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Strongly nonlinear elliptic unilateral problems without sign condition and with free obstacle in Musielak-Orlicz spaces

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Abstract. In this paper, we prove the existence of solutions to an elliptic problem containing two lower order terms, the first nonlinear term satisfying the growth conditions and without sign conditions and the second is a continuous function on $\mathbb{R}$.

Key words and phrases. Poincaré inequality, Musielak-Orlicz-Sobolev Spaces, Unilateral problems, Measurable obstacle, Lower order term.

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Resumen. En este artículo, demostramos la existencia de soluciones a un problema diferencial elíptico que contiene dos términos de bajo orden, donde el primer término no lineal satisface condiciones de crecimiento sin restricciones en el signo y el segundo es una función continua sobre $\mathbb{R}$.

Palabras y frases clave. desigualdad de Poincaré, espacios Musielak-Orlicz-Sobolev, problemas unilaterales, obstáculo medible, Término de orden inferior.

1. Introduction

In the present paper, we deal with an existence result for a nonlinear elliptic unilateral problems associated to the following equation:

$$A(u) - \operatorname{div}(\Phi(u)) + g(x, u, \nabla u) = f \quad \text{in } \Omega, \tag{1}$$

where Ω is a bounded Lipchitz open subset of $\mathbb{R}^N (N \geq 2)$ which satisfies the segment propierty and $A(u) = -\text{div}\, a(x, u, \nabla u)$ is a Leray-Lions operator defined on $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longrightarrow W^{-1} L_\psi(\Omega)$ where φ and ψ are two complementary Musielak-Orlicz functions. The lower order term Φ is a continuous function on $\mathbb{R}$, g is a nonlinearity with the following natural growth condition:

$$|g(x, s, \xi)| \leq b(|s|)\Big(c(x) + \varphi(x, |\xi|)\Big) \tag{2}$$

and which satisfies the classical sign condition $g(x, s, \xi)s \geq 0$, and the right hand side f is assumed to belong to $L^1(\Omega)$.

On Orlicz spaces and in the variational case, it is well known that Gossez and Mustonen solved in [15] the following obstacle problem:

$$\begin{cases} u \in K_\phi, \\ \langle A(u), u - v\rangle + \displaystyle\int_\Omega g(x, u)(u - v)\, dx \leq \int_\Omega f(u - v)\, dx, \\ \text{ for all } v \in K_\phi \cap L^\infty(\Omega), \end{cases} \tag{3}$$

with $f \in L^1(\Omega)$ and K_ϕ is a convex subset in $W_0^1 L_M(\Omega)$ given by $K_\phi = \{v \in W_0^1 L_M(\Omega) : v \geq \phi \text{ a.e in } \Omega\}$, with the obstacle ϕ is a measurable function satisfying some regularity condition. An existence result has been proved in [1] by Aharouch, Benkirane and Rhoudaf where the non-linearity g depend on x, u and ∇u and without assuming the Δ_2-condition on the N-function and also in [2] the authors were studied the problem (1) in the case where the non-linearity g depends only on x and u under the restriction that the N-function satisfies the Δ_2-condition.

In the framework of variable exponent Sobolev spaces, Azroul, Redwane and Yazough in [4] have shown the existence of solutions for the unilateral problem associated to (1) where $\Phi \equiv 0$ and the second member f is a integrable function, for more results in this topic see [5, 19].

In the setting of Musielak-Orlicz spaces and in the case where $\Phi \equiv 0$, Benkirane and Ait Khellou [20] proved the existence of solutions for the obstacle problem (1), they generalized the work of Gossez and Mustonen in [15].

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field. The generalized Orlicz (Musielak-Orlicz) spaces are of interest not only as the natural generalization of these important examples, but also in their own right. They have appeared in many problems in PDEs and the calculus of variations [3, 12] and have applications to image processing [11, 18] and fluid dynamics [16, 17].

Our purpose in this paper, then, is to study the strongly nonlinear unilateral problems associated to the equation (1) but without assuming any sign condition and any regularity on the obstacle. More precisely, we prove the existence of solutions for the following unilateral problem:

$$(\mathcal{P})\begin{cases} u \geq \Psi \text{ a.e. in } \Omega, \quad T_k(u-v) \in W_0^1 L_\varphi(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \Phi(u)\nabla T_k(u-v)\,dx \\ \displaystyle+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx, \\ \text{for all } v \in K_\Psi \cap L^\infty(\Omega), \quad \forall k \geq 0. \end{cases}$$

where $f \in L^1(\Omega)$ and $K_\Psi = \{u \in W_0^1 L_\varphi(\Omega) : u \geq \Psi \text{ a.e. in } \Omega\}$, with Ψ a measurable function on Ω.

To overcome this difficulty (due to the elimination of the sign condition) in the present paper, we modify the condition (2) by the following one

$$|g(x,s,\xi)| \leq c(x) + h(s)\varphi(x,|\xi|),$$

the model problem is to consider

$$g(x,u,\nabla u) = c(x) + |\sin\, u|e^{-u^2}\varphi(x,|\nabla u|),$$

where $c(x) \in L^1(\Omega)$.

Further works for the unilateral problem corresponding to (1) in the L^p case can be found in [10, 9, 22, 23].

This research is divided into several parts: In Section 2, we recall some well-known preliminaries, properties and results of Orlicz-Sobolev spaces. In Section 2.3, we prepare some auxiliary results to prove our theorem. In the final Section 3, we make precise all the assumptions on $a(.)$, Φ, g and f, we also give the main result of this paper (Theorem 3.1) concerning the existence of solutions.

2. Preliminaries

2.1. Musielak-Orlicz function:

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x,\cdot)$ is an N-function for all $x \in \Omega$ (i.e. convex, strictly increasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$, for all $t > 0$, $\displaystyle\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0$ and $\displaystyle\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty$),

(b) $\varphi(\cdot,t)$ is a measurable function.

The function φ is called a Musielak–Orlicz function.

For a Musielak-orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x, 2t) \leq k\, \varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{4}$$

When (4) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t \longrightarrow 0} \left(\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0, \quad (\text{resp. } \lim_{t \longrightarrow \infty} \left(\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0).$$

***Remark* 2.1.** [8] If $\gamma \prec\prec \varphi$ near infinity, then $\forall \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have

$$\gamma(x,t) \leq k(\varepsilon)\varphi(x, \varepsilon t), \qquad \text{for all } t \geq 0. \tag{5}$$

2.2. Musielak-Orlicz space:

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \ \rho_{\varphi,\Omega}(u) < \infty \Big\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \big\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable } \Big/ \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty, \text{ for some } \lambda > 0 \big\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}$,

ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)\, dx \leq 1\right\},$$

which is called the Luxemburg norm and the so–called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|\, dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent [21].

