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A novel iterative method to solve nonlinear wave-like equations of fractional order with variable coefficients

Un nuevo método iterativo para resolver ecuaciones onduladas no lineales de orden fraccionario con coeficientes variables

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Abstract. In this work, we suggest a novel iterative method to give approximate solutions of nonlinear wave-like equations of fractional order with variable coefficients. The advantage of the proposed method is the ability to combine two different methods: Shehu transform method and homotopy analysis method, in addition to providing an approximate solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. This method can be called Shehu homotopy analysis method (SHAM). Three different examples are presented to illustrate the preciseness and effectiveness of the proposed method. The numerical results show that the solutions obtained by SHAM are in good agreement with the solutions found in the literature. Furthermore, the results show that this method can be implemented in an easy way and therefore can be used to solve other nonlinear fractional partial differential equations.

Key words and phrases. Nonlinear wave-like equations with variable coefficients, Caputo fractional derivative, Shehu transform, homotopy analysis method, approximate solution.

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Resumen. En este trabajo, sugerimos un método iterativo novedoso para dar una solución aproximada de ecuaciones onduladas no lineales de orden fraccionario con coeficientes variables. La ventaja del método propuesto es la capacidad de combinar dos métodos diferentes: el método de transformación de Shehu y el método de análisis de homotopía, además de proporcionar una solución aproximada en forma de una serie convergente con componentes fácilmente computables, que no requieren linealización ni pequeñas perturbaciones. Este método se puede llamar método de análisis de homotopía Shehu (SHAM). Se presentan tres ejemplos diferentes para ilustrar la precisión y eficacia del método propuesto. Los resultados numéricos muestran que las soluciones obtenidas por SHAM están en buen acuerdo con las soluciones encontradas en la literatura. Además, los resultados muestran que este método es fácil de aplicar y, por lo tanto, se puede utilizar para resolver otras ecuaciones diferenciales parciales fraccionarias no lineales.

Palabras y frases clave. Ecuaciones onduladas no lineales con coeficientes variables, derivada fraccional de Caputo, transformada Shehu, método de análisis de homotopía, solución aproximada.

1. Introduction

In recent years many scientists and researchers have been interested in the topic of nonlinear fractional partial differential equations because of its broad applications in various fields, such as physics, mechanics, electrochemistry, viscoelasticity, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and other fields of science and engineering. See for example [1, 4, 5, 6, 14, 16].

Numerous semi-analytical methods such as: Adomian decomposition method (ADM) [25], fractional variational iteration method (FVIM) [26], fractional difference method (FDM) [22], reduced differential transform method (RDTM) [3], homotopy analysis method (HAM) [8], homotopy perturbation method (HPM) [11] are used to solve such nonlinear fractional problems.

Recently, other numerical and analytical methods have appeared to facilitate and improve the resolution speed of nonlinear fractional partial differential equations. They include the combination of Laplace transform, Sumudu transform or natural transform with the previously mentioned methods, among wich are: Laplace homotopy analysis method [29], Laplace decomposition method [13], Laplace variational iteration method [27], homotopy perturbation Sumudu transform method [28], homotopy analysis Sumudu transform method [18], variational iteration Sumudu transform method [2], natural transform homotopy perturbation method [20], natural decomposition method [23], homotopy analysis natural transform method [24], natural reduced differential transform method (NRDTM) [15].

Our main goal of this work is to propose a novel iterative method to solve the nonlinear wave-like equations of fractional order with variable coefficients called Shehu homotopy analysis method (SHAM). This method is a combination of two powerful methods: Shehu transform method and homotopy analysis method.

Consider the following nonlinear wave-like equations of fractional order with variable coefficients

$$D_t^\alpha u = \sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) + \sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X,t,u) + S(X,t), \tag{1}$$

subject to the initial conditions

$$u(X,0) = a_0(X), u_t(X,0) = a_1(X), \tag{2}$$

where D_t^α is the Caputo fractional derivative operator of order $1 < \alpha \leq 2$, $u = \{u(X,t), X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t \geq 0, n \in \mathbb{N}^*\}$, F_{1ij}, G_{1i} $i,j \in \{1,2,\ldots,n\}$ are nonlinear functions of X, t and u, F_{2ij}, G_{2i} $i,j \in \{1,2,\ldots,n\}$ are nonlinear functions of derivatives of u with respect to x_i and x_j $i,j \in \{1,2,\ldots,n\}$, respectively. Also H, S are nonlinear functions and k, m, p are nonnegative integer numbers.

In the case where $\alpha = 2$, equation (1) simplifies to classical nonlinear wave-like equations with variable coefficients. These kind of equations are one of the most widely used wave models to describe the evolution of stochastic systems for example, erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows and the stochastic behavior of exchange rates.

The paper is structured as follows. In Section 2, we present necessary definitions and preliminary results about fractional calculus and Shehu transform. In Sections 3 and 4, we present our results to solve the nonlinear wave-like equations of fractional order with variable coefficients (1) subject to the initial conditions (2) by the Shehu homotopy analysis method (SHAM). In Section 5, we present three numerical examples to show the accuracy and efficiency of this method and we present our obtained results (Graphs and Table), comparing them with their exact associated forms. These results were verified with Matlab (version R2016a). Finally, conclusions are drawn in Section 6.

2. Fundamental definitions

In this section, we present necessary definitions and preliminary results about fractional calculus and Shehu transform that will be applied in this paper.

Definition 2.1. [17] Let $f : [0,T] \longrightarrow \mathbb{R}$ be a continuous function. The left sided Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases} \tag{3}$$

where

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt, \alpha>0,$$

is the Euler gamma function.

