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A note on the $p-$adic Kozyrev wavelets basis

Una nota sobre la base de Kozyrev de wavelets $p-$ádicos

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Abstract. We present a basis of $p-$adic wavelets for Sobolev-type spaces consisting of eigenvectors of certain pseudodifferential operators. Our result extends a well-known result due to S. Kozyrev.

Key words and phrases. $p-$Adic numbers, $p-$Adic wavelets, Sobolev-type spaces.

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Resumen. Presentamos una base de wavelets $p-$ádica para espacios de tipo Sobolev que consiste de vectores propios de ciertos operadores pseudodiferenciales. Nuestro resultado extiende un conocido resultado debido a S. Kozyrev.

Palabras y frases clave. Números $p-$ádicos, wavelets $p-$ádicos, espacios tipo Sobolev.

1. Introduction

The field of p-adic numbers was introduced by the German mathematician Kurt Hensel in 1897. The construction of the field of p-adic numbers $\mathbb{Q}_p$ (here p is a fixed prime number) is very similar to the construction of the field of real numbers $\mathbb{R}$ starting from $\mathbb{Q}$. The field $\mathbb{Q}_p$ is constructed from the rational numbers $\mathbb{Q}$ as the completation with respect to the p-adic norm $|\cdot|_p$. The $p-$adic norm is non-Archimedean, i.e. $|x+y|_p \leq \max\left\{|x|_p, |y|_p\right\}$. As a consequence of this property the geometry of $\mathbb{Q}_p$ is completely different from the geometry of $\mathbb{R}$.

The theory of p-adic numbers has received great attention in the several areas of mathematics, including number theory, algebraic geometry, algebraic topology and analysis, among others. In the recent literature there are many articles where p-adic analysis is applied to other branches of the science, such as, physics, biology and psychology, among others.

The conventional description of the physical space-time uses the field $\mathbb{R}$ of real numbers, and there are many mathematical models based on $\mathbb{R}$ that successfully describe physical reality. Nevertheless, there are general arguments that suggest that one cannot make measurements in regions of extent smaller than the Planck length $\approx 10^{-33}$ cm, see e.g. [8]. This hypothesis conducts naturally to consider models involving geometry and analysis over $\mathbb{Q}_p$ instead of $\mathbb{R}$, as a possible alternative to describe the structure of space-time. In [13]-[14], I. Volovich posed the conjecture of the non-Archimedean nature of the space-time at the level of the Planck scale. This conjecture has originated a lot of research, for instance, in quantum mechanics, see e.g. [5], [10], [11], in string theory, see e.g. [4], [9]. For a further discussion on non-Archimedean mathematical physics, the reader may consult [5],[6],[12], [15] and the references therein.

In this article we present a basis of $p-$adic wavelets for Sobolev-type spaces $\mathcal{H}_l(\mathbb{C})$ with $l \in \mathbb{N}$, see Theorem 3.6. For $l = 0$ we have $\mathcal{H}_l(\mathbb{C}) = L^2$, and in this case our basis of wavelets agrees with the basis introduced by Albeverio and Kozyrev in [2]. Additionaly we show that these functions are eigenfunctions for a pseudodifferential operator with a radial symbol, see Theorem 3.8.

The spaces $\mathcal{H}_l(\mathbb{C})$ were introduced in [16], these spaces are the completion of the $\mathbb{C}$-vector space of Bruhat-Schwartz functions with respect to an inner product $\langle \cdot, \cdot \rangle_l$, $l \in \mathbb{N}$, (which coincides with the product of L^2 when $l = 0$). Furthermore, these spaces are very important in the construction of the non-Archimedean versions of the Kondratiev and Hida spaces, which in turn are useful in the construction of quantum field theories over a $p-$adic space-time, see [3].

This article is organized as follows. In Section 2, we present a brief review of the $p-$adic analysis necessary in this article. In Section 3, we introduce spaces $\mathcal{H}_l(\mathbb{C})$ and give wavelets bases for them, see Theorem 3.6. We finally show that the functions $\psi^{(l)}_{\gamma,\eta,\zeta}$, are eigenfunctions for a pseudodifferential operator with a radial symbol.

2. The field of p-adic numbers

In this section we collect some basic results about p-adic analysis that will be used along this article. For an in-depth review of the p-adic analysis the reader may consult [1], [7], [12].

Let p be fixed prime number. The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases} \tag{1}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order of* x.

Any $p-$adic number $x \neq 0$ has an unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j, \tag{2}$$

where $x_j \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } ord(x) \geq 0 \\ \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j & \text{if} \quad ord(x) < 0. \end{cases} \tag{3}$$

In addition, any non-zero $p-$adic number can be represented uniquely as $x = p^{ord(x)} ac\,(x)$ where $ac\,(x) = \sum_{j=0}^{\infty} x_j p^j$, $x_0 \neq 0$, is called the *angular component* of x. Notice that $|ac\,(x)|_p = 1$.