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega) = \big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } \big/ \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty, \text{ for all } \lambda > 0\big\}.$$

A Musielak function φ is called locally integrable on Ω if $\rho_\varphi(t\chi_E) < \infty$ for all $t > 0$ and all measurable $E \subset \Omega$ with $\mathrm{meas}(E) < \infty$.

Let φ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable [21].

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \left\{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m,\ D^\alpha u \in L_\varphi(\Omega)\right\}$$

and

$$W^m E_\varphi(\Omega) = \left\{u \in E_\varphi(\Omega) : \forall|\alpha| \leq m,\ D^\alpha u \in E_\varphi(\Omega)\right\}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Lemma 2.2. (See [21]). *Let*

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\right\}$$

for $u \in W^m L_\varphi(\Omega)$, these functionals difined a convex modular and a norm respectively on the Sobolev-Orlicz-Musielak space $W^m L_\varphi(\Omega)$.

Let us move to the completeness of the Sobolev-Orlicz-Musielak space $W^m L_\varphi(\Omega)$.

Lemma 2.3. (See [21]). *Let φ a Musielak function such that*

$$\textit{there exist a constant } c_0 > 0 \textit{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{6}$$

Then, the space $\left(W^m L_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\right)$ is a Banach space.

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\}.$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.$$

We introduce the following type of convergence which plays an important role in the proof of our results.

Definition 2.4. A sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is said to be convergent for the modular convergece to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

Now, we give the following key inequalities.

Lemma 2.5. (See [21]). *Let φ a Musielak-Orlicz function and ψ its complementary function. Then, we have*

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega. \tag{7}$$

This inequality implies that

$$\||u|\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{8}$$

We have also the relations between the norm and the modular in $L_\varphi(\Omega)$

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \textit{ if } \|u\|_{\varphi,\Omega} > 1, \tag{9}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \ \textit{if} \ \|u\|_{\varphi,\Omega} \leq 1. \tag{10}$$

Finally, we give the so called Hölder inequality

$$\left| \int_\Omega u(x)v(x)\,dx \right| \leq \|u\|_{\varphi,\Omega} \| |v| \|_{\psi,\Omega}. \tag{11}$$

2.3. Auxiliary results

This subsection is devoted to some auxiliary lemmas and key inequalities used later in the prove of our results.

Lemma 2.6. (See [24]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak–Orlicz functions which satisfy the following conditions:*

There exist a constant $c_0 > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c_0$. There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{log(\frac{1}{|x-y|}}\right)}, \quad \forall, \quad \textit{for all } t \geq 1. \tag{12}$$

If $D \subset \Omega$ is a bounded measurable set, then

$$\int_D \varphi(x,\lambda)\,dx < \infty, \quad \textit{for all } \lambda > 0. \tag{13}$$

There exist a constant $c_2 > 0$ such that $\psi(x,1) \leq c_2$ a.e in Ω.

Then, $D(\Omega)$ is dense in the both spaces $L_\varphi(\Omega)$ and $W_0^1 L_\varphi(\Omega)$ for their modular convergence and $D(\overline{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ the modular convergence in $W^1 L_\varphi(\Omega)$.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 2.7. [7] *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak–Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} \text{ a.e in } \{x \in \Omega : u(x) \in D\} \\ 0 \quad \text{a.e in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 2.8 (Poincaré inequality)**.** (See [24]). *Let φ a Musielak Orlicz function which satisfies the assumptions of Lemma 2.6, suppose that $\varphi(x,t)$ decreases with respect of one of coordinate of x. Then, that exists a constant $c > 0$ depends only of Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)\,dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)\,dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{14}$$

Lemma 2.9. *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

***Proof*.** Let $\lambda > 0$ be such that $\int_\Omega \varphi(x, \frac{u_n - u}{\lambda}) \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_\psi(\Omega)$. Multiplying v by a suitable constant, we can assume $\lambda v \in L_\psi(\Omega)$. By young's inequality,

$$|(u_n - u)v| \leq \varphi(x, \frac{u_n - u}{\lambda}) + \psi(x, \lambda v),$$

which implies, by Vitali's theorem, that $\int_\Omega |(u_n - u)v| \to 0$. ☑

Lemma 2.10. (See [7]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ which satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $||u_n||_\infty \leq (N+1)||u||_\infty$.

Lemma 2.11. (See [6]). *Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in (W_0^1 L_\varphi(\Omega))^N$ then*

$$\int_\Omega div\, u\, dx = 0.$$

Lemma 2.12. (See [20]) *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{15}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$.

Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^p = \prod \left\{ u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2} \right\}.$$

into $(L_\psi(\Omega))^q$ for the modular convergence.

Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^p$ to $(E_\gamma(\Omega))^q$.

3. Assumptions and main result

Throughout the paper, Ω will be a bounded Lipschitz subset of $\mathbb{R}^N$ $N \geq 2$, and let φ and γ two Musielak Orlicz functions such that φ satisfies the conditions of Lemma 2.8 and $\gamma \prec\prec \varphi$.

Given an obstacle measurable function $\Psi : \Omega \longrightarrow \mathbb{R}$, consider the set

$$K_\Psi = \big\{u \in W_0^1 L_\varphi(\Omega) \,:\, u \geq \Psi \text{ a.e. in } \Omega\big\}.$$

This convex set is sequentially $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed in $W_0^1 L_\varphi(\Omega)$ (see [8]).

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longrightarrow W^{-1} L_\psi(\Omega)$ be a mapping given by

$$A(u) = -\text{ div } a(x, u, \nabla u),$$

where ψ is the Musielak–Orlicz function complementary to φ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$|a(x,s,\xi)| \leq \beta\bigg(c(x) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\bigg), \tag{16}$$

$$\bigg(a(x,s,\xi) - a(x,s,\xi')\bigg)(\xi - \xi') > 0, \tag{17}$$

$$a(x,s,\xi).\xi \geq \alpha\varphi(x,|\xi|), \tag{18}$$

where $c(.)$ belongs to $E_\psi(\Omega)$, $c(.) \geq 0$ and $\alpha, \beta, \nu \in \mathbb{R}_+^*$.

$$\Phi : \mathbb{R} \longrightarrow \mathbb{R}^N \text{ is a continuous function.} \tag{19}$$

Furthermore, let $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Caratheodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the following growth condition

$$|g(x,s,\xi)| \leq \rho(x) + h(s)\varphi(x,|\xi|), \tag{20}$$

is satisfied, where $h : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and $\rho(x)$ belongs $L^1(\Omega)$.

$$\begin{gathered}\text{For each } v \in K_\Psi \cap L^\infty(\Omega), \text{ there exists a sequence}\\ v_n \in K_\Psi \cap W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega) \text{ such that}\\ v_n \longrightarrow v \text{ for the modular convergence.}\end{gathered} \tag{21}$$