Definition 2.2. [17] Let $f:[0,T]\longrightarrow\mathbb{R}$ be a continuous function. The left sided Caputo fractional derivative of order $\alpha\geq 0$ is defined by

$$D^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1}f^{(n)}(\xi)d\xi, n-1<\alpha<n,\\ f^{(n)}(t), \qquad \alpha=n,\end{cases} \tag{4}$$

where $n=[\alpha]+1$ with $[\alpha]$ being the integer part of α.

Definition 2.3. [21] The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions

$$A=\left\{f(t)/\exists N,\eta_1,\eta_2>0,|f(t)|<N\exp\left(\frac{|t|}{\eta_j}\right),\text{ if } t\in(-1)^j\times[0,\infty)\right\},$$

by the following integral

$$\mathbb{S}\left[f(t)\right]=F(s,v)=\int_0^\infty \exp\left(-\frac{st}{v}\right)f(t)dt. \tag{5}$$

Theorem 2.4. [7] *Let $n\in\mathbb{N}^*$ and $\alpha>0$ be such that $n-1<\alpha\leq n$ and $F(s,v)$ be the Shehu transform of the function $f(t)$, then the Shehu transform denoted by $F_\alpha(s,v)$ of the Caputo fractional derivative of $f(t)$ of order α, is given by*

$$\mathbb{S}\left[D^\alpha f(t)\right]=F_\alpha(s,v)=\frac{s^\alpha}{v^\alpha}F(s,v)-\sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-(k+1)}\left[D^k f(t)\right]_{t=0}. \tag{6}$$

3. SHAM to solve nonlinear wave-like equations of fractional order with variable coefficients

Theorem 3.1. *Consider the nonlinear wave-like equations of fractional order with variable coefficients (1) subject to the initial conditions (2). Then, by SHAM the accurate approximation solution of equations (1) and (2) is given in the form of infinite series as follows*

$$u(X,t)=\sum_{n=0}^\infty u_n(X,t).$$

Proof. We consider the following nonlinear wave-like equations of fractional order with variable coefficients (1) subject to the initial conditions (2).

First we define

$$\begin{aligned}
\mathcal{N}u &= \sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}),\\
\mathcal{M}u &= +\sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^{p}}{\partial x_i^p} G_{2i}(u_{x_i}),\\
\mathcal{K}u &= H(X,t,u).
\end{aligned}$$

Equation (1) is written in the form

$$D_t^{\alpha}u(X,t) = \mathcal{N}u(X,t) + \mathcal{M}u(X,t) + \mathcal{K}u(X,t) + S(X,t). \tag{7}$$

Applying the Shehu transform with respect to the time variable t on both sides of (7) and from Theorem 2.4 and the initial conditions (2), we obtain

$$\begin{aligned}
&\mathbb{S}\left[u(X,t)\right] - \left(\frac{v}{s}a_0(X) + \left(\frac{v}{s}\right)^2 a_1(X) + \frac{v^{\alpha}}{s^{\alpha}}\mathbb{S}\left[S(X,t)\right]\right) \\
&-\frac{v^{\alpha}}{s^{\alpha}}\mathbb{S}\left[\mathcal{N}u(X,t) + \mathcal{M}u(X,t) + \mathcal{K}u(X,t)\right] = 0.
\end{aligned} \tag{8}$$

Define the nonlinear operator

$$\begin{aligned}
R\left[\phi(X,t,q\right] = \mathbb{S}\left[\phi(X,t,q)\right] - \left(\frac{v}{s}a_0(X) + \left(\frac{v}{s}\right)^2 a_1(X) + \frac{v^{\alpha}}{s^{\alpha}}\mathbb{S}\left[S(X,t)\right]\right)\\
-\frac{v^{\alpha}}{s^{\alpha}}\mathbb{S}\left[\mathcal{N}\phi(X,t,q) + \mathcal{M}\phi(X,t,q) + \mathcal{K}\phi(X,t,q)\right].
\end{aligned}$$

By means of homotopy analysis method [19], we construct the so-called zero-order deformation equation

$$(1-q)\mathbb{S}\left[\phi(X,t,q) - \phi(X,t,0)\right] = qhH(X,t)R\left[\phi(X,t,q)\right], \tag{9}$$

where q, is an embedding parameter and $q \in [0;1]$, $h \neq 0$ is an auxiliary parameter, $H(X,t) \neq 0$ is an auxiliary function, $\phi(X,t,q)$ is an unknown function and $\mathbb{S}$ is an auxiliary linear Shehu operator. When $q = 0$ and $q = 1$, we have

$$\left\{\begin{array}{l}\phi(X,t,0) = u_0(X,t),\\ \phi(X,t,1) = u(X,t).\end{array}\right. \tag{10}$$

When q increases from 0 to 1, the $\phi(X,t,q)$ varies from $u_0(X,t)$ to $u(X,t)$. Expanding $\phi(X,t,q)$ in Taylor series with respect to q, we have

$$\phi(X,t,q) = u_0(X,t) + \sum_{n=1}^{+\infty} q^n u_n(X,t), \tag{11}$$

where

$$u_n(X,t) = \frac{1}{n!}\left.\frac{\partial^n \phi(X,t,q)}{\partial q^n}\right|_{q=0}. \tag{12}$$

When $q=1$, (11) becomes

$$u(X,t) = u_0(X,t) + \sum_{n=1}^{+\infty} u_n(X,t). \tag{13}$$

We define the vectors

$$\overrightarrow{u} = \{u_0(X,t),\ u_1(X,t),\ u_2(X,t),\dots,u_n(X,t)\}. \tag{14}$$

