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Summability of a Fourier series

Sumabilidad de una serie de Fourier

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Abstract. In this expository article we discuss the notion of summability, in a historical context, focusing on two methods, Cesàro's and Abel's. We apply these methods to Fourier series, analyzing in detail the summability results they provide.

***Key words*:** Cesàro summability, Abel summability, Fourier series, Fejér kernel, Poisson kernel, good kernels, Shakarchi's and Stein's conditions.

Resumen. En este artículo expositorio estudiamos la noción de sumabilidad en un contexto histórico, concentrándonos en dos métodos, el de Cesàro y el de Abel. Además aplicamos estos métodos a las series de Fourier, analizando en detalle los teoremas de sumabilidad que resultan.

***Palabras clave*:** Método de sumabilidad de Cesàro, método de sumabilidad de Abel, series de Fourier, núcleo de Fejér, núcleo de Poisson, núcleos buenos, las condiciones de Shakarchi y Stein.

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1 Introduction

The purpose of this expository article is to discuss, in a historical context, two summability methods, Cesàro's and Abel's, and to apply them to the Fourier series of a 2π-periodic function that is Riemann integrable on $[-\pi, \pi]$.

The present article continues with and extends the topics presented in [1], where we studied results of convergence *à la Cauchy*, for the Fourier series of a 2π-periodic function satisfying appropriate conditions.

The organization is as follows: In Section 2 we collect a number of historical observations on how the treatment of series evolved over time. Section 3 is dedicated to two summability methods, Cesàro's and Abel's. Their properties and the relation of one to the other are illustrated with numerous examples. Next, we apply these two summation methods to the Fourier series of a 2π-periodic function that is Riemann integrable on $[-\pi, \pi]$. As it is the case of convergence, each summation method is associated with an integral operator defined by an appropriate kernel, Fejér for Cesàro's method and Poisson for Abel's method. To calculate these two kernels is the subject of Section 3. In Section 4 we use the notion of good kernel, discussed by Rami Shakarchi and Elias M. Stein in [18], to prove that both, Cesàro's method and Abel's method, sum the Fourier series to the value of the function, at every point of continuity. We also show that the Fejér kernel and the Abel kernel satisfy the condition stated by Antoni Zygmund in ([21], p. 88). Finally, in the last section, we go over a brief discussion of convergence versus summability, for a Fourier series.

2 Convergent series and divergent series

We begin with a series $\sum_{j\geq 0} a_j$ of real terms a_j. It converges, or it is convergent, if the sequence of its partial sums $\sum_{0\leq j\leq n} a_j$ has a finite limit as the index n goes to infinity. In other words, if there is a real number a so that, for each $\varepsilon > 0$ there is $N = N(\varepsilon) \geq 1$ for which

$$\left| \sum_{j=0}^{n} a_j - a \right| < \varepsilon, \tag{1}$$

for all $n \geq N_\varepsilon$. If this is the case, we write

$$\sum_{j\geq 0} a_j = a$$

and we call a the sum of the series. The limit of a sequence, if it exists, is unique. Therefore, when a exists, it is unique. If a does not exist, we say that the series diverges, or that it is divergent. As a consequence of the definition of convergent series, the general term a_j of a convergent series goes to zero as $j \to \infty$.

These definitions and results appear in Augustin-Louis Cauchy's *Analyse Algébrique*, published in Paris in 1821.

It is permissible to say, for instance, that the series $\sum_{j\geq 1} j$ "goes" to infinity, or that it diverges to infinity, because its partial sums increase without bound as n increases. However, since ∞ is not a number, the series is divergent, according to Cauchy's definition.

The Cauchy's condition, which appears in p. 125 of *Analyse Algébrique* and is part of the Calculus canon, allows us to decide whether a series is convergent or not, without having to identify its sum.

According to ([20], p. 11), John Wallis had already formulated, in 1655, a definition of convergence equivalent to (1). The expression "convergent series" is due to James Gregory, who began to use it in 1668 ([20], p. 16). As for the expression “divergent series”, the same source states that it was coined by Nicolaus (I) Bernoulli in 1713.

Nevertheless, Cauchy was the first to make a rigorous study of series, focused on convergence ([17], Cauchy’s biography).

Divergent series puzzled and challenged mathematicians for many centuries. For instance, Isaac Newton and Gottfried Wilhelm Leibniz, who were the first to manipulate series systematically, had little inclination to deal with divergent series, although, as Godfrey Harold Hardy puts it ([8], p. 1), “Leibniz sometimes played with them.” Still, regardless of how careless their manipulations look today, even in Arquimedes’s time, mathematicians had a pretty good idea of whether a series was convergent or divergent. Moreover, the great masters just seemed to know what manipulations were “permissible”, no matter how devoid of meaning they appeared to be. As an example, Hardy uses the work of Leonhard Euler on the series $1-1+1-1+\cdots$ (see [8], p. 14):

For $0 \leq r < 1$,

$$\sum_{j\geq 0} (-1)^j \, r^j = \frac{1}{1+r}. \tag{2}$$

The right-hand side of (2) can be rightfully evaluated for $r = 1$. Therefore,

$$\lim_{r\to 1^-} \sum_{j\geq 0} (-1)^j \, r^j = \frac{1}{2}.$$

Taking $r = 1$ on the left-hand side of (2) we have, formally, the series $1-1+1-1+\cdots$ Euler’s conclusion is that

$$1-1+1-1+\ldots = \frac{1}{2}.$$

Leibniz had already this result, using “probability and metaphysics” ([8], p. 14).

Be that as it may, mathematicians of different periods did not fail to notice that reckless manipulations of divergent series often led to interesting conclusions, which sometimes could be verified by other means. Hardy cites Euler as saying ([8], p. 15) that “the controversies excited by the use of divergent series are largely ‘verbal’.” Hardy goes on to saying: “Here, as elsewhere, Euler was substantially right. The puzzles of the time about divergent series arose mostly, not from any particular mystery in divergent series as such, but from disinclination to give formal definitions and from the inadequacy of the current theory of functions.”

Indeed, before Cauchy insisted on the need for explicit definitions, even the most illustrious mathematicians were not inclined to ask “What *is the definition* of, say, $1-1+1-\cdots$?”, but rather they asked the entirely different question “What *is* $1-1+1-\cdots$?” ([8], p. 6).

After Euler, Joseph Fourier and Simeón Denis Poisson were the analysts who used divergent series most ([8], p. 17). Nevertheless, with Cauchy’s definition, the attention was placed on convergent series, with divergent series being gradually removed from analysis. However, in the last quarter of the nineteenth century, they made a dramatic reentrance, with the works of Henri Jules Poincaré, Thomas Jan Stieltjes and Ernesto Cesàro. Poincaré, in an article published in 1886 in *Acta Mathematica*, used particular divergent series to approximate the solutions of ordinary differential equations, near irregular points. Stieltjes, in his doctoral dissertation, published the same year in *Annales Scientifiques de l'École Normal Supérieur*, showed how certain divergent series could give excellent approximations for important special functions. It is in the work of Poincaré and Stieltjes, that the general notion of asymptotic expansion appeared. To be sure, quite a few mathematicians, among them Euler, Abraham de Moivre, James Stirling, Pierre-Simon Laplace and Adrien-Marie Legendre, had used asymptotic expansions in particular cases. Nevertheless, the formal concept of asymptotic expansion began with Stieltjes and Poincaré ([5], p. 1; [20], p. 151) and it is now used also in algebraic equations and partial differential equations, of interest in the applied sciences.

As for Ernesto Cesàro, he worked in a completely different direction, arguing, in an article published in 1890 in the *Bulletin des Sciences Mathématiques*, that summing a divergent series could mean something altogether different from Cauchy's definition.

It is Cesàro's approach that is relevant in our context. We will discuss it in the next section.

3 A divergent series can have a sum

We begin with a definition.