Finally, we assume that

$$K_\Psi \cap L^\infty(\Omega) \neq \varnothing \tag{22}$$

$$f \text{ is an element of } L^1(\Omega). \tag{23}$$

We define:

$$T_0^{1,\varphi}(\Omega) = \big\{u : \Omega \longrightarrow \mathbb{R} \ \text{ measurable such that } T_k(u) \in W_0^1 L_\varphi(\Omega)\ \forall k \geq 0\big\},$$

where $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ is the truncation at height k defined by:

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{24}$$

The aim of this paper is to prove the following existence result:

Theorem 3.1. *Assume that the assumptions (16)–(23) hold true, then there exists $u \in T_0^{1,\varphi}(\Omega)$ such that $u \geq \Psi$ and*

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega \Phi_n(u)\nabla T_k(u - v)\, dx$$

$$+\int_\Omega g(x, u, \nabla u)T_k(u - v)\, dx \leq \int_\Omega f T_k(u - v)\, dx,$$

for all $v \in K_\Psi \cap L^\infty(\Omega), \quad \forall k \geq 0.$

The proof of Theorem 3.1 is done in 5 steps.

Step 1: Approximate problem.

For $n \in \mathbb{N}^*$, let f_n be regular functions which strongly converge to f in $L^1(\Omega)$ such that $||f_n||_1 \leq c$ for some constant c and Φ_n is a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$ and set $g_n(x, s, \xi) = g(x, T_n(s), \xi)$.

Consider the approximate unilateral problem:

$$(\mathcal{P}_n) \begin{cases} u_n \in K_\Psi \cap D(A) \\ \displaystyle\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)\, dx + \int_\Omega \Phi(u_n)\nabla T_k(u_n - v)\, dx \\ \displaystyle+\int_\Omega g(x, u_n, \nabla u_n)T_k(u_n - v)\, dx \leq \int_\Omega f_n T_k(u_n - v)\, dx, \\ \text{for all } v \in K_\Psi. \end{cases}$$

For fixed $n > 0$, it's obvious to observe that $g_n(x, s, \xi)\xi \geq 0$, $|g_n(x, s, \xi)| \leq |g(x, s, \xi)|$ and $|g_n(x, s, \xi)| \leq n$, Since g_n is bounded for any fixed n, as a consequence, proving of a weak solution $u_n \in W_0^1 L_\varphi(\Omega)$ of $(\mathcal{P}_n)$ is an easy task (see e.g. [8, Theorem 8], [15, Proposition 1]).

Step 2 : A priori estimates.

The estimates derived in this step rely on usual techniques for problems of the type $(\mathcal{P}_n)$.

By (21) and (22), there exists $v_0 \in K_\Psi \cap W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega)$.

For η small enough, let $v = u_n - \eta \exp(G(u_n)) T_k(u_n - v_0)^+$ where $G(s) = \int_0^s \frac{h(r)}{\alpha} dr$ (the function h appears in (20)), choosing v as test function in problem $(\mathcal{P}_n)$, we have

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla(\exp(G(u_n)) T_k(u_n - v_0)^+)\, dx \\
&+ \int_\Omega \Phi_n(u_n) \nabla(\exp(G(u_n)) T_k(u_n - v_0)^+)\, dx \\
&+ \int_\Omega g_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_k(u_n - v_0)^+\, dx \\
&\le \int_\Omega f_n \exp(G(u_n)) T_k(u_n - v_0)^+\, dx.
\end{aligned} \tag{25}$$

Defining $\widetilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau) \nabla(\exp(G(u_n)) T_k(u_n - v_0)^+) d\tau$, one has $\widetilde{\Phi}_n(0) = 0$. As each component of $\widetilde{\Phi}_n$ is uniformly Lipschitz continuous, the Lemma 2 in [14] ensures that the function $\widetilde{\Phi}_n(u_n)$ belongs to $(W_0^1 L_\varphi(\Omega))^N$. So that by Lemma 2.11, we obtain

$$\int_\Omega \Phi_n(u_n) \nabla u_n\, dx = \int_\Omega \operatorname{div}\big(\widetilde{\Phi}_n(u_n)\big) = 0\, dx.$$

Moreover, from (20), one gets

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla(T_k(u_n - v_0)^+) \exp(G(u_n))\, dx \\
&+ \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \frac{h(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n - v_0)^+\, dx \\
&\le \int_\Omega h(u_n) \varphi(x, |\nabla u_n|) \exp(G(u_n)) T_k(u_n - v_0)^+\, dx \\
&\quad + \int_\Omega \big(f_n + \rho(x)\big) \exp(G(u_n)) T_k(u_n - v_0)^+\, dx.
\end{aligned} \tag{26}$$

By using (18) and the fact that $||f_n||_{L^1(\Omega)} \le ||f||_{L^1(\Omega)}$, $\rho \in L^1(\Omega)$, we have

$$\begin{aligned}
&\int_{\{0 \le u_n - v_0 \le k\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n))\, dx \\
&\qquad \le \int_{\{0 \le u_n - v_0 \le k\}} a(x, u_n, \nabla u_n) \nabla v_0 \exp(G(u_n))\, dx + c_1 \\
&\qquad \le c \int_{\{0 \le u_n - v_0 \le k\}} a(x, u_n, \nabla u_n) \frac{\nabla v_0}{c} \exp(G(u_n))\, dx + c_1,
\end{aligned} \tag{27}$$

where c_1 is a positive constant independent of n and $0 < c < 1$.

Using (17), we have

$$\begin{aligned}\int_{\{0\leq u_n-v_0\leq k\}} & a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx \\ & \leq c\Big\{\int_{\{0\leq u_n-v_0\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx \\ & -\int_{\{0\leq u_n-v_0\leq k\}} a\Big(x,u_n,\frac{\nabla v_0}{c}\Big)\Big(\nabla u_n-\frac{\nabla v_0}{c}\Big)\exp(G(u_n))\,dx + c_1\Big\},\end{aligned} \tag{28}$$

which implies that,

$$\begin{aligned}(1-c)&\int_{\{0\leq u_n-v_0\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx \\ & \leq c\int_{\{0\leq u_n-v_0\leq k\}} \Big|a\Big(x,u_n,\frac{\nabla v_0}{c}\Big)\Big|\,\Big|\Big(\nabla u_n-\frac{\nabla v_0}{c}\Big)\Big|\exp(G(u_n))\,dx + c_1 \\ & \leq c\int_{\{0\leq u_n-v_0\leq k\}} \Big|a\Big(x,u_n,\frac{\nabla v_0}{c}\Big)\Big|\,\Big|\frac{\nabla v_0}{c}\Big|\exp(G(u_n))\,dx \\ & \quad + c\int_{\{0\leq u_n-v_0\leq k\}} \Big|a\Big(x,u_n,\frac{\nabla v_0}{c}\Big)\Big|\,\Big|\nabla u_n\Big|\exp(G(u_n))\,dx + c_1.\end{aligned} \tag{29}$$

Since $\frac{\nabla v_0}{c} \in (E_\varphi(\Omega))^N$, then by using the Young's inequality and the condition (16) we have,

$$\begin{aligned}(1-c)&\int_{\{0\leq u_n-v_0\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx \\ & \leq \frac{\alpha(1-c)}{2}\int_{\{0\leq u_n-v_0\leq k\}} \varphi(x,|\nabla u_n|)\exp(G(u_n))\,dx + c_2(k),\end{aligned} \tag{30}$$

where $c_2(k)$ is a positive constant which depends only on k.

