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On an elementary functional equation

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In the Internet page *Favorite Mathematical Constants* [**2**], maintained by Steven Finch, the following question was posed [(†)]: If φ is the Golden ratio, and $g(x) = \varphi - \sqrt{\varphi^2 - x}$, is the solution of the functional equation

$$\begin{aligned} 2\varphi F(g(x)) &= F(x) \\ F(0) &= 0 \\ F'(0) &= 1 \end{aligned} \tag{0}$$

unique?

In this short note we show that both the existence and uniqueness of the solution of (0) follow as a particular case of a more general result (Theorem below), whose proof is based in the theory of iterated functions, the same technique used by R. B. Paris to prove the existence of a solution of (0) in [**3**]. This author considers equation (0) while proving that

$$\varphi - \varphi_n \sim \frac{2C}{(2\varphi)^n} \qquad \text{as} \qquad n \to \infty ,$$

[(†)] We thank professor Víctor Albis for calling up our attention on these questions .

where $\varphi_1 = 1$, $\varphi_n = \sqrt{1+\varphi_{n-1}}$ for $n \geq 2$, and

$$C = \varphi F\left(\frac{1}{\varphi}\right) \approx 1.098630 \ .$$

Theorem. *Let $g(x)$ be a continuous increasing function defined on the closed interval $[a, b]$, $a < 0 < b$, satisfying the following conditions:*

$$\text{(i)} \quad \begin{cases} g(x) < x & \text{if} \quad 0 < x \leq b \\ g(x) > x & \text{if} \quad a \leq x < 0 \\ g(0) = 0 \ , \end{cases}$$

(ii) $g'(0) = \alpha$, $0 < \alpha < 1$, *and* $g''(0)$ *exists.*

Then the functional equation

$$F(g(x)) = \alpha F(x) \ , \tag{1}$$

with

$$F(0) = 0, \quad F'(0) = 1 \ , \tag{2}$$

has a unique solution in the interval $[a, b]$

Proof. On $[a, b]$ let us define recursively the following sequence of functions

$$X_1(t) = t, \quad X_{n+1}(t) = g(X_n(t)), \\ n = 1, 2, \cdots \tag{3}$$

It is well known that

$$X_n(t) \to 0 \quad (n \to \infty) \tag{4}$$

monotonically. Furthermore [**1**], the limit

$$\lim_{n\to\infty} \frac{X_n(t)}{\alpha^{n-1}} = C(t) \tag{5}$$

exists, with $C(0) = 0$. If the function $F(x)$ satisfies (1), then we must have

$$F(g(X_n(t))) = F(X_{n+1}(t)) = \alpha F(X_n(t)) \ , \tag{6}$$

for all n. From this it follows easily that

$$F(X_{n+1}(t)) = \alpha^n F(X_1(t))) = \alpha^n F(t). \tag{7}$$

According to conditions (2) of the theorem, we must choose the value of $F(t)$ so that

$$1 = F'(0) = \lim_{n\to\infty} \frac{F(X_{n+1}(t))}{X_{n+1}(t)} = \lim_{n\to\infty} \frac{\alpha^n F(t)}{X_{n+1}(t)}$$
$$= \lim_{n\to\infty} \frac{F(t)}{X_{n+1}(t)/\alpha^n} = \frac{F(t)}{C(t)} .$$

Then

$$F(t) = C(t) . \tag{8}$$

On the other hand, from (5) we get

$$C(g(t)) = \lim_{n\to\infty} \frac{X_n(g(t))}{\alpha^{n-1}} = \lim_{n\to\infty} \frac{X_{n+1}(t)}{\alpha^{n-1}}$$
$$= \alpha \cdot \lim_{n\to\infty} \frac{X_{n+1}(t)}{\alpha^n} = \alpha C(t) . \tag{9}$$

Therefore, (8) guarantees the existence of a solution for (1).