Differentiating (9) $n-$times with respect to q, then setting $q=0$ and finally dividing them by $n!$, we obtain the so-called n^{th} order deformation equation

$$\mathbb{S}\left[u_n(X,t) - \chi_n u_{n-1}(X,t)\right] = hH(X,t)\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(X,t)\right], \tag{15}$$

where

$$\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(X,t)\right] = \frac{1}{(n-1)!}\left.\frac{\partial^{n-1} R\left[\phi(X,t,q)\right]}{\partial q^{n-1}}\right|_{q=0}, \tag{16}$$

and

$$\chi_n = \begin{cases} 0, n \leq 1, \\ 1, n > 1. \end{cases}$$

Applying the inverse Shehu transform on both sides of (15), we get

$$u_n(X,t) = \chi_n u_{n-1}(X,t) + \mathbb{S}^{-1}\left[hH(X,t)\mathcal{R}_n\overrightarrow{u}_{n-1}(X,t)\right]. \tag{17}$$

The n^{th} deformation equation (17) is linear and it can be easily solved. So, the N^{th}-order approximation of $u(X,t)$ is given by

$$u(X,t) = \sum_{n=0}^{N} u_n(X,t).$$

When $N \to \infty$, the accurate approximation solution of (7), is give by

$$u(X,t) = \sum_{n=0}^{+\infty} u_n(X,t). \tag{18}$$

The proof is complete. ☑

4. Convergence of the SHAM

Assume that $\mathcal{B} = \left(C\left(\Omega\right), \|.\|\right)$ is the Banach space, the space of all continuous functions on $\Omega \subset \mathbb{R}^n \times \mathbb{R}^+$ with the norm, $\|u(X,t)\|_{\mathcal{B}} = \sup\limits_{(X,t)\in\Omega} |u(X,t)|$.

Theorem 4.1. *Let $u_n(X,t)$ and $u(X,t)$ be defined in Banach space $\mathcal{B}$, then the series $\sum\limits_{n=0}^{+\infty} u_n(X,t)$ converges to the solution $u(X,t)$ of equation (1) if there exists $0 < \theta < 1$ such that*

$$\|u_n(X,t)\| \leq \theta \|u_{n-1}(X,t)\|, \forall n \in \mathbb{N}. \tag{19}$$

Proof. Define that $\{S_n(X,t)\}_{n\geq 0}$ is the sequence of partial sums of the series (18), as

$$S_n(X,t) = \sum_{k=0}^{n} u_k(X,t), \tag{20}$$

and we need to show that $\{S_n(X,t)\}_{n\geq 0}$ is a Cauchy sequence in Banach space $\mathcal{B}$.

For this purpose, we consider

$$\begin{aligned} \|S_{n+1}(X,t) - S_n(X,t)\| &\leq \|u_{n+1}(X,t)\| \leq \theta \|u_n(X,t)\| \\ &\leq \theta^2 \|u_{n-1}(X,t)\| \leq \cdots \leq \theta^{n+1} \|u(X,t)\|. \end{aligned} \tag{21}$$

For every, $n, m \in \mathbb{N}, n \geq m$, by using (21) and triangle inequality successively, we have

$$\begin{aligned} \|S_n(X,t) - S_m(X,t)\| &= \|S_n(X,t) - S_{n-1}(X,t) + S_{n-1}(X,t) - S_{n-2}(X,t) \\ &\quad + \cdots + S_{m+1}(X,t) - S_m(X,t)\| \\ &\leq \|S_n(X,t) - S_{n-1}(X,t)\| + \|S_{n-1}(X,t) - S_{n-2}(X,t)\| + \\ &\quad \cdots + \|S_{m+1}(X,t) - S_m(X,t)\| \\ &\leq \theta^n \|u_0(X,t)\| + \theta^{n-1} \|u_0(X,t)\| + \cdots + \theta^{m+1} \|u_0(X,t)\| \\ &= \theta^{m+1} \left(1 + \theta + \cdots + \theta^{n-m-1}\right) \|u_0(X,t)\| \\ &\leq \theta^{m+1} \left(\frac{1-\theta^{n-m}}{1-\theta}\right) \|u_0(X,t)\|. \end{aligned} \tag{22}$$

Since $0 < \theta < 1$, we have $1 - \theta^{n-m} < 1$, then

$$\|S_n(X,t) - S_m(X,t)\| \leq \frac{\theta^{m+1}}{1-\theta} \|u_0(X,t)\|. \tag{23}$$

So $\|S_n(X,t) - S_m(X,t)\| \to 0$ as $n, m \to \infty$ as $u_0(X,t)$ is bounded.

Thus $\{S_n(X,t)\}_{n\geq 0}$ is a Cauchy sequence in Banach space and hence convergent. Therefore, there exists $u(X,t) \in \mathcal{B}$ such that

$$\sum_{n=0}^{\infty} u_n(X,t) = u(X,t). \tag{24}$$

The proof is complete. ☑

Corollary 4.2. T*he maximum absolute truncation error of the series solution (18) for equations (1) and (2) is estimated to be*

$$\left\| u(X,t) - \sum_{k=0}^{N} u_k(X,t) \right\| \leq \frac{\theta^{N+1}}{1-\theta} \left\| u_0(X,t) \right\|. \tag{25}$$

***Proof*.** From Theorem 4.1 and (23), we have

$$\left\| S_n(X,t) - S_N(X,t) \right\| \leq \frac{\theta^{N+1}}{1-\theta} \left\| u_0(X,t) \right\|. \tag{26}$$

But we assume that $S_n(X,t) = \sum\limits_{k=0}^{n} u_k(X,t)$ and since $n \to +\infty$, we obtain $S_n(X,t) \to u(X,t)$, so (26) can be rewritten as

$$\left\| u(X,t) - S_N(X,t) \right\| = \left\| u(X,t) - \sum_{k=0}^{N} u_k(X,t) \right\| \leq \frac{\theta^{N+1}}{1-\theta} \left\| u_0(X,t) \right\|. \tag{27}$$

The proof is complete. ☑

5. Illustrative Examples

In this section, we apply the proposed Shehu homotopy analysis method to solve three examples of nonlinear wave-like equations of fractional order with variable coefficients in order to establish the applicability and the accuracy of the method.