Finally, from (17), we can conclude that,

$$\int_{\{0\leq u_n-v_0\leq k\}} \varphi(x,|\nabla u_n|)\exp(G(u_n))\,dx \leq c_3(k). \tag{31}$$

Since $\exp(G(-\infty)) \leq \exp(G(u_n)) \leq \exp(G(+\infty))$ and $\exp(G(\pm\infty)) \leq \exp\left(\frac{||h||_{L^1(\Omega)}}{\alpha}\right)$, we get

$$\int_{\{0\leq u_n-v_0\leq k\}} \varphi(x,|\nabla u_n|)\,dx \leq c_4(k). \tag{32}$$

Similarly, taking $v = u_n + \exp(-G(u_n))T_k(u_n - v_0)^-$ as test function in $(\mathcal{P}_n)$, we obtain

$$(1-c)\int_{\{-k\leq u_n-v_0\leq 0\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(-G(u_n)\,dx \leq \frac{\alpha(1-c)}{2}\int_{\{-k\leq u_n-v_0\leq 0\}} \varphi(x,|\nabla u_n|)\exp(-G(u_n)dx + c_5(k), \tag{33}$$

and then

$$\int_{\{-k\leq u_n-v_0\leq 0\}} \varphi(x,|\nabla u_n|)\,dx \leq c_6(k). \tag{34}$$

Combining (32) and (34), we deduce that,

$$\int_{\{|u_n-v_0|\leq k\}} \varphi(x,|\nabla u_n|)\,dx \leq c_7(k). \tag{35}$$

Since $\{x\in\Omega; |u_n|\leq k\} \subset \{x\in\Omega; |u_n - v_0| \leq k + ||v_0||_\infty\}$,

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx = \int_{\{|u_n|\leq k\}} \varphi(x,|\nabla u_n|)\,dx \leq \int_{\{|u_n-v_0|\leq k+||v_0||_\infty\}} \varphi(x,|\nabla u_n|)\,dx,$$

thus

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \leq C(k+||v_0||_\infty). \tag{36}$$

Thanks to Lemma 2.8, there exists a constant $\lambda > 0$ depends only of Ω such that

$$\int_\Omega \varphi(x,|v|)\,dx \leq \int_\Omega \varphi(x,\lambda|\nabla v|)\,dx \quad \forall v\in W_0^1L_\varphi(\Omega). \tag{37}$$

Taking $v = \frac{1}{\lambda}|T_k(u_n)|$ in (37) and using (36), one has

$$\int_\Omega \varphi(x,\frac{1}{\lambda}|T_k(u_n)|)\,dx \leq \int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \leq C(k+||v_0||_\infty). \tag{38}$$

Then we deduce by using (38), that

$$\begin{aligned} meas\{|u_n|>k\} &\leq \frac{1}{\inf_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}\int_{\{|u_n|>k\}} \varphi(x,\frac{k}{\lambda})\,dx \\ &\leq \frac{1}{\inf_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}\int_\Omega \varphi(x,\frac{1}{\lambda}|T_k(u_n)|)\,dx \\ &\leq \frac{C(k+||v_0||_\infty)}{\inf_{x\in\Omega}\varphi(x,\frac{k}{\lambda})} \quad \forall n, \quad \forall k\geq 0. \end{aligned} \tag{39}$$

For any $\beta > 0$, we have

$$meas\{|u_n - u_m| > \beta\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\}$$

so that

$$meas\{|u_n - u_m| > \beta\} \leq \frac{2C(k + ||v_0||_\infty)}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\}. \tag{40}$$

By using (38), we deduce that $(T_k(u_n))$ is bounded in $W_0^1 L_\varphi(\Omega)$, and then we can assume that $(T_k(u_n))$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$ then by (40) and the fact that $\dfrac{2C(k + ||v_0||_\infty)}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} \to 0$ as $k \to +\infty$ there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

This proves that u_n is a Cauchy sequence in measure, thus, u_n converges almost everywhere to some measurable function u.

Finally, by (36) and Lemma 4.4 of [13], we obtain for all $k > 0$

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow T_k(u) & \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{41}$$

Next step, we will use Banach-Steinhaus Theorem to prove the following proposition but first let reamrk that for all $s \in \mathbb{R}$ we have

$$a(x, T_k(u_n), \nabla T_k(u_n)) = \begin{cases} a(x, u_n, \nabla u_n) & \text{if } |s| \leq k, \\ 0 & \text{if } |s| > k. \end{cases} \tag{42}$$

Proposition 3.2. *Let u_n be a solution of the approximate problem ($\mathcal{P}_n$), then*

$$\Big(a(x, T_k(u_n), \nabla T_k(u_n))\Big)_n \text{ is bounded in } (L_\psi(\Omega))^N. \tag{43}$$

Proof. Let $w \in (E_\varphi(\Omega)^N$ with $\|w\|_{\varphi,\Omega} \leq 1$. Thanks to (17) we can write

$$\Big(a(x, u_n, \nabla u_n) - a(x, u_n, w)\Big)\Big(\nabla u_n - w\Big) > 0,$$

hence

$$\int_{\{|u_n|\leq k\}} a\Big(x, u_n, \nabla u_n\Big) w \, dx \leq \int_{\{|u_n|\leq k\}} a\Big(x, u_n, \nabla u_n\Big)\nabla u_n \, dx - \int_{\{|u_n|\leq k\}} a(x, u_n, w)\Big(\nabla u_n - w\Big) \, dx. \tag{44}$$

Using (16) and since $T_k(u_n)$ is bounded in $W_0^1L_\varphi(\Omega)$, one easily deduces that

$$\int_\Omega a\Big(x, T_k(u_n), \nabla T_k(u_n)\Big)\nabla T_k(u_n) \le c_8(k). \tag{45}$$

Combining the fact that $T_k(u_n)$ is bounded in $W_0^1L_\varphi(\Omega)$, (44) and (46), we get

$$\int_\Omega a\Big(x, T_k(u_n), \nabla T_k(u_n)\Big)w \le c_9(k). \tag{46}$$

Hence, thanks to the Banach-Steinhaus theorem, the sequence $(a(x, u_n, \nabla u_n))_n$ is bounded in $(L_\psi(\Omega))^N$. ☑

Step 3: Almost every where convergence of gradients.