Definition 1. *A series $\sum_{j\geq 0} a_j$ is summable according to Cesàro's method, or it is Cesàro summable, if there is a number a such that*

$$\lim_{n\to\infty} \frac{s_0+s_1+s_2+\cdots+s_n}{n+1} = a, \tag{3}$$

where $s_n = \sum_{0\leq j\leq n} a_j$ for $n \geq 0$.

If the series is Cesàro summable, we write

$$\sum_{j\geq 0} a_j = a \ (C,1),$$

and we say that a is the Cesàro sum of the series, or the $(C,1)$ sum of the series.

Since a is the limit of a sequence, a is uniquely determined.

Let us observe that the average

$$c_n = \frac{s_0+s_1+s_2+\cdots+s_n}{n+1}$$

for $n \geq 0$, provides a way of finding the terms of the sequence $\{c_n\}_{n\geq 0}$ recursively:

$$\begin{aligned} c_0 &= s_0 = a_0, \\ c_n &= \frac{n}{n+1}c_{n-1} + \frac{s_n}{n+1} = \frac{nc_{n-1}+s_n}{n+1} \text{ for } n \geq 1. \end{aligned} \tag{4}$$

Example 1. *The partial sums of the series* $\sum_{j\geq 0}(-1)^j = 1-1+1-1+\cdots$ *form the sequence* $\{1,0,1,0,\ldots\}$*, which does not converge. By using (4) with* $c_0 = a_0 = 1$*, we have*

n	c_n	n	c_n	n	c_n
1	$\frac{1}{2}$	3	$\frac{1}{2}$	5	$\frac{1}{2}$
2	$\frac{2}{3}$	4	$\frac{3}{5}$	6	$\frac{4}{7}$

and so on. Therefore, we guess that

$$c_n = \begin{cases} \frac{1}{2} & \textit{for} \quad n = 2k+1,\, k\geq 0 \\ \frac{k+1}{2k+1} & \textit{for} \quad n = 2k,\, k \geq 1 \end{cases},$$

which can be verified by a simple inductive argument on k.

Since the sequence $\{c_n\}_{n\geq 0}$ *converges to* $\frac{1}{2}$ *as* $n\to\infty$*, we conclude that*

$$\sum\nolimits_{j\geq 0}(-1)^j = \frac{1}{2}\ (C,1),$$

which gives a rigorous justification to Leibniz's result.

Example 2. *For the series* $\sum_{j\geq 0} j$*, which diverges to infinity,*

$$c_n = \frac{1}{n+1}\sum_{j=0}^{n} j \underset{induction}{=} \frac{n(n+1)}{2(n+1)} = \frac{n}{2}$$

for $n\geq 0$.

Thus, $c_n\to\infty$ *as* $n\to\infty$*. According to Definition 1, the series* $\sum_{j\geq 0} j$ *is not* $(C,1)$ *summable.*

Example 3. *We consider the series* $\sum_{j\geq 0}(-1)^j(j+1) = 1-2+3-4+\cdots$

The sequence of its partial sums is $\{1,-1,2,-2,3,-3,\ldots\}$*, which diverges.*

From the values

n	c_n	n	c_n	n	c_n
0	1	4	$\frac{3}{5}$	8	$\frac{5}{9}$
1	0	5	0	9	0
2	$\frac{2}{3}$	6	$\frac{4}{7}$	10	$\frac{6}{11}$
3	0	7	0	11	0

,

we guess that

$$c_n = \begin{cases} \frac{k+1}{2k+1} & \textit{for} \quad n = 2k,\, k\geq 0 \\ 0 & \textit{for} \quad n = 2k+1,\, k \geq 0, \end{cases}$$

which can be verified by induction.

The sequence $\{c_{2k}\}_{k\geq 0}$ *converges to* $\frac{1}{2}$ *as* $k\to\infty$*, while the sequence* $\{c_{2k+1}\}_{k\geq 0}$ *is identically zero. So, the sequence* $\{c_n\}_{n\geq 0}$ *diverges, from which we conclude that the series* $\sum_{j\geq 0}(-1)^j(j+1)$ *is not* $(C,1)$ *summable.*

Hardy attributes to Cauchy the following result ([8], p. 10):

If a series $\sum_{j\geq 0} a_j$ converges to a, the average $\frac{s_0+s_1+\cdots+s_n}{n+1}$ also converges to a as $n \to \infty$. In the context of $(C,1)$ summability, Cauchy's result is stated as follows:

Theorem 1. *Cesàro's method is regular. That is, if* $\sum_{j\geq 0} a_j$ *converges to* a*, then* $\sum_{j\geq 0} a_j = a\ (C,1)$.

Proof. If we fix $n_0 \geq 1$ and consider $n > n_0$, we can write

$$\begin{aligned} |c_n - a| &\leq \frac{|s_0 - a| + \cdots + |s_{n_0} - a|}{n+1} + \frac{1}{n+1}\sum_{j=n_0+1}^{n} |s_j - a| \\ &\leq \frac{n_0+1}{n+1}\max_{0\leq j\leq n_0} |s_j - a| + \frac{n-n_0}{n+1}\sup_{j\geq n_0+1} |s_j - a| \\ &\leq C\frac{n_0+1}{n+1} + \sup_{j\geq n_0+1} |s_j - a|, \end{aligned}$$

where $C = \sup_{j\geq 0} |s_j - a|$, which is finite because the sequence $\{s_j\}_{j\geq 0}$, being convergent, is also bounded.

If we fix $\varepsilon > 0$, according to (1) there is $n_0 = n_0(\varepsilon) \geq 1$ so that

$$\sup_{j\geq n_0+1} |s_j - a| \leq \varepsilon.$$

Therefore,

$$|c_n - a| \leq \underbrace{C\frac{n_0+1}{n+1}}_{(i)} + \varepsilon,$$

for all $n > n_0$.

Since there is $N = N(\varepsilon) > n_0$ such that $(i) \leq \varepsilon$ for $n \geq N$, we can say that

$$|c_n - a| < 2\varepsilon$$

for all $n \geq N$.

This completes the proof of the theorem. □

Cauchy's definition of convergence can be viewed as a particular summability method, which we call Cauchy's method.

As we showed in Example 2, the series $\sum_{j\geq 0} j$ is not $(C,1)$ summable. Nevertheless, it constitutes an example of the following general phenomenon ([8], p. 10):

If the series $\sum_{j\geq 0} a_j$ diverges to infinity, then the sequence $\{c_n\}_{n\geq 0}$ goes to infinity as $n \to \infty$. That is, Cesàro's method is regular in this extended sense, called complete regularity.

The method also enjoys other natural and useful properties.

Theorem 2. *(for the proof, see [8], p. 95, Theorem 40).*

1. *The* $(C,1)$ *method is linear: If* $\sum_{j\geq 0} a_j = a\ (C,1)$ *and* $\sum_{j\geq 0} b_j = b\ (C,1)$*, then*
$$\alpha \sum_{j\geq 0} a_j + \beta \sum_{j\geq 0} b_j = \alpha a + \beta b\ \ (C,1),$$
for all numbers α, β.
2. *The* $(C,1)$ *method is stable:* $\sum_{j\geq 0} a_j = a\ (C,1)$ *if, and only if,* $\sum_{j\geq 1} a_j = a - a_0$ $(C,1)$.

As a consequence of 2) in Theorem 2, we can conclude, inductively, that

$$\sum_{j\geq 0} a_j = a\ \ (C,1)\ \text{ if, and only if, } \sum_{j\geq k+1} a_j = a - a_0 - \cdots - a_k\ \ (C,1), \tag{5}$$

for any $k \geq 0$ fixed.

The convergence of a series is linear and stable. Furthermore, the convergence of a series and its sum, or its divergence, does not change if we “dilute” the series. That is, if we insert any number of zeros as terms of the series, in any way ([8], p. 59, Section 3.9).

