that $\overline{\mathfrak{L}}(\overline{X})$ is an internal S-continuous local martingale on $\mathfrak{g}$. Thus by Theorem 5.2.(b) in [11], $st(\overline{\mathfrak{L}}(\overline{X})) = \mathfrak{L}(st(\overline{X}))$ is a local martingale on $\mathfrak{g}$. This implies that $st(\overline{X})$ is a G-martingale on G. □

Acknowledgments. I deeply thank the referee for his valuable help in the preparation of the final version of this work.

References

1. S. Albeverio, J.E. Fenstad, R. Høegh-Krohn, T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, 1986.
2. R.M. Anderson, *A Nonstandard representation for brownian motion and Ito integration*, Israel Journal of Mathematics **25** (1976), 15 – 46.
3. J.M. Bismut, *Mécanique Aléatoire*, L.N.M. 866, Springer Verlag, 1981.

4. W.M. Boothby, *An Introduction to Differentiable Manifolds and Riemanian Geometry*, Academic Press, 1975.
5. R.W.R. Darling, *Martingales in manifolds*, Sém. Prob. XVI, Suplément, **L.N.M.921** (1982), 217–236.
6. K.D. Elworthy, *Stochastic Differential Equations on Manifolds*, Cambridge University Press, 1982.
7. M. Emery, *Stabilité des solutions des équations différentielles stochastiques, aplications aux intégrales multiplicatives stochastiques*, Z. Wahrsch. Ver. Geb. **41** (1978), 241–262.
8. A. Estrade, *Calcul stochastique discontinu sur les groupes de Lie*, Doctoral Thesis Université D'Orleans (1990).
9. M. Hakim-Dowek and D. Lépingle, *L'exponential stochastique des groupes de Lie*, Sém. de Prob. XX, L.N.M. **1.204**, 352–376.
10. S. Helgason, *Differential Geometry, Lie Groups and Symmetric spaces*, Academic Press, 1978.
11. D.N.Hoover and E. Perkins, *Nonstandard Construction of the Stochastic Integral and Applications to stochastic Differential Equations I, II*, Trans. Amer. Math. Soc. **275-1** (1983), 1 – 58.
12. M. Ibero, *Intégrales stochastiques multiplicatives et construction de diffusions sur un groupe de Lie*, Bull. Sc. Math. **2e. Serie 100** (1976), 175–195.
13. N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusions Processes*, North Holland, 1981.
14. K.Ito, *Brownian motions on a Lie group.*, Proc. Japan Acad. **26** (1950), 4–10.
15. K.Ito, *On a formula concerning stochastic differentials*, Nagoya Math. J. **3** (1951), 55–65.
16. K. Ito and H.P. McKean, *Diffusion Processes and their sample paths*, Springer Verlag, 1974.
17. R.L Karandikar, *Approximation results for multiplicative stochastic integrals*, Sém. de Prob. XVI **L.N.M. 920** (1980/81), 384–391, Springer Verlag.
18. R.L Karandikar, *Girsanov type formula for a Lie group valued Brownian Motion*, Sém. de Prob. XVII **L.N.M. 986** (1981/82), 198–204.
19. R.L Karandikar, *Multiplicative stochastic integration,*, North Holland Publishing Co.
20. T.L. Lindstrϕm, *Hyperfinite stochastic integration I, II , III*, Math. Scand **46** (1980), 265–333.
21. P.Loeb, *Conversion from non standard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122.
22. P. Malliavin, *Géométrie Différentielle Stochastique*, Presse de l'Université de Montreal, 1978.
23. H.P. McKean, *Brownian motion on the 3-dimensional rotation group*, Mem. of the Coll. Sci. Univ. of Kyoto, Serie A **XXXIII Math. N1** (1960).
24. H.P. McKean, *Stochastic Integrals*, Academic Press, 1969.
25. P.A. Meyer, *Géométrie stochastique sans larmes*, Sém. Prob. XV, **L.N.M. 850** (1981), 44–102, Springer Verlag.
26. P.A. Meyer, *Géométrie différentielle stochastique (bis)*, Sém. Prob. XVI Suplément. **L.N.M. 921** (1982), 165–207, Springer Verlag.
27. M. Muñoz de Özak, *Nonstandard construction of Brownian Motion and martingales on Lie groups*, Thesis (1995), Ruhr-Universität- Bochum. Germany.
28. P.J. Olver, *Applications of the Lie Groups to Differential Equations*, Springer Verlag, 1986.

29. C.G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales. Vol2. Ito Calculus*, John Wiley & Sons, 1987.
30. L. Schwartz, *Semi-martingales sur des Variétés et Martingales uniformes sur des Variétés Analytiques Complexes*, Springer Verlag L.N.M. 780, Heidelberg Berlin New York, 1980.
31. L. Schwartz, *Géométrie différentiel de 2ème ordre, semi-martingales et équations différentielles stochastiques sur une variété différentielle*, L.N.M. 921 **Sém. Proba. XVI Suplément** (1982), 1–148, Springer Verlag.
32. K. Yosida, *Brownian motion on a homogeneous space*, Pacific Journal of Mathematics **2** (1952), 263–296.

(Recibido en noviembre de 1997; revisado en mayo de 1998)

Myriam Muñoz de Özak
Departamento de Matemáticas y Estadística
Universidad Nacional de Colombia, Bogotá
COLOMBIA
e-mail:mymunoz@matematicas.unal.edu.co