Example 5.1. Consider the 2-dimensional nonlinear wave-like equation of fractional order with variable coefficients

$$D_t^{\alpha} u = \frac{\partial^2}{\partial x \partial y}(u_{xx} u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xy u_x u_y) - u, \tag{28}$$

subject to the initial conditions

$$u(x,y,0) = e^{xy},\ u_t(x,y,0) = e^{xy}, \tag{29}$$

where D_t^{α} is the Caputo fractional derivative operator of order $1 < \alpha \leq 2$, and u is a function of $x, y, t \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{+}$.

The exact solution of equations (28) and (29) for $\alpha = 2$, is given by

$$u(x,y,t) = (\sin t + \cos t)\, e^{xy}.$$

Taking the Shehu transform on both sides of (28) and from Theorem 2.4 and the initial conditions (29), we obtain

$$\mathbb{S}\left[u\right] - \left(\frac{v}{s}e^{xy} + \left(\frac{v}{s}\right)^2 e^{xy}\right) - \frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x \partial y}(u_{xx}u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyu_x u_y) - u\right] = 0. \tag{30}$$

We take the nonlinear part as

$$\begin{aligned} R\left[\phi(x,y,t,q\right] = \mathbb{S}\left[\phi\right] - \left(\frac{v}{s}e^{xy} + \left(\frac{v}{s}\right)^2 e^{xy}\right) \\ - \frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x \partial y}(\phi_{xx}\phi_{yy}) - \frac{\partial^2}{\partial x \partial y}(xy\phi_x\phi_y) - \phi\right]. \end{aligned}$$

In view of the HAM technique and, assuming $H(x,y,t) = 1$, we construct the so-called zero-order deformation equation as follows

$$(1-q)\mathbb{S}\left[\phi(x,y,t,q) - \phi(x,y,t,0)\right] = qhR\left[\phi(x,y,t,q)\right]. \tag{31}$$

When $q = 0$ and $q = 1$, we get

$$\begin{cases} \phi(x,y,t,0) = u_0(x,y,t), \\ \phi(x,y,t,1) = u(x,y,t). \end{cases}$$

Thus, we obtain the n^{th} order deformation equation

$$\mathbb{S}\left[u_n(x,y,t) - \chi_n u_{n-1}(x,y,t)\right] = h\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(x,y,t)\right]. \tag{32}$$

Applying the inverse Shehu transform on both sides of equation (32), we get

$$u_n(x,y,t) = \chi_n u_{n-1}(x,y,t) + \mathbb{S}^{-1}\left[h\mathcal{R}_n\overrightarrow{u}_{n-1}(x,y,t)\right]. \tag{33}$$

From (33), we have

$$\begin{aligned} u_1(x,y,t) &= h\mathbb{S}^{-1}\left[\mathcal{R}_1\overrightarrow{u}_0(x,y,t)\right], \\ u_2(x,y,t) &= u_1(x,y,t) + h\mathbb{S}^{-1}\left[\mathcal{R}_2\overrightarrow{u}_1(x,y,t)\right], \\ u_3(x,y,t) &= u_2(x,y,t) + h\mathbb{S}^{-1}\left[\mathcal{R}_3\overrightarrow{u}_2(x,y,t)\right], \\ &\vdots \end{aligned} \tag{34}$$

where

$$
\begin{aligned}
\mathcal{R}_1 \overrightarrow{u}_0(x,y,t) &= \mathbb{S}\left[u_0\right] - \left(\frac{v}{s}e^{xy} + \left(\frac{v}{s}\right)^2 e^{xy}\right) \\
&\quad - \frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x \partial y}\left(\left(u_0\right)_{xx}\left(u_0\right)_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\left(u_0\right)_x\left(u_0\right)_y\right) - u_0\right], \\
\mathcal{R}_2 \overrightarrow{u}_1(x,y,t) &= \mathbb{S}\left[u_1\right] - \frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x \partial y}\left(\left(u_1\right)_{xx}\left(u_0\right)_{yy} + \left(u_0\right)_{xx}\left(u_1\right)_{yy}\right)\right. \\
&\quad \left. - \frac{\partial^2}{\partial x \partial y}\left(xy\left(\left(u_1\right)_x\left(u_0\right)_y + \left(u_0\right)_x\left(u_1\right)_y\right)\right) - u_1\right], \qquad (35) \\
\mathcal{R}_3 \overrightarrow{u}_2(x,y,t) &= \mathbb{S}[u_2] - \frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x \partial y}\left(\left(u_2\right)_{xx}\left(u_0\right)_{yy} + \left(u_1\right)_{xx}\left(u_1\right)_{yy} + \left(u_0\right)_{xx}\left(u_2\right)_{yy}\right)\right. \\
&\quad \left. - \frac{\partial^2}{\partial x \partial y}\left(xy\left(\left(u_2\right)_x\left(u_0\right)_y + \left(u_1\right)_x\left(u_1\right)_y + \left(u_0\right)_x\left(u_2\right)_y\right)\right) - u_2\right], \\
&\ \ \vdots
\end{aligned}
$$