From (5), the $(C,1)$ summability of the series $\sum_{j\geq 0} a_j$ does not change if we add a finite number of zeros.

Indeed, let us suppose that the series $\sum_{j\geq 0} a_j$ is $(C,1)$ summable to a and that all the zeros are inserted before a_0 and between a_0 and a_k for some $k \geq 0$. According to (5), the series $\sum_{j\geq k+1} a_j$ is $(C,1)$ summable to $a - a_0 - \cdots - a_k$, which remains the same if we reinsert all the zeros in the appropriate places.

However, adding an infinite number of zeros, can destroy the summability or change the sum, of a $(C,1)$ summable series ([8], p. 60). For example,

Example 4. *We claim that*

$$1 - 1 + 0 + 1 - 1 + 0 + \cdots = \frac{1}{3}\ (C,1), \tag{6}$$

while we showed in Example 1 that $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}\ (C,1)$.

To verify (6), we begin by calculating a few partial sums

n	s_n	n	s_n	n	s_n	n	s_n
0	1	3	1	6	1	9	1
1	0	4	0	7	0	10	0
2	0	5	0	8	0	11	0

,

from which it can be proved by induction on k *that*

$$s_n = \begin{cases} 1 & \textit{if} \quad n = 3k \quad \textit{for} \quad k \geq 0 \\ 0 & \textit{if} \quad n = 3k+1 \quad \textit{for} \quad k \geq 0 \\ 0 & \textit{if} \quad n = 3k+2 \quad \textit{for} \quad k \geq 0. \end{cases}$$

Therefore, if $n = 3k$ *for* $k \geq 0$,

$$c_n = \frac{s_0 + \cdots + s_n}{n+1} = \frac{k+1}{3k+1} \underset{k\to\infty}{\to} \frac{1}{3},$$

if $n = 3k + 1$ *for* $k \geq 0$,

$$c_n = \frac{s_0 + \cdots + s_n}{n+1} = \frac{k+1}{3k+1} \underset{k\to\infty}{\to} \frac{1}{3},$$

and, if $n = 3k + 2$ *for* $k \geq 0$,

$$c_n = \frac{s_0 + \cdots + s_n}{n+1} = \frac{k+1}{3k+2} \underset{k\to\infty}{\to} \frac{1}{3}.$$

The sets $\{3k\}_{k\geq 0}$, $\{3k+1\}_{k\geq 0}$ *and* $\{3k+2\}_{k\geq 0}$ *consist of the numbers that, when divided by* 3*, have a remainder equal to* 0*,* 1*, and* 3*, respectively. That is to say, these sets are the three congruence classes modulo* 3*, which, therefore, form a partition of the natural numbers.*

Hence, there is

$$\lim_{n\to\infty} c_n = \frac{1}{3}.$$

Daniel Bernoulli had already applied the $(C, 1)$ method, in 1713, to some specific type of series ([8], p. 8). The method had also been used by Ferdinand George Frobenius in an article published in the *Journal für die reine und angewandte Mathematik* (Crelle's Journal) in 1880 ([8], pp. 8 and 389). Actually, it was already known in Euler's time that averaging the terms of a sequence, could improve its behavior ([15], p. 4). Still, all these observations were made in specific cases. The idea of defining the sum of a general divergent series rests firmly with Cesàro.

Cesàro's method is denoted $(C, 1)$, rather than (C), because in its 1890 article, Cesàro actually defines a whole family of summation methods, (C, k), by iteration. More precisely, the method (C, k) for $k \geq 2$ consists of iterating the partial sum k times, with the result being divided by a number depending on n and k, that is equal to $n + 1$ when $k = 1$ (see [8], Section 5.4). Since we will restrict ourselves to the method $(C, 1)$, no more will be said about the (C, k) method for $k \neq 1$.

Euler's work on the series $1 - 1 + 1 - 1 + \cdots$ suggests the following summability method.

Definition 2. *If the series* $\sum_{j\geq 0} a_j r^j$ *converges for* $0 \leq r < 1$ *with sum* $f(r)$ *and there is* $\lim_{r\to 1^-} f(r) = a$*, we say that the series* $\sum_{j\geq 0} a_j$ *is Abel summable with sum* a *or*

$$\sum\nolimits_{j\geq 0} a_j = a \ (A).$$

For instance,

$$\sum\nolimits_{j\geq 0} (-1)^j (j+1) = \frac{1}{4} \ (A), \tag{7}$$

which is a particular case of the following example:

Example 5. *We claim that, for each $n \geq 1$,*

$$\sum_{j \geq 0} (-1)^j (j+1)^n = -\frac{\sum\limits_{j=1}^{n} \left(\sum\limits_{i=0}^{j} (-1)^i \binom{n+1}{i} (j-i)^n \right) (-1)^j}{2^{n+1}} \; (A). \tag{8}$$

For $0 < |c| < 1$ we consider the series

$$\sum_{j \geq 0} (j+1)^n c^j = \frac{1}{c} \sum_{k \geq 1} k^n c^k.$$

The series $\sum_{k \geq 1} k^n c^k$ has a long and distinguished history, beginning with Euler's investigations on the series

$$-\sum_{j \geq 0} \frac{(-1)^s}{j^s}$$

for $s = -1, -2, -3, \ldots$, which was later named Dirichlet's eta function, after the mathematician Peter Gustav Lejeune-Dirichlet.

The sum of the series $\sum_{k \geq 1} k^n c^k$, in the sense of convergence, was calculated in [2], using purely analytic methods:

$$\sum_{k \geq 1} k^n c^k = \frac{\sum\limits_{j=1}^{n} \left(\sum\limits_{i=0}^{j} (-1)^i \binom{n+1}{i} (j-i)^n \right) c^j}{(1-c)^{n+1}} \tag{9}$$

for $n \geq 1$, $|c| < 1$.

Such a proof might not be expected. Indeed, the answer involves the so-called Euler's polynomials, whose coefficients have a combinatorial meaning in the context of permutations, as it is explained in ([2], Section 5). Therefore, the traditional proof of (9) is combinatorial in nature.

In any case, we do have the sum of the series and that is all we need here.

Now, we can write, for $c = -r$, $0 < r < 1$,

$$\sum_{j \geq 0} (j+1)^n (-r)^j = -\frac{\sum\limits_{j=1}^{n} \left(\sum\limits_{i=0}^{j} (-1)^i \binom{n+1}{i} (j-i)^n \right) (-r)^j}{r(1+r)^{n+1}}. \tag{10}$$

The right-hand side of (10) has limit, when $r \to 1^-$, equal to

$$-\frac{\sum\limits_{j=1}^{n} \left(\sum\limits_{i=0}^{j} (-1)^i \binom{n+1}{i} (j-i)^n \right) (-1)^j}{2^{n+1}}.$$

Therefore, we have verified (8). Finally, when $n = 1$, we have

$$\begin{aligned}\sum_{j=1}^{n}\left(\sum_{i=0}^{j}(-1)^i\binom{n+1}{i}(j-i)^n\right)(-r)^j &= -\sum_{i=0}^{1}(-1)^i\binom{2}{i}(1-i)\\ &= -1,\end{aligned}$$

which gives us (7).

It seems strange that a summation method would carry the name of Niels Erik Abel, an ardent admirer of Cauchy, and a fierce opponent to the use of divergent series.

Indeed, in Abel's words, written in 1828, "divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever." ([8], preface by John Edensor Littlewood).

The reason for the association is Abel's theorem (for the proof see, for instance, [6], p. 330, Theorem 7.26 and p. 331, Corollary 7.28):

Theorem 3. *If the series $\sum_{j\geq 0} a_j$ converges, then the series $\sum_{j\geq 0} a_j r^j$ converges for $0 \leq r \leq 1$ and*

$$\sum_{j\geq 0} a_j = \lim_{r\to 1^-}\sum_{j\geq 0} a_j r^j.$$

Abel's theorem guarantees that the Abel method is regular, in the sense of Theorem 1.