We will introduce the following function of one real variable s, which is defined as

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le j \\ 0 & \text{if } |s| \ge j+1 \\ j+1-s & \text{if } j \le |s| \le j+1 \\ j+1+s & \text{if } -(j+1) \le |s| \le -j \end{cases}$$

with j a nonnegative real parameter.

Let $\Omega_s = \{x \in \Omega : |\nabla T_k(u(x))| \le s\}$ and denote by χ_s the characteristic function of Ω_s. Clearly, $\Omega_s \subset \Omega_{s+1}$ and $\text{meas}(\Omega\backslash\Omega_s) \to 0$ as $s \to \infty$.

In order to prove the modular convergence of truncation $T_k(u_n)$, we shall show the following assertions:

Assertion (i).

$$\lim_{j\to\infty} \limsup_{n\to\infty} \int_{\{j \le |u_n| \le j+1\}} a(x, u_n, \nabla u_n)\nabla u_n \, dx = 0. \tag{47}$$

Assertion (ii).

$$T_k(u_n) \to T_k(u) \text{ in } W_0^1L_\varphi(\Omega) \text{ for the modular convergence } \forall k > 0. \tag{48}$$

Proof. **of Assertion (i).** If we take $v = u_n + \exp(-G(u_n))T_1(u_n - T_j(u_n))^-$ as test function in $(\mathcal{P}_n)$, we get,

$$\begin{aligned} &\int_{\{-(j+1) \le u_n \le -j\}} a(x, u_n, \nabla u_n)\nabla u_n \exp(-G(u_n))\, dx \\ &\le \int_\Omega \big(-f_n + \rho(x)\big)\exp(-G(u_n))T_1(u_n - T_j(u_n))^- \, dx. \end{aligned} \tag{49}$$

Using the fact that

$$\exp(G(-\infty)) \leq \exp(-G(u_n)) \leq \exp(G(+\infty))$$

we deduce

$$\begin{aligned}&\int_{\{-(j+1)\leq u_n\leq -j\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx\\&\leq -c_{10}\int_\Omega \big(f_n-\rho(x)\big)\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx.\end{aligned}\tag{50}$$

Since $f_n \to f$ in $L^1(\Omega)$ and $|f_n\exp(-G(u_n))T_1(u_n-T_j(u_n))^-| \leq \exp\left(\frac{||h||_{L^1(\Omega)}}{\alpha}\right)$ $|f_n|$ then Vitali's Theorem permits us to confirm that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega f_n\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx = 0. \tag{51}$$

Similarly, since $\rho \in L^1(\Omega)$, we obtain

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega \rho\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx = 0. \tag{52}$$

Putting together the results from equations (50), (51), (52), we conclude that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{-(j+1)\leq u_n\leq -j\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{53}$$

On the other hand, taking $v = u_n - \eta\exp(G(u_n))T_1(u_n-T_j(u_n))^+$ as test function in $(\mathcal{P}_n)$ and reasoning as in the proof of (53), we deduce that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{j\leq u_n\leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{54}$$

Thus (47) follows from (53) and (54). ☑

***Proof*. of Assertion (ii).** Let $k \geq ||v_0||_\infty$. By using (21) there exists a sequence there exists $v_j \in K_\Psi \cap W_0^1E_\varphi(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1E_\varphi(\Omega)$. ☑

Let $v = u_n - \eta \exp(G(u_n))(T_k(u_n) - T_k(v_i))^+ h_j(u_n)$ as test function in $(\mathcal{P}_n)$, we obtain by using (18) and (20)

$$\begin{aligned}&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(v_i))h_j(u_n)\,dx\\ &\quad - \int_{\{j\leq u_n\leq j+1\}} \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-T_k(v_i))^+\,dx\\ &\quad \leq \int_\Omega \rho(x)(T_k(u_n)-T_k(v_i))^+ h_j(u_n)\exp(G(u_n))\,dx\\ &\qquad + \int_\Omega f_n(x)(T_k(u_n)-T_k(v_i))^+ h_j(u_n)\exp(G(u_n))\,dx.\end{aligned} \tag{55}$$

Thanks to (54), the second integral tends to zero as n and j tend to infinity, and by Lebesgue Theorem, we deduce that the right–hand side converges to zero as n and i tend to infinity.

Then the least inequality becomes,

$$\begin{aligned}&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(v_i))h_j(u_n)dx\\ &-\int_{\{T_k(u_n)-T_k(v_i)\geq 0,\,|u_n|\geq k\}} \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla T_k(v_i)h_j(u_n)dx\leq \epsilon(n,i,j).\end{aligned} \tag{56}$$

Now, observe that

$$\begin{aligned}&\Big|\int_{\{T_k(u_n)-T_k(v_i)\geq 0,\,|u_n|\geq k\}} \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla T_k(v_i)h_j(u_n)\,dx\Big|\\ &\leq c_{11}\int_{\{|u_n|\geq k\}} |a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))|\,|\nabla v_i|\,dx.\end{aligned} \tag{57}$$

On the one hand, since $(|a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))|)_n$ is bounded in $(L_\psi(\Omega))^N$, we get for a subsequence, $a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n)) \rightharpoonup l_j$ weakly in $(L_\psi(\Omega))^N$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$ with $l_j \in (L_\psi(\Omega))^N$ and since $|\nabla v_i|\chi_{\{|u_n|\geq k\}}$ converges strongly to $|\nabla v_i|\chi_{\{|u|\geq k\}}$ in $E_\varphi(\Omega)$ we have by letting $n\to\infty$

$$\int_{\{|u_n|\geq k\}} |a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))|\,|\nabla v_i|\,dx \to \int_{\{|u_n|\geq k\}} l_j|\nabla v_i|\,dx.$$

Now, we use the modular convergence of $(v_i)_i$, which leads to

$$\int_{\{|u_n|\geq k\}} l_j|\nabla v_i|\,dx \to \int_{\{|u_n|\geq k\}} l_j|\nabla T_k(u)|\,dx.$$

Since $\nabla T_k(u) = 0$ on the subset $\{x \in \Omega : |u(x)| > k\}$. we deduce that

$$\int_{\{|u_n| \geq k\}} |a(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))| \, |\nabla v_i| \, dx = \epsilon(n, i, j).$$

Combining this with (56) and (57) we obtain.

$$\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) a(x, T_k(u_n), \nabla T_k(u_n)) \nabla(T_k(u_n) - T_k(v_i)) h_j(u_n) dx \leq \epsilon(n, i, j). \tag{58}$$

On the other side, we have

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) a(x, T_k(u_n), \nabla T_k(u_n)) \nabla(T_k(u_n) - T_k(v_i)) h_j(u_n) dx \\
&\geq \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)] dx \\
&\times [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)] h_j(u_n) \, dx \\
&+ \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i) [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)] h_j(u_n) dx \\
&- c_{12} \int_{\Omega \setminus \Omega^s} |a(x, T_k(u_n), \nabla T_k(u_n))| \, |\nabla v_i| \, dx,
\end{aligned} \tag{59}$$

where χ_s^j denotes the characteristic function of the subset $\Omega_s^j = \{x \in \Omega : |\nabla T_k(v_i)| \leq s\}$.