Using the initial condition (29) and the iteration formulas (34) and (35), we obtain

$$
\begin{aligned}
u_0(x,y,t) &= (1+t)e^{xy}, \\
u_1(x,y,t) &= h\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy}, \\
u_2(x,y,t) &= \left(h+h^2\right)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy} \\
&\quad + h^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}, \\
u_3(x,y,t) &= \left(h+h^2\right)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy} \\
&\quad + 3(h^2+h^3)\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy} \\
&\quad + h^3\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right)e^{xy}, \\
&\ \ \vdots
\end{aligned}
$$

Finally, according to the SHAM, the approximate solution of (28) and (29) is

$$
\begin{aligned}
u(x,y,t) = &\left(1 + t + (3h+2h^2)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) + (4h^2+3h^3)\right. \\
&\left. \times\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) + h^3\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) + \cdots\right)e^{xy}. \qquad (36)
\end{aligned}
$$

When $h=-1$, the approximate solution of equations (28) and (29) can be written as

$$u(x,y,t) = \left(1 + t - \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\ \left. + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots \right) e^{xy}. \tag{37}$$

For the special case $\alpha = 2$, we obtain from (37)

$$\begin{aligned} u(x,y,t) &= \left(1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} - \frac{t^7}{7!} + \cdots \right) e^{xy} \\ &= (\sin t + \cos t)\, e^{xy}, \end{aligned}$$

which is the exact solution and is the same as obtained via ADM [9], RDTM [10] and HPM [12].

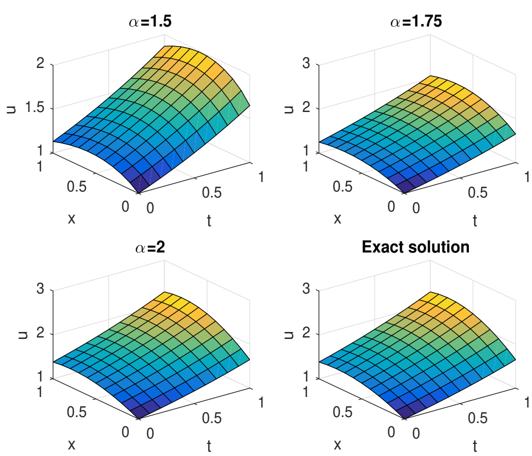

FIGURE 1. 3D plots for the 4−term approximate solution by SHAM and exact solution for equations (28) and (29) when $h=-1$ and $y=0.5$

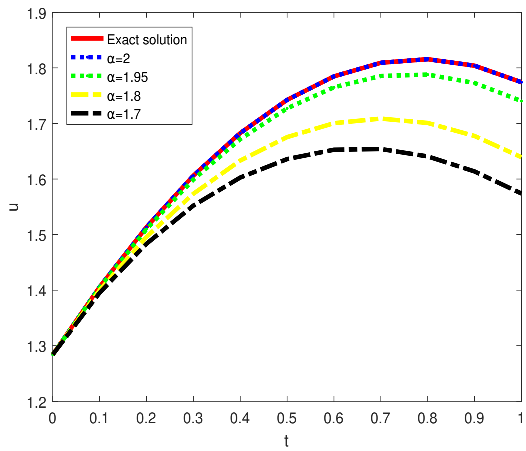

FIGURE 2. 2D plots for the 4−term approximate solution by SHAM and exact solution for equations (28) and (29) when $h = -1$ and $x = y = 0.5$

t	u_{ADM}	u_{RDTM}	u_{HPM}	$u_{SHAM}(\alpha = 2)$	u_{exact}	$\lvert u_{exact} - u_{SHAM}\rvert$
0.1	1.4058	1.4058	1.4058	1.4058	1.4058	3.2196×10^{-13}
0.3	1.6061	1.6061	1.6061	1.6061	1.6061	2.1569×10^{-9}
0.5	1.7424	1.7424	1.7424	1.7424	1.7424	1.3095×10^{-7}
0.7	1.8093	1.8093	1.8093	1.8093	1.8093	1.9680×10^{-6}
0.9	1.8040	1.8040	1.8040	1.8040	1.8040	1.4947×10^{-5}

TABLE 1. Comparison of ADM, RDTM, HPM and SHAM solution for the first four approximations with exact solution for equations (28) and (29) at $h = -1$ and $x = y = 0.5$

Example 5.2. Consider the following nonlinear wave-like equation of fractional order with variable coefficients

$$D_t^{\alpha} u = u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2 \frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u, \tag{38}$$

subject to the initial conditions

$$u(x, 0) = e^x, u_t(x, 0) = e^x, \tag{39}$$

where D_t^{α} is the Caputo fractional derivative operator of order $1 < \alpha \leq 2$ and u is a function of $x, t \in]0, 1[\times \mathbb{R}^+$.