A result which assures the convergence of some kind of average assuming the summability of a certain series by Cauchy's method of convergence, is called an Abelian theorem.

The converse of an Abelian theorem is usually false. In fact, a summation method for which the converse is true is a trivial method since it only sums convergent series. However, modified versions of the converse can be true and of great interest. As an example, we mention a result due to Hardy.

Theorem 4. *If $\sum_{j\geq 0} a_j = a\ (C, 1)$ and the sequence $\{ja_j\}_{j\geq 0}$ is bounded, then $\sum_{j\geq 0} a_j$ converges to a.*

Hardy proves Theorem 4 in [8] as a particular case of a more general result ([8], p. 121, Theorem 63).

A proof of Theorem 4 attributed to Littlewood, is given in ([20], p. 156).

Modified converses of Abelian theorems are called Tauberian theorems, because it was the mathematician Alfred Tauber who proved the first one:

Theorem 5. *If $\sum_{j\geq 0} a_j = a\ (A)$ and the sequence $\{ja_j\}_{j\geq 0}$ converges to zero as $j \to \infty$, then $\sum_{j\geq 0} a_j$ converges to a.*

A proof of this result is in ([8], p. 149, Theorem 85).

Examples 3 and 5 show that there are (A) summable series that are not $(C, 1)$ summable. On the other hand,

Theorem 6. *(for the proof see, for instance, [8], p. 108, Section 5.12) Cesàro summability* $(C, 1)$ *implies Abel summability* (A)*, with the same sum.*

That is, (A) summability is strictly stronger that $(C, 1)$ summability, meaning that it sums more series.

Having investigated the two methods that interest us, it is now time to turn our attention to Fourier series, as an excellent ground where to test them. To be sure, much more can be said about summation methods, for which we refer to [8], as well as to the book by Lloyd Leroy Smail [19] which discusses every one of the many results on summability published up to 1925.

For completeness, we begin our discussion of Fourier series recounting briefly the genesis of the subject and the first convergence result (for more details, see [1]).

4 Fourier series and Dirichlet's result

Before Joseph Fourier, the nature of heat was not well understood. Indeed, in 1736, the French Academy called for essays on the topic "The nature and the propagation of 'fire'", where the word 'fire' was meant to signify 'heat'. All the submissions, including Euler's, missed the point and attempted to explain how fires develop ([10], p. 5).

Nevertheless, according to Umberto Bottazzini ([3], p. 59), by the end of the eighteenth century, heat was starting to be perceived as a form of energy that could aid in production. There was an ever increasing use of steam engines in industrial processes, particularly in England and France. "But if it is the *practical interests* that are best expressed in the English textile mills, it is the *theoretical aspects* that particularly engaged the French scientists." ([3], p. 59).

Under the title *The Analytical Theory of Heat*, Fourier published in 1822 two pieces, written in 1807 and 1811. In a radical departure from the work of others, Fourier developed a mathematical model for the propagation of heat, a differential equation known as heat equation.

To solve the heat equation, Fourier used certain series, now called Fourier series. At the time, the heat equation was viewed as Fourier's crowning achievement, while the series "were considered a disgrace." ([10], p. 6).

The topic of Fourier series basically rests upon the formulas

$$f(x) = a_0 + \sum_{n\geq 1} \left(a_n \cos nx + b_n \sin nx\right), \tag{11}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx, \tag{12}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx,\ n \neq 0, \tag{13}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx,\ n \neq 0. \tag{14}$$

Fourier shows in several particular cases, that the series converges to the function f, meaning that the series converges pointwise to $f(x)$, for each x. Then, he proceeds to state that "all the series converge". Later on, he says "we must remark that our demonstration applies to an entirely arbitrary function." ([10], p. 12).

In spite of these rather exuberant statements, the following question persisted: Does the series on the right hand side of (11) really converge to $f(x)$ for all x? After several mathematicians of the time, including Cauchy, produced more or less faulty proofs, Dirichlet showed pointwise convergence under rather general conditions. His work was published in 1829 in *Journal für die reine und angewandte Mathematik* (Crelle's Journal).

In Jean-Pierre Kahane's words ([10], p. 31), "The article of Dirichlet on Fourier series is a turning point in the theory and also in the way mathematical analysis is approached and written. Its intention is simply to give a correct statement and a correct proof of the convergence of Fourier series. The result is a paradigm of what is correctness in analysis."

Kahane reproduces the full article in pages 36 to 46 of [10]. A discussion of Dirichlet's work, and much more, is found in the 2009 reprint of a monograph by Henri Lebesgue [16]. It makes for an instructive reading, since "[the] book reproduces the text of the original edition. The content and language reflect the beliefs, practices and terminology of their time, and have not been updated."

Here is Dirichlet's result (for the proof see, for instance, [15], Theorem 16.4, p. 61):

Theorem 7. *(Dirichlet) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function that is continuous and has a bounded continuous derivative, except, possibly, at a finite number of points in each interval of length 2π. Then, the equality (11) holds at every $x \in \mathbb{R}$ where f is continuous.*

That is to say, under these assumptions, the Fourier series of f is summable to f in the sense of pointwise convergence, what we called before, Cauchy's method. This result and its ramifications, were discussed at length in [1], so we proceed now to the core of our work.

5 Fejér and Abel meet Fourier

For quite sometime after Dirichlet proved his convergence result, there was hope that the hypotheses could be weaken to the extent of proving the convergence of the Fourier series at any point of continuity.

However, these hopes were dashed in 1873, when Paul du Bois-Reymond constructed a 2π-periodic and continuous function whose Fourier series does not converge at zero (for the details see, for instance, [15], Chapter 18). Consequently, a new question was posed: If a function is 2π-periodic and continuous, is there a way of recovering the function from the coefficients a_n and b_n given by (12), (13) and (14)?

At the tender age of nineteen, Leopold Fejér showed, in a note published in 1900 in the *Comptes-Rendus de l'Académie des Sciences de Paris*, that the answer is yes. The full article appeared in *Mathematische Annalen* in 1904.

Theorem 8. *(Fejér) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function that is Riemann integrable on $[-\pi, \pi]$. Then,*

$$a_0 + \sum_{n\geq 1} (a_n \cos nx + b_n \sin nx) = f(x) \quad (C, 1),$$

at every $x \in \mathbb{R}$ where f is continuous.

If f is continuous on $\mathbb{R}$, the convergence of the Cesàro means $\{C_k(x)\}_{k\geq 0}$ to $f(x)$ is uniform on $x \in \mathbb{R}$. That is,

$$\sup_{x\in\mathbb{R}} |C_k(x) - f(x)| \underset{k\to\infty}{\to} 0.$$

Theorem 6 gives us immediately the following result:

Theorem 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function that is Riemann integrable on $[-\pi, \pi]$. Then,*

$$a_0 + \sum_{n\geq 1} (a_n \cos nx + b_n \sin nx) = f(x) \quad (A),$$

at every $x \in \mathbb{R}$ where f is continuous.

As in the case of Dirichlet's convergence result, the proof of Theorem 8 rests upon the possibility of having a convenient representation for the Cesàro means, as an integral operator of the form

$$f \to \int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt. \tag{15}$$

Likewise, a direct proof of Theorem 9 depends on having a similar representation for the Abel means.

Remark 1. *The function $\mathcal{K}_n : \mathbb{R} \to \mathbb{R}$ is called the kernel of the operator (15). Later on, we will be able to work under the assumption that, for each $n = 0, 1, 2, \ldots$, the kernel $\mathcal{K}_n$ is 2π-periodic, and, at least, continuous. So, from now on, we will make such assumption. In doing so, we follow Yitzhak Katznelson (see [12], p. 9, Definition 2.2).*

We state now a lemma concerning the formula appearing in (15).