Reasoning as above, we get

$$\int_{\Omega \setminus \Omega^s} |a(x, T_k(u_n), \nabla T_k(u_n))| \, |\nabla v_i| \, dx = \int_{\Omega \setminus \Omega^s} l_k \, |\nabla T_k(u)| \, dx + \epsilon(n, i, j). \tag{60}$$

For what concerns the second term of the right hand side of the (59) we can write,

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i) [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)] h_j(u_n) dx \\
&= \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i) \nabla T_k(u_n)) \, dx \\
&- \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n)) a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i) \nabla T_k(v_i)\chi_s^i) h_j(u_n) \, dx.
\end{aligned} \tag{61}$$

Starting with the first term of the last equality, we have by letting $n \to \infty$,

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} &\exp(G(u_n))a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)\nabla T_k(u_n))\,dx\\ = \int_{\{T_k(u)-T_k(v_i)\geq 0\}} &\exp(G(u))a(x, T_k(u), \nabla T_k(v_i)\chi_s^i)\nabla T_k(u)\,dx + \epsilon(n),\end{aligned}$$

since

$$\begin{aligned}&\exp(G(u_n))a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)\chi_{\{T_k(u_n)-T_k(v_i)\geq 0\}}\\ &\to \exp(G(u))a(x, T_k(u), \nabla T_k(v_i)\chi_s^i)\chi_{\{T_k(u)-T_k(v_i)\geq 0\}}\end{aligned}$$

strongly in $(E_\psi(\Omega))^N$ by using Lemma 2.12 while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.

Letting again $i \to \infty$, one has, since

$a(x, T_k(u), \nabla T_k(v_i)\chi_s^i)\chi_{\{T_k(u)-T_k(v_i)\geq 0\}} \to a(x, T_k(u), \nabla T_k(u)\chi_s)$ strongly in $((E_\psi(\Omega))^N$ by using the modular convergence of v_i and Lebesgue theorem,

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} &\exp(G(T_k(u_n)))a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)\nabla T_k(u_n))\,dx\\ &= \int_\Omega \exp(G(u_n))a(x, T_k(u), \nabla T_k(u)\chi_s)\nabla T_k(u))\,dx + \epsilon(n, i, j).\end{aligned}$$

In the same way, we have

$$\begin{aligned}-\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} &\exp(G(T_k(u_n)))a(x, T_k(u_n),\nabla T_k(v_i)\chi_s^i)\nabla T_k(v_i))\chi_s^i h_j(u_n)\,dx\\ &= -\int_\Omega \exp(G(u_n))a(x, T_k(u), \nabla T_k(u)\chi_s)\nabla T_k(u))\chi_s\,dx + \epsilon(n, i, j).\end{aligned}$$

Adding the two equalities we conclude that

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} &\exp(G(u_n))a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)\\ &\times [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)]h_j(u_n)\,dx\\ &= \epsilon(n, i, j).\end{aligned} \tag{62}$$

Combining (58)–(60) and (62), we then conclude

$$\begin{aligned}&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)]\\ &\times [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)]h_j(u_n)\,dx\\ &\leq c_{13}\int_{\Omega\setminus\Omega^s} l_k\,|\nabla T_k(u)|\,dx + \epsilon(n, i, j).\end{aligned} \tag{63}$$

Now, takin $v = u_n + \exp(-G(u_n))(T_k(u_n) - T_k(v_i))^- h_j(u_n)$ as test function $(\mathcal{P}_n)$ and reasoning as in (63) it is possible to conclude that,

$$\begin{aligned} &\int_{\{T_k(u_n)-T_k(v_i)\le 0\}} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)] \\ &\times [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)] h_j(u_n)\, dx \\ &\le c_{14} \int_{\Omega\setminus\Omega^s} l_k\, |\nabla T_k(u)|\, dx + \epsilon(n, i, j). \end{aligned} \tag{64}$$

Finally by using (63) and (64), we get

$$\begin{aligned} &\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)] \\ &\times [\nabla T_k(u_n)) - \nabla T_k(v_i)\chi_s^i)] h_j(u_n)\, dx \\ &\le c_{15} \int_{\Omega\setminus\Omega^s} l_k\, |\nabla T_k(u)|\, dx + \epsilon(n, i, j). \end{aligned} \tag{65}$$

On the other hand, we have

$$\begin{aligned} &\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_j(u_n)\, dx \\ &- \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)][\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i] h_j(u_n)\, dx \\ &= \int_\Omega a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i] h_j(u_n)\, dx \\ &- \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_j(u_n)\, dx \\ &+ \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(v_i)\chi_s^i - \nabla T_k(u)\chi_s] h_j(u_n)\, dx, \end{aligned} \tag{66}$$

and, as it can be easily seen, each integral of the right-hand side of the form $\epsilon(n, i, j)$ implying that

$$\begin{aligned} &\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\ &[\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_j(u_n)\, dx \\ &= \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)] \\ &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i] h_j(u_n)\, dx + \epsilon(n, i, j). \end{aligned} \tag{67}$$

Furthermore, using (65) and (67), we have

$$\begin{aligned}&\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_j(u_n)\,dx\\&\le c_{16}\int_{\Omega\setminus\Omega^s} l_k\,|\nabla T_k(u)|\,dx+\epsilon(n,i,j).\end{aligned}\tag{68}$$

Now, we remark that

$$\begin{aligned}&\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\&=\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_j(u_n)\,dxw\\&+\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s][1-h_j(u_n)]\,dx.\end{aligned}\tag{69}$$

Since $1-h_j(u_n)=0$ in $\{|u_n(x)|\le j\}$, then for j large enough the second term of the right hand side of (69) can be written as follows

$$\begin{aligned}&\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s][1-h_j(u_n)]\,dx\\&=-\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u)\chi_s)[1-h_j(u_n)]\,dx\\&\qquad+\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s[1-h_j(u_n)]\,dx.\end{aligned}\tag{70}$$

Thanks to $a(x,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_\psi(\Omega))^N$ uniformly on n while $\nabla T_k(u)\chi_s(1-h_j(u_n))$ converges to zero strongly in $(L_\varphi(\Omega))^N$, hence the first term of the right-hand side of (70) converges to zero as n goes to infinity.