The exact solution of equations (38) and (39) for $\alpha = 2$, is given by

$$u(x, t) = e^{x+t}.$$

Taking the Shehu transform on both sides of (38) and from Theorem 2.4 and the initial conditions (39), we obtain

$$\mathbb{S}\left[u\right]-\left(\frac{v}{s}e^{x}+\left(\frac{v}{s}\right)^{2}e^{x}\right)-\frac{v^{\alpha}}{s^{\alpha}}\mathbb{S}\left[u^{2}\frac{\partial^{2}}{\partial x^{2}}(u_{x}u_{xx}u_{xxx})+u_{x}^{2}\frac{\partial^{2}}{\partial x^{2}}(u_{xx}^{3})-18u^{5}+u\right]=0. \tag{40}$$

We take the nonlinear part as

$$\begin{aligned} R\left[\phi(x,t,q\right] &= \mathbb{S}\left[\phi\right]-\left(\frac{v}{s}e^{x}+\left(\frac{v}{s}\right)^{2}e^{x}\right) \\ &\quad -\frac{v^{\alpha}}{s^{\alpha}}\mathbb{S}\left[\phi^{2}\frac{\partial^{2}}{\partial x^{2}}(\phi_{x}\phi_{xx}\phi_{xxx})+\phi_{x}^{2}\frac{\partial^{2}}{\partial x^{2}}(\phi_{xx}^{3})-18\phi^{5}+\phi\right]. \end{aligned}$$

In view of the HAM technique and assuming $H(x,t)=1$, we construct the so-called zero-order deformation equation as follows

$$(1-q)\mathbb{S}\left[\phi(x,t,q)-\phi(x,t,0)\right]=qhR\left[\phi(x,t,q)\right]. \tag{41}$$

When $q=0$ and $q=1$, we get

$$\begin{cases} \phi(x,t,0)=u_{0}(x,t), \\ \phi(x,t,1)=u(x,t). \end{cases}$$

Thus, we obtain the n^{th} order deformation equation

$$\mathbb{S}\left[u_{n}(x,t)-\chi_{n}u_{n-1}(x,t)\right]=h\mathcal{R}_{n}\left[\overrightarrow{u}_{n-1}(x,t)\right]. \tag{42}$$

Applying the inverse Shehu transform on both sides of equation (42), we get

$$u_{n}(x,t)=\chi_{n}u_{n-1}(x,t)+\mathbb{S}^{-1}\left[h\mathcal{R}_{n}\overrightarrow{u}_{n-1}(x,t)\right]. \tag{43}$$

From (43), we have

$$\begin{aligned} u_{1}(x,t) &= h\mathbb{S}^{-1}\left[\mathcal{R}_{1}\overrightarrow{u}_{0}(x,t)\right], \\ u_{2}(x,t) &= u_{1}(x,t)+h\mathbb{S}^{-1}\left[\mathcal{R}_{2}\overrightarrow{u}_{1}(x,t)\right], \\ &\ \ \vdots \end{aligned} \tag{44}$$

where

$$\mathcal{R}_1 \overrightarrow{u}_0(x,t) = \mathbb{S}\left[u_0\right] - \left(\frac{v}{s}e^x + \left(\frac{v}{s}\right)^2 e^x\right)$$
$$- \frac{v^\alpha}{s^\alpha}\mathbb{S}\left[u_0^2 \frac{\partial^2}{\partial x^2}\left[\left(u_0\right)_x \left(u_0\right)_{xx} \left(u_0\right)_{xxx}\right] + \left(u_0\right)_x^2 \frac{\partial^2}{\partial x^2}\left(u_0\right)_{xx}^3 - 18u_0^5 + u_0\right], \tag{45}$$

$$\mathcal{R}_2 \overrightarrow{u}_1(x,t) = \mathbb{S}[u_1] - \frac{v^\alpha}{s^\alpha}\mathbb{S}\Big[2u_0u_1 \frac{\partial^2}{\partial x^2}\left[(u_0)_x(u_0)_{xx}(u_0)_{xxx}\right] + u_0^2 \frac{\partial^2}{\partial x^2}\left[(u_1)_x(u_0)_{xx}(u_0)_{xxx}\right.$$
$$+ \left(u_0\right)_x \left(u_1\right)_{xx} \left(u_0\right)_{xxx} + \left(u_0\right)_x \left(u_0\right)_{xx} \left.\left(u_1\right)_{xxx}\right] + 2\left(u_0\right)_x \left(u_1\right)_x \frac{\partial^2}{\partial x^2}\left(u_0\right)_{xx}^3$$
$$+ \left(u_0\right)_x^2 \frac{\partial^2}{\partial x^2}\left[3\left(u_0\right)_{xx}^2 \left(u_1\right)_{xx}\right] - 90u_0^4 u_1 + u_1\Big],$$
$$\vdots$$

Using the initial condition (39) and the iteration formulas (44) and (45), we obtain

$$u_0(x,t) = (1+t)e^x,$$
$$u_1(x,t) = -h\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^x,$$
$$u_2(x,t) = -\left(h+h^2\right)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^x$$
$$- h\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^x,$$
$$\vdots$$

Finally, according to the SHAM, the approximate solution of (38) and (39 is

$$u(x,t) = \left(1 + t - (2h+h^2)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right.$$
$$\left. -h\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) + \cdots\right)e^x.$$

When $h = -1$, the approximate solution of equations (38) and (39) can be written as

$$u(x,t) = e^x\left(1 + t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right). \tag{46}$$

For the special case $\alpha = 2$, we obtain from (46)

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right)e^x = e^{x+t},$$

which is the exact solution and is the same as obtained via ADM [9], RDTM [10] and HPM [12].