Lemma 1. *The following statements hold:*

1. *Let $F : \mathbb{R} \to \mathbb{R}$ be 2π-periodic. Then, for each $k \in \mathbb{Z}$ different from zero, $2k\pi$ is also a period.*
2. *Let $F : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, and Riemann integrable on $[-\pi, \pi]$. If $x \in \mathbb{R}$ is fixed, the function $y \to F(x-y)$ is 2π-periodic, and Riemann integrable on $[-\pi, \pi]$.*
3. *Let $F : \mathbb{R} \to \mathbb{R}$ be 2π-periodic. Then, F takes the same values on any interval of length 2π.*

4. *Let $F:\mathbb{R}\to\mathbb{R}$ be 2π-periodic, and Riemann integrable on $[-\pi,\pi]$. Then,*

$$\int_{-\pi}^{\pi} F(y)\,dy = \int_{-\pi}^{\pi} F(x-y)\,dy$$

for every $x\in\mathbb{R}$ fixed.

5. *Let $f,g:\mathbb{R}\to\mathbb{R}$ be 2π-periodic functions that are Riemann integrable on $[-\pi,\pi]$. Then, for each $x\in\mathbb{R}$ fixed, the functions $y\to f(x-y)\,g(y)$ and $y\to f(y)\,g(x-y)$ are Riemann integrable on $[-\pi,\pi]$ and*

$$\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy = \int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy.$$

The statements made in Lemma 1 about Riemann integrability, can be found in many textbooks (for instance, see [7], Chapter 5, and [18], Appendix, pp. 280-288). As for the other statements, we sketch the proof next.

Proof. By the 2π-periodicity of F, if $y\in\mathbb{R}$,

$$F(y) = F(y-2\pi+2\pi) = F(y-2\pi),$$

$$F(y+2k\pi) = F(y+2(k-1)\pi+2\pi) = F(y+2(k-1)\pi)$$

for $k=2,3,\ldots$, and

$$F(y+2k\pi) = F(y+2(k+1)\pi-2\pi) = F(y+2(k+1)\pi)$$

for $k=-2,-3,\ldots$

Therefore, 1) follows by induction. As for 2), if $x\in\mathbb{R}$ is fixed,

$$F(x-(y+2\pi)) = F(x-y-2\pi) = F(x-y).$$

So, we have 2).

For 3), we fix an arbitrary interval $[-\pi+a,\pi+a]$ for $a\in\mathbb{R}$ fixed. We only need to observe that given $t\in[-\pi+a,\pi+a]$, there is $y\in[-\pi,\pi]$ so that $F(t)=F(y)$ and reciprocally.

To prove 4), we fix $x\in\mathbb{R}$. Then,

$$\int_{-\pi}^{\pi} F(y)\,dy \underset{(i)}{=} \int_{-\pi+x}^{\pi+x} F(y)\,dy \underset{y\to s=-y}{=} -\int_{\pi-x}^{-\pi-x} F(-s)\,ds$$
$$= \int_{-\pi-x}^{\pi-x} F(-s)\,ds \underset{s\to y=x+s}{=} \int_{-\pi}^{\pi} F(x-y)\,dy,$$

where we have used 3) in (i).

Finally, if $x \in \mathbb{R}$ is fixed, let $F : \mathbb{R} \to \mathbb{R}$ be the function $y \to f(x-y)\, g(y)$. As we proved in 4),

$$\int_{-\pi}^{\pi} F(y)\, dy = \int_{-\pi}^{\pi} F(x-y)\, dy,$$

or

$$\int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy = \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

which is 5).

This completes the proof of the lemma. □

Part 5) in Lemma 1, shows that the operation

$$(f, g) \to \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy$$

is commutative. It is called the periodic convolution of f and g. It gives a function, denoted $f * g$, which is 2π-periodic, and continuous from $\mathbb{R}$ into $\mathbb{R}$ (see [18], pp. 45-48, Proposition 3.1 (v), Lemma 3.2). Under the assumptions in Remark 1, Lemma 1 applies to the formula appearing in (15).

6 The Fejér kernel and the Poisson kernel

Let us recall that the nth partial sum S_n in Cauchy's method for the series in (11) is

$$S_n(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx),$$

where the coefficients are given by the formulas (12), (13) and (14), with $f : \mathbb{R} \to \mathbb{R}$ a 2π-periodic function, Riemann integrable on $[-\pi, \pi]$.

For convenience, we will use complex exponentials, although "they were not used in Fourier series until well into the twentieth century" ([10], p. 2). The identities

$$\begin{aligned} \cos jx &= \frac{e^{ijx} + e^{-ijx}}{2}, \\ \sin jx &= \frac{e^{ijx} - e^{-ijx}}{2i} \end{aligned}$$

give

$$S_n(x) = \sum_{j=0}^{n} \frac{1}{2}\left(a_j + \frac{b_j}{i}\right) e^{ijx} + \sum_{j=0}^{n} \frac{1}{2}\left(a_j - \frac{b_j}{i}\right) e^{-ijx},$$

where

$$\begin{aligned} \frac{1}{2}\left(a_j + \frac{b_j}{i}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ijt} f(t)\, dt, \\ \frac{1}{2}\left(a_j - \frac{b_j}{i}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ijt} f(t)\, dt, \end{aligned}$$

for $j \geq 1$.

Therefore,

$$S_n(x) = \int_{-\pi}^{\pi} \left[\frac{1}{2\pi} \sum_{j=-n}^{n} e^{ij(x-t)} \right] f(t), \tag{16}$$

As it was proved in [1], manipulating the expression $\frac{1}{2\pi} \sum_{j=-n}^{n} e^{ij(x-t)}$ we can write

$$S_n(x) = \int_{-\pi}^{\pi} \mathcal{D}_n(x-t) f(t)\, dt$$

where the real function $\mathcal{D}_n(t)$, called the Dirichlet kernel, is

$$\begin{matrix} \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} & \text{if} & t \neq 0 \\ \frac{2n+1}{2\pi} & \text{if} & t = 0. \end{matrix} \tag{17}$$

Definition 3. *We state here the definition of partial sum for each of the two summation methods, Cesàro's and Abel's.*

1. *The nth partial sum C_n in Cesàro's method is*

$$C_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} S_j(x).$$

2. *Abel's method is a little different. The partial sum is defined as*

$$A_r(x) = a_0 + \sum_{j \geq 1} (a_j \cos jx + b_j \sin jx)\, r^j,$$

indexed by a continuous parameter r, $0 \leq r < 1$. As we will see, this difference does not cause any trouble and actually can be avoided.

Now, we are ready to prove that each of these partial sums can be written as an integral operator of the form (15). We will assume that the function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and Riemann integrable on $[-\pi, \pi]$.