The second term converges to zero because $\nabla T_k(u)\chi_s(1-h_j(u_n))\rightharpoonup\nabla T_k(u)$ $\chi_s(1-h_j(u))=0$ strongly in $E_\varphi(\Omega)$ and by the continuity of the Nymetskii operator $a(x,T_k(u_n),\nabla T_k(u)\chi_s)$ converges strongly to $a(x,T_k(u),\nabla T_k(u)\chi_s)$. Finally, we deduce that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s][1-h_j(u_n)]\,dx=0.\end{aligned}\tag{71}$$

Combining (68), (69) and (71), we get

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx$$
$$\leq c_{16}\int_{\Omega\setminus\Omega^s} l_k\,|\nabla T_k(u)|\,dx + \epsilon(n,i,j). \tag{72}$$

Letting n, i, j and s tend to infinity, we deduce

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \to 0$$

as $n \to \infty$ and as $s \to \infty$.

As in [20], we deduce that there exists a subsequence, still denoted by u_n, such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega, \tag{73}$$

which implies that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,T_k(u),\nabla T_k(u)) \text{ weakly in } (L_\psi(\Omega))^N \tag{74}$$
$$\text{for } \sigma(\Pi L_\psi, \Pi E_\varphi),\ \forall k > 0.$$

Step 4: Equi-integrability of the non-linearities.

We shall prove that $g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u)$ strongly in $L^1(\Omega)$, by using Vitali's theorem. Since $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ a.e. in Ω, thanks to (41) and (73), it suffices to prove that $g_n(x,u_n,\nabla u_n)$ are uniformly equi–integrable in Ω.

On the one hand, let $v = u_n + \exp(-G(u_n)) \int_{u_n}^0 h(s)\chi_{\{s<-\ell\}}\,ds$. Since $v \in W_0^1 L_\varphi(\Omega)$ and $v \geq \Psi$, v is an admissible test function in $(\mathcal{P}_n)$. Then, we obtain by using (20), that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{h(u_n)}{\alpha}\exp(-G(u_n))\int_{u_n}^0 h(s)\chi_{\{s<-\ell\}}\,ds\,dx \\
&\quad + \int_\Omega a(x,(u_n),\nabla u_n)\nabla u_n \exp(-G(u_n))h(u_n)\chi_{\{s<-\ell\}}\,dx \\
&\quad \leq \int_\Omega \rho(x)\exp(-G(u_n))\int_{u_n}^0 h(s)\chi_{\{s<-\ell\}}\,ds\,dx \\
&\qquad + \int_\Omega h(x)\varphi(x,|\nabla u_n|)\exp(-G(u_n))\int_{u_n}^0 h(s)\chi_{\{s<-\ell\}}\,ds\,dx \\
&\qquad - \int_\Omega f_n \exp(-G(u_n))\int_{u_n}^0 h(s)\chi_{\{s<-\ell\}}\,ds\,dx.
\end{aligned}$$

Using (18) and since $\int_{u_n}^{0} h(s)\chi_{\{s<-\ell\}}\,ds \le \int_{-\infty}^{-\ell} h(s)\,ds$, we get

$$\begin{aligned}&\int_\Omega a(x,(u_n),\nabla u_n)\nabla u_n \exp(-G(u_n))h(u_n)\chi_{\{u_n<-\ell\}}\,dx\\ &\le \exp\Big(\frac{||h||_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{-\infty}^{-\ell} h(s)\,ds(||\rho||_{L^1(\Omega)}+||f_n||_{L^1(\Omega)}).\end{aligned}$$

Using again (18), we obtain

$$\int_{\{u_n<-\ell\}} h(x)\varphi(x,|\nabla u_n|)\,dx \le c_{17}\int_{-\infty}^{-\ell} h(s)\,ds.$$

And since $h\in L^1(\mathbb{R})$, we deduce that,

$$\lim_{\ell\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-\ell\}} h(x)\varphi(x,|\nabla u_n|)\,dx = 0. \tag{75}$$

On the other hand, let $M=\exp(||h||_{L^1(\mathbb{R})})\int_0^{+\infty} h(s)\,ds$ and $\ell\ge M+||v_0||_{L^\infty(\Omega)}$. Consider $v = u_n - \exp(G(u_n))\int_0^{u_n} h(s)\chi_{\{s>l\}}\,ds$. Since $v\in W_0^1L_\varphi(\Omega)$ and $v\ge\Psi$, v is an admissible test function in $(\mathcal{P}_n)$. Then, similarly to (75), we deduce that

$$\lim_{\ell\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>\ell\}} h(x)\varphi(x,|\nabla u_n|)\,dx = 0. \tag{76}$$

Combining (73), (75) and (76) and Vitali's Theorem,we conclude that $g(x,u,\nabla u)\in L^1(\Omega)$ and we can easily to see that

$$g_n(x,u_n,\nabla u_n)\longrightarrow g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{77}$$

Step 5: Passing to the limit.

Let $v\in K_\Psi\cap W_0^1L_\varphi(\Omega)\cap L^\infty(\Omega)$, we take $u_n - T_k(u_n-v)$ as test function in $(\mathcal{P}_n)$, we can write

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx + \int_\Omega \Phi_n(u_n)\nabla T_k(u_n-v)\,dx\\ &\quad+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\\ &\quad\le \int_\Omega f_nT_k(u_n-v)\,dx,\end{aligned} \tag{78}$$

which implies that

$$\begin{aligned}\int_{\{|u_n-v|\leq k\}} & a(x,u_n,\nabla u_n)\nabla u_n\,dx \\ & - \int_{\{|u_n-v|\leq k\}} a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\nabla v\,dx \\ & + \int_\Omega \Phi_n(T_{k+||v||_\infty}(u_n))\nabla T_k(u_n-v)\,dx \\ & + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx \\ & \leq \int_\Omega f_n T_k(u_n-v)\,dx.\end{aligned} \tag{79}$$

By Fatou's lemma and the fact that

$$a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n)) \rightharpoonup a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))$$

weakly in $(L_\psi(\Omega))^N$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$, one easily sees that

$$\begin{aligned}\int_{\{|u_n-v|\leq k\}} & a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\nabla T_{k+||v||_\infty}(u_n)\,dx \\ & - \int_{\{|u_n-v|\leq k\}} a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\nabla v\,dx \\ & \geq \int_{\{|u-v|\leq k\}} a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))\nabla T_{k+||v||_\infty}(u)\,dx \\ & - \int_{\{|u-v|\leq k\}} a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))\nabla v\,dx.\end{aligned} \tag{80}$$

Furthermore, for n large enough $(n > k + ||v||_\infty)$

$$\begin{aligned}\int_\Omega \Phi_n(T_{k+||v||_\infty}(u_n))\nabla T_k(u_n-v)\,dx &= \int_\Omega \Phi_n(T_{k+||v||_\infty}(u_n))\nabla T_k(u-v)\,dx \\ &= \int_\Omega \Phi(T_{k+||v||_\infty}(u_n))\nabla T_k(u-v)\,dx \\ &\to \int_\Omega \Phi(T_k(u))\nabla T_k(u-v)\,dx.\end{aligned} \tag{81}$$