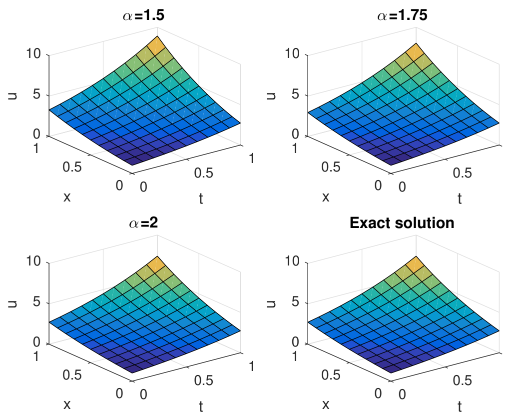

FIGURE 3. 3D plots for the 4−term approximate solution by SHAM and exact solution for equations (38) and (39) when $h = -1$

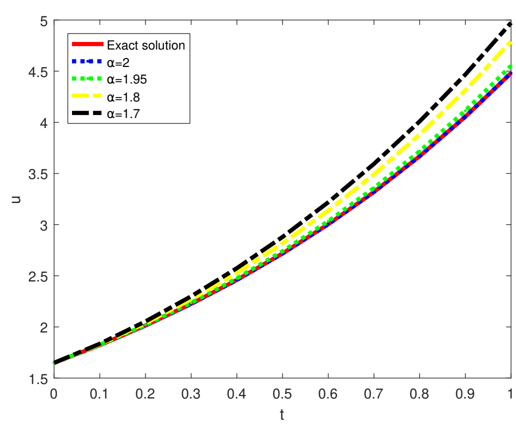

FIGURE 4. 2D plots for the 4−term approximate solution by SHAM and exact solution for equations (38) and (39) when $h = -1$ and $x = 0.5$

t	u_{ADM}	u_{RDTM}	u_{HPM}	$u_{SHAM}(\alpha=2)$	u_{exact}	$\lvert u_{exact}-u_{SHAM}\rvert$
0.1	1.8221	1.8221	1.8221	1.8221	1.8221	4.1350×10^{-13}
0.3	2.2255	2.2255	2.2255	2.2255	2.2255	2.7750×10^{-9}
0.5	2.7183	2.7183	2.7183	2.7183	2.7183	1.6907×10^{-7}
0.7	3.3201	3.3201	3.3201	3.3201	3.3201	2.5543×10^{-6}
0.9	4.0552	4.0552	4.0552	4.0552	4.0552	1.9535×10^{-5}

TABLE 2. Comparison of ADM, RDTM, HPM and SHAM solution for the first four approximations with exact solution for equations (38) and (39) at $h=-1$ and $x=0.5$

Example 5.3. Consider the following one dimensional nonlinear wave-like equation of fractional order with variable coefficients

$$D_t^\alpha u = x^2\frac{\partial}{\partial x}(u_x u_{xx}) - x^2(u_{xx})^2 - u, \tag{47}$$

subject to the initial conditions

$$u(x,0)=0,\ u_t(x,0)=x^2, \tag{48}$$

where D_t^α is the Caputo fractional derivative operator of order $1<\alpha\leq 2$ and u is a function of $(x,t)\in\,]0,1[\,\times\mathbb{R}^+$.

The exact solution of equations (47) and (48) for $\alpha=2$, is given by

$$u(x,t)=x^2\sin t.$$

Taking the Shehu transform on both sides of (47) and from Theorem 2.4 and the initial conditions (48), we obtain

$$\mathbb{S}\left[u\right]-\left(\frac{v}{s}\right)^2x^2-\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[x^2\frac{\partial}{\partial x}(u_xu_{xx})-x^2(u_{xx})^2-u\right]=0. \tag{49}$$

We take the nonlinear part as

$$R\left[\phi(x,t,q\right]=\mathbb{S}\left[\phi\right]-\left(\frac{v}{s}\right)^2x^2-\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[x^2\frac{\partial}{\partial x}(\phi_x\phi_{xx})-x^2(\phi_{xx})^2-\phi\right].$$

In view of the HAM technique and assuming $H(x,t)=1$, we construct the so-called zero-order deformation equation as follows

$$(1-q)\mathbb{S}\left[\phi(x,t,q)-\phi(x,t,0)\right]=qhR\left[\phi(x,t,q)\right]. \tag{50}$$

When $q=0$ and $q=1$, we get

$$\begin{cases}\phi(x,t,0)=u_0(x,t),\\ \phi(x,t,1)=u(x,t).\end{cases}$$

Thus, we obtain the n^{th} order deformation equation

$$\mathbb{S}\left[u_n(x,t)-\chi_n u_{n-1}(x,t)\right]=h\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(x,t)\right]. \tag{51}$$

Applying the inverse Shehu transform on both sides of equation (51), we get

$$u_n(x,t)=\chi_n u_{n-1}(x,t)+\mathbb{S}^{-1}\left[h\mathcal{R}_n\overrightarrow{u}_{n-1}(x,t)\right]. \tag{52}$$

From (52), we have

$$\begin{aligned} u_1(x,t)&=h\mathbb{S}^{-1}\left[\mathcal{R}_1\overrightarrow{u}_0(x,t)\right],\\ u_2(x,t)&=u_1(x,t)+h\mathbb{S}^{-1}\left[\mathcal{R}_2\overrightarrow{u}_1(x,t)\right],\\ u_3(x,t)&=u_2(x,t)+h\mathbb{S}^{-1}\left[\mathcal{R}_3\overrightarrow{u}_2(x,t)\right],\\ &\vdots \end{aligned} \tag{53}$$

where

$$\begin{aligned} \mathcal{R}_1\overrightarrow{u}_0(x,t)&=\mathbb{S}\left[u_0\right]-\left(\frac{v}{s}\right)^2x^2-\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[x^2\frac{\partial}{\partial x}\left((u_0)_x(u_0)_{xx}\right)-x^2\left((u_0)_{xx}\right)^2-u_0\right],\\ \mathcal{R}_2\overrightarrow{u}_1(x,t)&=\mathbb{S}\left[u_1\right]-\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[x^2\frac{\partial}{\partial x}\left((u_0)_x(u_1)_{xx}+(u_1)_x(u_0)_{xx}\right)\right.\\ &\qquad\left.-2x^2(u_0)_{xx}(u_1)_{xx}-u_1\right],\\ \mathcal{R}_3\overrightarrow{u}_2(x,t)&=\mathbb{S}\left[u_2\right]-\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[x^2\frac{\partial}{\partial x}\left((u_0)_x(u_2)_{xx}+(u_1)_x(u_1)_{xx}+(u_2)_x(u_0)_{xx}\right)\right.\\ &\qquad\left.-x^2((u_1)_{xx}^2+2(u_0)_{xx}(u_2)_{xx})-u_2\right],\\ &\vdots \end{aligned} \tag{54}$$