Lemma 2. *The nth partial sum C_n in Cesàro's method can be written as*

$$C_n(x) = \int_{-\pi}^{\pi} \mathcal{F}_n(x-t) f(t)\, dt,$$

where the real function $\mathcal{F}_n(t)$, called Fejér kernel, is, for $|t| \leq \pi$,

$$\mathcal{F}_n(t) = \begin{cases} \frac{1}{2\pi(n+1)} \frac{1-\cos(n+1)t}{1-\cos t} & \text{if} \quad t \neq 0 \\ \frac{n+1}{2\pi} & \text{if} \quad t = 0 \end{cases} \tag{18}$$

or, equivalently,

$$\mathcal{F}_n(t) = \begin{cases} \frac{1}{2\pi(n+1)} \frac{\sin^2 \frac{n+1}{2} t}{\sin^2 \frac{t}{2}} & \text{if} \quad t \neq 0 \\ \frac{n+1}{2\pi} & \text{if} \quad t = 0. \end{cases} \tag{19}$$

Proof. We have, for $0 < |t| \leq \pi$,

$$\mathcal{F}_n(t) = \frac{1}{n+1}\sum_{j=0}^{n}\mathcal{D}_j(t) = \frac{1}{2\pi(n+1)}\sum_{j=0}^{n}\frac{\sin\left(j+\frac{1}{2}\right)t}{\sin\frac{t}{2}}.$$

Using the identity

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\ \sin\beta$$

with $\alpha = \left(j+\frac{1}{2}\right)t$ and $\beta = \frac{t}{2}$, we can write

$$\begin{aligned}\frac{1}{2\pi(n+1)}\sum_{j=0}^{n}\frac{\sin\left(j+\frac{1}{2}\right)t}{\sin\frac{t}{2}} &= \frac{1}{2\pi(n+1)}\sum_{j=0}^{n}\frac{\sin\left(j+\frac{1}{2}\right)t\ \sin\frac{t}{2}}{\sin^2\frac{t}{2}}\\ &= -\frac{1}{4\pi(n+1)\sin^2\frac{t}{2}}\sum_{j=0}^{n}(\cos(j+1)t - \cos jt)\\ &= \frac{1}{4\pi(n+1)}\frac{1-\cos(n+1)t}{\sin^2\frac{t}{2}}.\end{aligned}$$

Since

$$\cos t = \cos\left(\frac{t}{2}+\frac{t}{2}\right) = \cos^2\frac{t}{2} - \sin^2\frac{t}{2} = 1 - 2\sin^2\frac{t}{2},$$

we have

$$\mathcal{F}_n(t) = \frac{1}{2\pi(n+1)}\frac{1-\cos(n+1)t}{1-\cos t}.$$

Alternatively, if we write

$$\cos(n+1)t = \cos\left(\frac{n+1}{2}t + \frac{n+1}{2}t\right) = 1 - 2\sin^2\frac{n+1}{2}t,$$

we have

$$\mathcal{F}_n(t) = \frac{1}{2\pi(n+1)}\frac{\sin^2\frac{n+1}{2}t}{\sin^2\frac{t}{2}}.$$

By L'Hôpital's rule, we define

$$\mathcal{F}_n(0) = \lim_{t\to 0}\mathcal{F}_n(t) = \frac{n+1}{2\pi}.$$

The proof of the lemma is complete. □

Remark 2. *Extended by periodicity, the function $\mathcal{F}_n$, from $\mathbb{R}$ to $\mathbb{R}$, is 2π-periodic, non-negative, even, and it is continuous with continuous derivatives of all orders.*

Lemma 3. *The partial sum A_r in Abel's method can be written as*

$$A_r(x) = \int_{-\pi}^{\pi} \mathcal{P}_r(x-t) f(t)\, dt,$$

where the real function $\mathcal{P}_r(t)$, called Poisson kernel after the mathematician Siméon Denis Poisson, is, for $|t| \leq \pi$,

$$\mathcal{P}_r(t) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos t + r^2}.$$

Proof. Using (16), we write, for $0 \leq r < 1$,

$$\begin{aligned} A_r(x) &= \sum_{j\geq 0} (a_j \cos jx + b_j \sin jx) r^j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt \\ &+ \sum_{j\geq 1} \frac{r^j}{2\pi} \int_{-\pi}^{\pi} e^{-ij(x-t)} f(t)\, dt + \sum_{j\geq 1} \frac{r^j}{2\pi}\int_{-\pi}^{\pi} e^{ij(x-t)} f(t)\, dt. \end{aligned} \tag{20}$$

The general term of each series is bounded by Br^j, uniformly on t, where

$$B = \sup_{|t|\leq\pi} |f(t)|. \tag{21}$$

Therefore, using Weierstrass's M-test (see, for instance, [7], p. 219, Theorem 7.2), we can interchange the series with the integral. Thus,

$$\begin{aligned} A_r(x) &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\left(\sum_{j\geq 1} e^{-ij(x-t)} r^j + \sum_{j\geq 0} e^{ij(x-t)} r^j\right) f(t)\, dt \\ &\underset{(i)}{=} \int_{-\pi}^{\pi} \frac{1}{2\pi}\left(\frac{re^{-i(x-t)}}{1-re^{-i(x-t)}} + \frac{1}{1-re^{i(x-t)}}\right) f(t)\, dt \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{re^{-i(x-t)}\left(1-re^{i(x-t)}\right) + 1 - re^{-i(x-t)}}{\left(1-re^{-i(x-t)}\right)\left(1-re^{i(x-t)}\right)} f(t)\, dt \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(x-t)+r^2} f(t)\, dt, \end{aligned}$$

where we have used in (i) the formula for the sum of a geometric series.

This completes the proof of the lemma. □

Remark 3. *Extended by periodicity, the function $\mathcal{P}_n$, from $\mathbb{R}$ to $\mathbb{R}$, is 2π-periodic, positive, even, and it is continuous with continuous derivatives of all orders.*

In the next section we introduce the notion of good kernel, discussed by Rami Shakarchi and Elias M. Stein in [18], and investigate its importance.

7 Good kernels

We begin with the following definition:

Definition 4. *([18], p. 48) Given an integral operator of the form (15), the kernel $\mathcal{K}_n$ is called a good kernel if it satisfies the following conditions:*

1. $$\int_{-\pi}^{\pi} \mathcal{K}_n(t)\, dt = 1$$
   for all $n \geq 0$.
2. *There is $C > 0$ so that*
   $$\int_{-\pi}^{\pi} |\mathcal{K}_n(t)|\, dt \leq C$$
   for all $n \geq 0$.
3. *For each $0 < \delta < \pi$ fixed, there is*
   $$\lim_{n\to\infty} \int_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(t)|\, dt = 0.$$

The significance of Definition 4 is shown in the result that follows.

Theorem 10. *([18], p. 49, Theorem 4.1) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function that is Riemann integrable on $[-\pi, \pi]$. Then,*

a) *if $\mathcal{K}_n$ is a good kernel, there is*
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt = f(x)$$
at each $x \in \mathbb{R}$ where the function f is continuous, and

b) *the limit is uniform on $x \in \mathbb{R}$, when f is continuous everywhere.*

Proof. According to Lemma 1,

$$\int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt \underset{t\to s=x-t}{=} -\int_{x+\pi}^{x-\pi} \mathcal{K}_n(s) f(x-s)\, ds$$
$$= \int_{x-\pi}^{x+\pi} \mathcal{K}_n(s) f(x-s)\, ds,$$

Also by Lemma 1, the above is equal to

$$\int_{-\pi}^{\pi} \mathcal{K}_n(s) f(x-s)\, ds. \tag{22}$$

Therefore,

$$\left|\int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt - f(x)\right| \underset{(ii)}{=} \left|\int_{-\pi}^{\pi} \mathcal{K}_n(s) f(x-s)\, ds - f(x)\int_{-\pi}^{\pi} \mathcal{K}_n(s)\, ds\right|$$
$$= \left|\int_{-\pi}^{\pi} \mathcal{K}_n(s)\left(f(x-s) - f(x)\right) ds\right|,$$

where we have used 1) in Definition 4, in (ii).

If the function f is continuous at x, given $\varepsilon > 0$, there is $\delta = \delta(x, \varepsilon) > 0$, which we can choose smaller than π, so that

$$|f(x-s) - f(x)| \le \varepsilon$$

for $|s| < \delta$.

Then, we can write

$$\begin{aligned}\left|\int_{-\pi}^{\pi} \mathcal{K}_n(s)\left(f(x-s) - f(x)\right) ds\right| &\le \varepsilon \int_{|s|<\delta} |\mathcal{K}_n(s)|\, ds \\ &\quad + 2 \sup_{|t|\le\pi} |f(t)| \int_{\delta\le|t|\le\pi} |\mathcal{K}_n(s)|\, ds \\ &\underset{(iii)}{\le} C\varepsilon + 2B \int_{\delta\le|t|\le\pi} |\mathcal{K}_n(s)|\, ds,\end{aligned}$$

where $B = \sup_{|t|\le\pi} |f(t)|$ and we have used 2) in Definition 4, in the first term of (iii).