Since $T_k(u_n-v) \to T_k(u-v)$ weakly in $L^\infty(\Omega)$ as $n\to\infty$ we have

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx \to \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \text{ as } n\to\infty, \tag{82}$$

and

$$\int_\Omega f_n \nabla T_k(u_n - v)\,dx \to \int_\Omega f\,\nabla T_k(u_n - v)\,dx \text{ as } n \to \infty. \tag{83}$$

Combining (79)–(83), we have

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx &+ \int_\Omega \Phi(u)\nabla T_k(u-v)\,dx\\ &+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \le \int_\Omega fT_k(u-v)\,dx.\end{aligned} \tag{84}$$

Now, let $v \in K_\Psi \cap L^\infty(\Omega)$ by condition (21) there exists $v_j \in K_\Psi \cap W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v modular, let $\ell > \max(||v_0||_\infty, ||v||_\infty)$, taking $v = T_\ell(v_j)$ in (84), we have

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_\ell(v_j))\,dx + \int_\Omega \Phi(u)\nabla T_k(u - T_\ell(v_j))\,dx\\ &\quad+\int_\Omega g(x,u,\nabla u)T_k(u - T_\ell(v_j))\,dx\\ &\quad\le \int_\Omega fT_k(u - T_\ell(v_j))\,dx.\end{aligned} \tag{85}$$

We can easily pass to the limit as $j \to +\infty$ to get

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_\ell(v))\,dx + \int_\Omega \Phi(u)\nabla T_k(u - T_\ell(v)\,dx\\ &\quad+\int_\Omega g(x,u,\nabla u)T_k(u - T_\ell(v))\,dx\\ &\quad\le \int_\Omega fT_k(u - T_\ell(v))\,dx, \qquad \forall v \in K_\Psi \cap L^\infty(\Omega).\end{aligned} \tag{86}$$

Finally, letting ℓ ($\ell > \max(||v_0||_\infty, ||v||_\infty)$) to the infinity we deduce

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx &+ \int_\Omega \Phi(u)\nabla T_k(u-v)\,dx\\ &+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx = \int_\Omega fT_k(u-v)\,dx,\end{aligned}$$

$\forall v \in K_\Psi \cap L^\infty(\Omega)$ and $\forall k > 0$.

Thus the proof of Theorem 3.1 is complete.

References

[1] L. Aharouch, A. Benkirane, and M. Rhoudaf, *Strongly nonlinear elliptic variational unilateral problems in orlicz spaces*, Abstr. Appl. Anal. Art. ID 46867(20), (2006), 1–20.

[2] L. Aharouch, J. Bennouna, and A. Touzani, *Existence of Renormalized Solution of Some Elliptic Problems in Orlicz Spaces*, Rev. Mat. Complut. **22** (2009), no. 1, 91–110.

[3] M. Avci and A. Pankov, *Multivalued elliptic operators with nonstandard growth*, Advances in Nonlinear Analysis **0** (2016), no. 0, 1–14.

[4] E. Azroul, H. Redwane, and C. Yazough, *Stronglyn onlinear non homogeneous elliptic unilateral problems with l^1 data and no sign conditions*, Electron. J. Differ. Equ. **79** (2012), 1–20.

[5] M. Bendahmane and P. Wittbold, *Renormalized solutions for nonlinear elliptic equations with variable exponents and l^1 data*, Nonlinear Anal. **70** (2009), 567–583.

[6] A. Benkirane and J. Bennouna, *Existence of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms in orlicz spaces*, In: Partial Differential Equations. Lect. Notes Pure Appl. Math. Dekker, New York **229** (2002), no. 0, 251–138.

[7] A. Benkirane and M. Sidi El Vally (Ould Mohamedhen Val), *Some approximation properties in Musielak-Orlicz-Sobolev spaces*, Thai J. Math. **10** (2012), no. 2, 371–381.

[8] ______, *Variational inequalities in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin **21** (2014), 787–811.

[9] L. Boccardo, S. Segura de león, and C. Trombetti, *Bounded and unbounded solutions for a class of quasi–linear elliptic problems with a quadratic gradient term*, J. Math. Pures Appl. **9** (2001), no. 80, 919–940.

[10] L. Boccardo, F. Murat, and J. P. Puel, *l^1 estimate for some nonlinear elliptic partial differential equations and application to an existence result*, SIAM J. Math. Anal. **2** (1992), 326–333.

[11] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. (electronic) **66** (2006), no. 4, 1383–1406.

[12] F. Giannetti and A. Passarelli di Napoli, *Regularity results for a new class of functionals with nonstandard growth conditions*, J. Differential Equations **254** (2013), no. 3, 1280–1305.

[13] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. **190** (1974), 163–205.

[14] ______, *A strongly nonlinear elliptic problem in orlicz-sobolev spaces*, Proc. Symp. Pure Math. **45** (1986), 455–462.

[15] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Anal., Theory Methods Appl. **11** (1987), 379–392.

[16] P. Gwiazda and A. Swierczewska-Gwiazda, *On non-newtonian fluids with a property of rapid thickening under different stimulus*, Math. Models Methods Appl. Sci. **18** (2008), no. 7, 1073–1092.

[17] ______, *On steady non-newtonian fluids with growth conditions in generalized orlicz spaces*, Topol. Methods Nonlinear Anal. **32** (2008), no. 1, 103–114.

[18] P. Harjulehto, P. Hastö, V. Latvala, and O. Toivanen, *Critical variable exponent functionals in image restoration*, Appl. Math. Lett. **26** (2013), no. 1, 56–60.

[19] B. Karim, B. Zerouali, and O. Chakrone, *Existence and Multiplicity Results for Steklov Problems with $p(.)-$Growth Conditions*, Bull. Iran. Math. Soc. **44** (2018), 819–836.

[20] M. Ait Khellou and A. Benkirane, *Elliptic inequalities with L^1 data in Musielak-Orlicz spaces*, Monatsh Math **183** (2017), 1–33.

[21] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer, Berlin, 1983.

[22] A. Porretta, *Nonlinear equations with natural growth terms and measure data*, Electron. J. Differ. Equ. Conf. **9** (2002), 181–202.

[23] ______, *Uniqueness of solutions for some nonlinear Dirichlet problems*, NoDEA Nonlinear differ. equ. appl. **11** (2004), 407–430.

[24] A. Talha, A. Benkirane, and M. S. B. Elemine Vall, *Entropy solutions for nonlinear parabolic inequalities involving measure data in musielak-orlicz-sobolev spaces*, Bol. Soc. Paran. Mat. **36** (2018), no. 2, 199–230.

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