Using the initial condition (48) and the iteration formulas (53) and (54), we obtain

$$\begin{aligned} u_0(x,t)&=tx^2,\\ u_1(x,t)&=hx^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\\ u_2(x,t)&=\left(h+h^2\right)x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+h^2x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},\\ u_3(x,t)&=\left(h+2h^2+h^3\right)x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+2(h^2+h^3)x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+h^3x^2\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},\\ &\vdots \end{aligned}$$

Finally, according to the SHAM, the approximate solution of (47) and (48) is

$$u(x,t)=\Bigg(t+\left(3h+3h^{2}+h^{3}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+(3h^{2}+2h^{3})\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\ +h^{3}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\cdots\Bigg)x^{2}.$$

When $h=-1$, the approximate solution of equations (47) and (48) can be written as

$$u(x,t)=x^{2}\left(t-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\cdots\right). \tag{55}$$

For the special case $\alpha=2$, we obtain from (55)

$$\begin{aligned} u(x,t)&=x^{2}\left(t-\frac{t^{3}}{3!}+\frac{t^{5}}{5!}-\frac{t^{7}}{7!}+\cdots\right)\\ &=x^{2}\sin t, \end{aligned}$$

which is the exact solution and is the same as obtained via ADM [9], RDTM [10] and HPM [12].

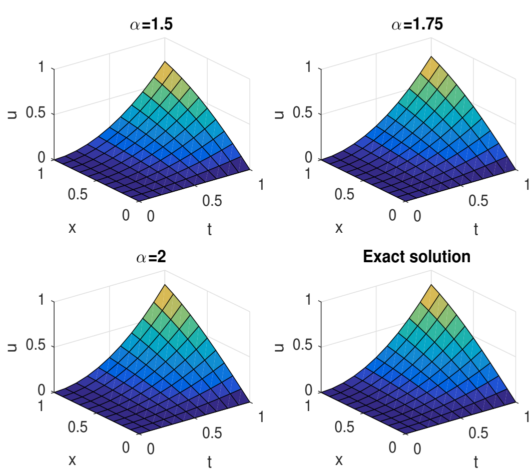

FIGURE 5. 3D plots for the 4−term approximate solution by SHAM and exact solution for equations (47) and (48) when $h=-1$

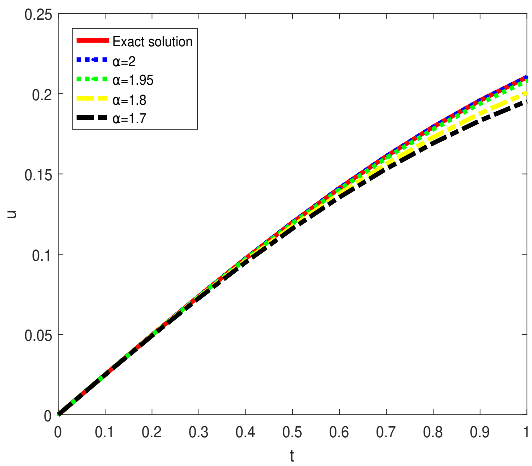

FIGURE 6. 2D plots for the 4−term approximate solution by SHAM and exact solution for equations (47) and (48) when $h = -1$ and $x = 0.5$

t	u_{ADM}	u_{RDTM}	u_{HPM}	$u_{SHAM}(\alpha = 2)$	u_{exact}	$\lvert u_{exact} - u_{SHAM}\rvert$
0.1	0.02496	0.02496	0.02496	0.02496	0.02496	6.8887×10^{-16}
0.3	0.07388	0.07388	0.07388	0.07388	0.07388	1.3549×10^{-11}
0.5	0.11986	0.11986	0.11986	0.11986	0.11986	1.3425×10^{-9}
0.7	0.16105	0.16105	0.16105	0.16105	0.16105	2.7677×10^{-8}
0.9	0.19583	0.19583	0.19583	0.19583	0.19583	2.6495×10^{-7}

TABLE 3. Comparison of ADM, RDTM, HPM and SHAM solution for the first four approximations with exact solution for equations (47) and (48) at $h = -1$ and $x = 0.5$

***Remark* 5.4.** The numerical results presented in the figures 1-6 and tables 1-3 show that the present method approximates the exact solution very well.

6. Conclusion

In this work, a novel iterative method called Shehu homotopy analysis method (SHAM) is proposed to obtain an approximate solution for nonlinear wave-like equations of fractional order with variable coefficients . The SHAM were used in a direct way without using linearization, perturbation or restrictive assumptions. Numerical solutions are obtained in a form of rapidly convergent series with easily computable components. Numerical results show the effectiveness and good accuracy of the proposed method. It is observed that the proposed method is highly suitable for such problems. It may be concluded that the SHAM is very powerful and efficient in finding the solutions for a large class of nonlinear partial differential equations of fractional order.

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