Finally, 3) tells us that there is $N = N(\varepsilon) \ge 1$ so that

$$\int_{\delta\le|t|\le\pi} |\mathcal{K}_n(s)|\, ds \le \varepsilon$$

for $n \ge N$.

This completes the proof of a).

As for b), we only need to observe that when f is continuous everywhere, it is uniformly continuous on $[-\pi, \pi]$, and also on $\mathbb{R}$ because f is periodic. Then, δ can be chosen independently of x and, therefore,

$$\sup_{x\in\mathbb{R}} \left|\int_{-\pi}^{\pi} \mathcal{K}_n(s)\left(f(x-s) - f(x)\right) ds\right| \le (C + 2B)\,\varepsilon.$$

So, we have proved b).

This completes the proof of the theorem. □

Remark 4. *The three conditions in Definition 4, only require that the function $\mathcal{K}_n$ is Riemann integrable on $[-\pi, \pi]$. In fact, that is the only assumption made by Shakarchi and Stein, with no reference to periodicity. As a consequence, they are forced to define the* *convolution as in (22), which is no longer a commutative operation. In our setting, under the assumptions stated in Remark 1, the expression (22) is an easy consequence of how the convolution $\mathcal{K}_n * f$ is defined.*

The conditions in Definition 4, are already considered in ([12], p. 9), where kernels satisfying those conditions are called summability kernels. In ([21], pp. 85-86), the third condition in Definition 4, appears in the following form:

For each $0 < \delta < \pi$ fixed, there is

$$\lim_{n\to\infty} \sup_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(t)| = 0. \tag{23}$$

Actually, in [12], 3) in Definition 4 is written in the following manner:

For each $0 < \delta < \pi$ fixed, there is

$$\lim_{n\to\infty} \int_{\delta}^{2\pi-\delta} |\mathcal{K}_n(t)|\, dt = 0.$$

We claim that for each $0 < \delta < \pi$ fixed,

$$\int_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(t)|\, dt = \int_{\delta}^{2\pi-\delta} |\mathcal{K}_n(t)|\, dt$$

assuming, as we have done, that $\mathcal{K}_n$ is 2π-periodic. Indeed,

$$\begin{aligned}\int_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(t)|\, dt &= \int_{-\pi}^{\pi} |\mathcal{K}_n(t)|\, dt - \int_{|t|<\delta} |\mathcal{K}_n(t)|\, dt \\ &\underset{(iv)}{=} \int_{-\pi+\pi-\delta}^{\pi+\pi-\delta} |\mathcal{K}_n(t)|\, dt - \int_{|t|<\delta} |\mathcal{K}_n(t)|\, dt \\ &= \int_{-\delta}^{2\pi-\delta} |\mathcal{K}_n(t)|\, dt - \int_{-\delta}^{\delta} |\mathcal{K}_n(t)|\, dt \\ &= \int_{\delta}^{2\pi-\delta} |\mathcal{K}_n(t)|\, dt,\end{aligned}$$

where we have used Lemma 1 in (iv).

In view of Theorem 10, it will be of interest to test Definition 4, on each of the kernels $\mathcal{F}_n$ and $\mathcal{P}_r$.

Lemma 4. *The kernel $\mathcal{F}_n$ is a good kernel.*

Proof. We start by observing that the kernel $\mathcal{D}_n$ satisfies 1) in Definition 4. Indeed, when the function f is identically equal to one,

$$S_n(t) = a_0$$

for every $n \geq 0$. Since

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt = 1,$$

we have

$$\int_{-\pi}^{\pi} \mathcal{D}_n(t)\, dt = 1 \tag{24}$$

for every $n \geq 0$.

Hence,

$$\int_{-\pi}^{\pi} \mathcal{F}_n(t)\, dt = \int_{-\pi}^{\pi} \frac{1}{n+1} \sum_{j=0}^{n} \mathcal{D}_j(t)\, dt = \frac{1}{n+1} \sum_{j=0}^{n} \int_{-\pi}^{\pi} \mathcal{D}_j(t)\, dt \underset{(i)}{=} 1,$$

where we have used (24), in (i). Therefore, the kernel $\mathcal{F}_n$ satisfies 1). Since $\mathcal{F}_n$ is a non-negative function, 2) follows.

As for 3), if we fix $0 < \delta < \pi$, we have $\cos t \leq \cos \delta$ for $\delta \leq |t| \leq \pi$.

Using (18),

$$0 \leq \mathcal{F}_n(t) \leq \frac{1}{2\pi(1-\cos\delta)} \frac{1}{(n+1)} \tag{25}$$

for $\delta \leq |t| \leq \pi$. So,

$$\int_{\delta \leq |t| \leq \pi} \mathcal{F}_n(t)\, dt \leq \frac{2}{(1-\cos\delta)} \frac{1}{n+1} \underset{n\to\infty}{\to} 0$$

and 3) holds.

This completes the proof of the lemma. □

Lemma 5. *The kernel $\mathcal{P}_r$ is a good kernel.*

Proof. Since the definition of good kernel has been formulated for kernels depending on a discrete parameter n, a clarification is in order. We can proceed in two ways:

We could make an obvious reinterpretation of Definition 4 and Theorem 10 in terms of the parameter r, for $0 \leq r < 1$. Or, we could fix an arbitrary sequence $\{r_n\}_{n\geq 1}$ with $0 \leq r_n < 1$ for all n, converging to one as $n \to \infty$, testing Definition 4 on the kernel $\mathcal{P}_{r_n}$.

We choose the first option.

According to (20), the partial sum A_r is identically one when f is identically one. So,

$$1 = \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(x-t)+r^2} dt = \int_{-\pi}^{\pi} \mathcal{P}_r(x-t)\, dt = \int_{-\pi}^{\pi} \mathcal{P}_r(t)\, dt.$$

Therefore, 1) is satisfied. Since $\mathcal{P}_r$ is positive, 2) is satisfied as well.

To verify 3), we begin by observing that

$$\mathcal{P}_r(t) \leq \mathcal{P}_r(\delta) \tag{26}$$

for $\delta \leq t \leq \pi$. Indeed,

$$\mathcal{P}'_r(t) = -\frac{\left(1-r^2\right)2r\sin t}{\left(1-2r\cos t+r^2\right)^2},$$

which is non-positive, for $\delta \leq t \leq \pi$.

Since $\mathcal{P}_r$ is an even function, (26) is true for $\delta \leq |t| \leq \pi$.

Furthermore, there is

$$\lim_{r\to 1^-} \mathcal{P}_r(t) = 0, \tag{27}$$

for each $0 < |t| < \pi$.

Because (26) holds for $\delta \leq |t| \leq \pi$, the limit in (27) is uniform on $\delta \leq |t| \leq \pi$. So, we can take the limit under the integral sign, and 3) holds.

This completes the proof of the lemma. □

The fact that $\mathcal{D}_n$ is not a good kernel while $\mathcal{F}_n$ and $\mathcal{P}_r$ are, sets pointwise convergence apart from Cesàro summability and Abel summability.

Remark 5. *If $\mathcal{K}_n$ is a good kernel, given $0 < \delta < \pi$ fixed,*

$$\int_0^\delta |\mathcal{K}_n(t)|\,dt = \int_0^\pi |\mathcal{K}_n(t)|\,dt - \int_\delta^\pi |\mathcal{K}_n(t)|\,dt$$

can be made arbitrarily close to $\frac{C}{2}$, for n large enough, where C is the constant in 2) of Definition 4.

This observation applies, in particular, to the kernels $\mathcal{F}_n$ and $\mathcal{P}_r$ but, as observed in ([1], Remark 4), it does not apply to the kernel $\mathcal{D}_n$.