To finish the proof we have to show that $F(t)$ defined by (5) and (8) has a derivative at 0 and that $F'(0) = 1$. In order to accomplish this we have to show first that the limit in (5) holds "uniformly". From $g(0) = 0$ and $g'(0) = \alpha$, we get

$$\frac{g(x)}{x} = \alpha + \tau(x) \tag{10}$$

where $\lim_{x\to 0} \tau(x) = 0$. Replacing in (10) x by $X_n(t)$ we obtain

$$X_{n+1}(t) = g(X_n(t)) = X_n(t)\,[\alpha + \tau(X_n(t))]$$
$$= \alpha X_n(t) \left[1 + \frac{\tau(X_n(t))}{\alpha}\right] ,$$

and from this

$$X_{n+1}(t) = \alpha^n t \prod_{k=1}^{n} \left[1 + \frac{\tau(X_k(t))}{\alpha}\right] . \tag{11}$$

Since $g''(0)$ exists, there are $\delta > 0$ and $M > 0$ such that

$$|\tau(x)| \le M \cdot |x| \quad \text{for all} \quad x \in (-\delta, \delta) . \tag{12}$$

On the other hand, by Dini's theorem the sequence of functions $((X_n(t))_{n\ge 1}$ converges uniformly to 0, that is, there is an integer N such that

$$X_n(t) \in (-\delta, \delta) \quad \text{for all} \quad t, \quad \text{and all} \quad n \ge N . \tag{13}$$

Therefore, the following inequality holds:

$$\sum_{k=N}^{\infty} |\tau(X_k(t))| \le M \sum_{k=N}^{\infty} |X_k(t)| \ . \tag{14}$$

The series in the member of the right of (14) converges uniformly. In fact,

$$\left|\frac{X_{k+1}(t)}{X_k(t)}\right| = \left|\frac{g(X_k(t))}{X_k(t)}\right| \le \alpha + |\tau(X_k(t))|$$
$$\le \alpha + M \cdot |X_k(t)| \ ,$$

which converges uniformly to $\alpha < 1$, as $k \to \infty$. Therefore, the infinite product

$$\prod_{k=1}^{\infty} \left[1 + \frac{\tau(X_k(t))}{\alpha}\right]$$

converges uniformly. Finally we get

$$C(t) = \lim_{n\to\infty} \frac{X_{n+1}(t)}{\alpha^n} = t \prod_{k=1}^{\infty} \left[1 + \frac{\tau(X_k(t))}{\alpha}\right] \ , \tag{15}$$

the convergence being uniform. Note that the function $C(t)$ is continuous. Also, the uniform convergence of the infinite product in (15) implies that $C(t)/t$ is continuous. Therefore

$$\lim_{t\to 0} \frac{F(t)}{t} = \lim_{t\to 0} \frac{C(t)}{t} = 1 \ ,$$

i.e. $F'(0) = 1$. □

Let us remark that the above argument does not prove that $F \in C^\infty$. However, an inductive argument on the order of the derivative, too lengthy and cumbersome to be included here, allows us to assert that F indeed is of class C^∞.

Now, let us go back to the original problem, where

$$g(x) = \varphi - \sqrt{\varphi^2 - x} \quad (\frac{1}{2} < \varphi < 1) \ .$$

The function $g(x)$ has two fixed points: $x = 0$ and $x = 2\varphi - 1 = \sqrt{5}(< \varphi^2)$. Furthermore,

$$g'(x) = \frac{1}{2\sqrt{\varphi^2 - x}} > 0, \quad g'(0) = \frac{1}{2\varphi} < 1$$

and

$$g''(0) = \frac{1}{4\varphi^3} > 0 \ .$$

The function $g(x)$ satisfies conditions (i) and (ii) of the theorem in the interval $(-\infty, 2\varphi - 1) = (-\infty, \sqrt{5})$, with $\alpha = \dfrac{1}{2\varphi} < 1$. Also, it is easy to verify that in this case the functional equation (1) does not have a solution in the interval $(2\varphi - 1, \varphi^2) = (\sqrt{5}, \varphi^2)$. Which shows that the convergence radius of the expansion of $F(t)$ as a power series about the origin, if it converges, is $\leq \sqrt{5}$, a sharper bound that the one proposed by Paris: φ^2 [**3**].

References

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