Remark 6. *A careful perusal of Lemma 4, specifically of estimate (25), would reveal that the kernel $\mathcal{F}_n$ satisfies (23). That is,*

For each $0 < \delta < \pi$, there is

$$\lim_{n\to\infty} \sup_{\delta\leq|t|\leq\pi} |\mathcal{F}_n(t)| = 0. \tag{28}$$

Likewise, since $\mathcal{P}_r$ is even, using (26) in Lemma 5, there is, for each $0 < \delta < \pi$,

$$\lim_{r\to 1^-} \sup_{\delta\leq|t|\leq\pi} |\mathcal{P}_r(t)| = 0. \tag{29}$$

It should be clear that each of (28) and (29) implies the appropriate version of 3) in Definition 4.

Lemma 4 and Lemma 5 tell us that the kernels $\mathcal{F}_n$ and $\mathcal{P}_n$ satisfy the conclusion of Theorem 10, therefore proving Theorem 8 and Theorem 9.

Theorem 4 (resp. Theorem 5) provides a qualified converse for Theorem 8 (resp. Theorem 9).

Theorem 10 can be extended to certain points of discontinuity. More precisely,

Definition 5. *A 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous if, on each interval of length 2π, is continuous except at a finite number of points, and at each of the points x of discontinuity, the $\lim_{t\to x^-} f(t)$ and the $lim_{t\to x^+} f(t)$ exist, with finite values denoted $f(x^-)$ and $f(x^+)$, respectively.*

It is plain that a 2π-periodic and piecewise continuous function $f : \mathbb{R} \to \mathbb{R}$ is bounded, and it is Riemann integrable on $[-\pi,\pi]$ and, of course, on any interval of length 2π.

Here is the version of Theorem 10 that goes with Definition 5. It should be clear that it holds true for the kernels $\mathcal{F}_n$ and $\mathcal{P}_r$.

Theorem 11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic and piecewise continuous function.*

A) *Let $\mathcal{K}_n : \mathbb{R} \to \mathbb{R}$ be a good kernel that is also even and non-negative.*
Then, there is

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt = \frac{f(x^-) + f(x^+)}{2} \tag{30}$$

at each $x \in \mathbb{R}$.

b) *The limit is uniform on $x \in \mathbb{R}$, when f is continuous everywhere.*

Proof. It should be clear that A) implies a) in Theorem 10, at each point x where the function f is continuous. Moreover, b) has already been proved, as part of Theorem 10. So, we fix a point $x \in \mathbb{R}$ where f is discontinuous in the sense of Definition 5.

The parity of the kernel implies that

$$\int_{-\pi}^{0} \mathcal{K}_n(t)\, dt = \int_{0}^{\pi} \mathcal{K}_n(t)\, dt = \frac{1}{2},$$

so,

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt - \frac{f(x^-) + f(x^+)}{2} \right| &= \left| \int_{-\pi}^{0} \mathcal{K}_n(t) \left(f(x-t) - f\left(x^+\right) \right) dt \right. \\
&\quad \left. + \int_{0}^{\pi} \mathcal{K}_n(t) \left(f(x-t) - f\left(x^-\right) \right) dt \right| \\
&\leq \int_{-\pi}^{0} \mathcal{K}_n(t) \left| f(x-t) - f\left(x^+\right) \right| dt \\
+ \int_{0}^{\pi} \mathcal{K}_n(t) \left| f(x-t) - f\left(x^-\right) \right| dt &= (i) + (ii).
\end{aligned}$$

For $0 < \delta < \pi$ fixed, we write

$$(i) = \left(\int_{-\pi}^{-\delta} + \int_{-\delta}^{0} \right) \mathcal{K}_n(t) \left| f(x-t) - f\left(x^+\right) \right| dt,$$

where

$$\left| \int_{-\delta}^{0} \mathcal{K}_n(t) \left| f(x-t) - f\left(x^+\right) \right| dt \right| = \int_{0}^{\delta} \mathcal{K}_n(t) \left| f(x+t) - f\left(x^+\right) \right| dt.$$

Given $\varepsilon > 0$, there is $\delta = \delta(x, \varepsilon) > 0$ so that $|f(x+t) - f(x^+)| \leq \varepsilon$ when $0 < t < \delta$. Therefore, for this value of δ, we can write

$$\int_{0}^{\delta} \mathcal{K}_n(t) \left| f(x+t) - f\left(x^+\right) \right| dt \leq \varepsilon.$$

Since f is bounded, using 3) in Definition 4,

$$\begin{aligned} & \int_{-\pi}^{-\delta} \mathcal{K}_n(t) \left| f(x-t) - f\left(x^+\right) \right| dt \\ \leq \quad & 2 \sup_{t \in \mathbb{R}} |f(t)| \int_{\delta \leq |t| \leq \pi} \mathcal{K}_n(t)\, dt \underset{n \to 0}{\to} 0. \end{aligned}$$

Hence, there is $N = N(\varepsilon) \geq 1$ such that

$$\int_{-\pi}^{-\delta} \mathcal{K}_n(t) \left| f(x-t) - f\left(x^+\right) \right| dt \leq 2 \sup_{t \in \mathbb{R}} |f(t)| \varepsilon,$$

for $n \geq N$.

With a very similar argument,

$$(ii) \leq \left(1 + 2 \sup_{t \in \mathbb{R}} |f(t)| \right) \varepsilon$$

for $n \geq N$.

This completes the proof of the theorem. □

Remark 7. *If we assign to f the value $\frac{f(x^-)+f(x^+)}{2}$ at each point x of discontinuity, we can say that the limit in (30) is $f(x)$, for every $x \in \mathbb{R}$.*

8 Summability versus convergence

As we have seen, that the kernels $\mathcal{F}_n$ and $\mathcal{P}_n$ are good in the sense of Definition 4, implies that Theorems 8 and 9 have fairly straightforward proofs as particular cases of Theorem 10. On the other hand, it was proved in [1] that the Dirichlet kernel does not satisfy 2) nor 3) in Definition 4, therefore showing that $\mathcal{D}_n$ is definitely not a good kernel. This badness justifies the difficulty, that persisted for a very long time, of establishing pointwise convergence results for the Fourier series, under minimal conditions. The problem was settled by Lennart Carleson, in the 1960s.

Theorem 12. *[4] Let $f : [-\pi, \pi] \to \mathbb{R}$ be a Lebesgue square integrable function on $[-\pi, \pi]$. Then, there is a set $E \subset [-\pi, \pi]$ of Lebesgue measure zero so that*

$$a_0 + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx) = f(x)$$

for $x \in [-\pi, \pi] \setminus E$.

In the *Mathematical Reviews*, **MR** 199631, Kahane refers to the results in Carleson's article as "spectacular" and catalogs the proofs as "very difficult" and "very delicate". Carleson's result was extended to p-integrable functions on $[-\pi, \pi]$, for $1 < p \leq \infty$, by Richard A. Hunt [9]. Let us observe that Hölder's inequality reduces the case $2 < p \leq \infty$ to Carleson's theorem.

As for the case $p = 1$, Andrey Kolmogorov had constructed already in [14] a function, Lebesgue integrable on $[-\pi, \pi]$, whose Fourier series diverges almost everywhere. Kolmogorov's result was improved by Yitzhak Katznelson (see [12], p. 59), who constructed a function, Lebesgue integrable on $[-\pi, \pi]$, for which the Fourier series diverges everywhere.

It is natural to wonder about the nature of the null sets where the Fourier series of 2π-periodic and continuous functions can diverge. In this respect, Katznelson [13], and Kahane and Katznelson [11], proved that given $E \subset [-\pi, \pi]$ of Lebesgue measure zero, there is a function $f : \mathbb{R} \to \mathbb{R}$, 2π-periodic and continuous, whose Fourier series diverges at every point of E.

In closing, we refer to ([21], Chapter VIII) for other results and examples on the divergence of Fourier series.

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