We extend the $p-$adic norm to $\mathbb{Q}_p^N$ by taking

$$||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \text{ for } x = (x_1, \ldots, x_N)\mathbb{Q}_p^N. \tag{4}$$

We define $ord(x) = \min_{1 \leq i \leq N}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a complete ultrametric space. For $r \in \mathbb{Z}$, denote by $B_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x - a||_p \leq p^r\}$ *the ball of radius* p^r *with center at* $a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$, and take $B_r^N(0) := B_r^N$. Note that $B_r^N(a) = B_r(a_1) \times \cdots \times B_r(a_N)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^N equals the product of N copies of $B_0 = \mathbb{Z}_p$, *the ring of $p-$adic integers of* $\mathbb{Q}_p$. We also denote by $S_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x - a||_p = p^r\}$ *the sphere of radius* p^r *with center at* $a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$, and take $S_r^N(0) := S_r^N$. We notice that $S_0^1 = \mathbb{Z}_p^{\times}$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^{\times}\right)^N \subsetneq S_0^N$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^N$. In addition, two balls in $\mathbb{Q}_p^N$ are either disjoint or one is contained in the other.

As a topological space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is totally disconnected, i.e., the only connected subsets of $\mathbb{Q}_p^N$ are the empty set and the points. A subset of $\mathbb{Q}_p^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^N$, see e.g. [12, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Thus $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a locally compact topological space.

We use $\Omega\left(p^{-r}||x-a||_p\right)$ to denote the characteristic function of the ball $B_r^N(a)$. For other sets, we use the notation 1_A for the characteristic function of a set A. Along the article $d^N x$ will denote a Haar measure on $\left(\mathbb{Q}_p^N, +\right)$ normalized by the condition $\int_{\mathbb{Z}_p^N} d^N x = 1$.

2.1. Some function spaces

2.1.1. The Bruhat-Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_p^N$ is called *locally constant* if for any $x \in \mathbb{Q}_p^N$ there exist a positive integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x+x') = \varphi(x) \text{ for any } x' \in B_{l(x)}^N. \tag{5}$$

Denote by $\mathcal{E}\left(\mathbb{Q}_p^N\right)$ the linear space of locally constant $\mathbb{C}$-valued functions on $\mathbb{Q}_p^N$. A function $\varphi : \mathbb{Q}_p^N \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D} := \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N) := \mathcal{D}(\mathbb{Q}_p^N)$.

Definition 2.1. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest number $l = l(\varphi)$ satisfying (5) is called the parameter of constancy of the function φ. Let us denote by $\mathcal{D}_M^l(\mathbb{Q}_p^N)$ the finite-dimensional space of test functions having supports in the ball B_M^N and with parameters of constancy $\geq l$.

Given $\rho \in [0, \infty)$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^N\right) := L^\rho\left(\mathbb{Q}_p^N, d^N x\right)$, the $\mathbb{C}-$vector space of all the complex valued functions g satisfying $\int_{\mathbb{Q}_p^N} |g(x)|^\rho d^N x < \infty$. The corresponding $\mathbb{R}$-vector spaces are denoted as $L_{\mathbb{R}}^\rho := L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^N\right) = L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^N, d^N x\right)$, $1 \leq \rho \leq \infty$.

2.2. The Fourier transform of test functions

Set $\chi_p(y) := \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e., a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$. The isomorphism is given by $\xi \to \chi_p(\xi x)$, see e.g. [1, Section 2.3].

Given $\xi = (\xi_1, \ldots, \xi_N)$ and $y = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$, we set $\xi \cdot x := \sum_{j=1}^N \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x)\varphi(x) d^N x \quad \text{for } \xi \in \mathbb{Q}_p^N, \tag{6}$$

where $d^N x$ is the normalized Haar measure on $\mathbb{Q}_p^N$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see e.g. [1], [12].

We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

If $f \in L^1$ its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x) f(x) d^N x, \quad \text{for } \xi \in \mathbb{Q}_p^N. \tag{7}$$

If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{||x||_p \le p^k} \chi_p(\xi \cdot x) f(x) d^N x, \quad \text{for } \xi \in \mathbb{Q}_p^N, \tag{8}$$

where the limit is taken in L^2. We recall that the Fourier transform is unitary on L^2, i.e. $||f||_{L^2} = ||\mathcal{F}f||_{L^2}$ for $f \in L^2$ and $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$ is also valid in L^2, see e.g. [7, Chapter III, Section 2].

3. A Wavelet basis for the spaces $\mathcal{H}_l(\mathbb{C})$

3.1. The spaces $\mathcal{H}_l(\mathbb{C})$

We denote the set of non-negative integers by $\mathbb{N}$, and set $[\xi]_p := [\max(1, \|\xi\|_p)]$ for $\xi \in \mathbb{Q}_p^N$. We define for φ, $\theta \in \mathcal{D}(\mathbb{Q}_p^N)$, and $l \in \mathbb{N}$, the following scalar product:

$$\langle \varphi, \theta \rangle_l = \int_{\mathbb{Q}_p^N} [\xi]_p^{2l}\, \widehat{\varphi}(\xi)\, \overline{\widehat{\theta}(\xi)} d^N \xi, \tag{9}$$

where the overbar denotes the complex conjugate. We also set $\|\varphi\|_l^2 := \langle \varphi, \varphi \rangle_l$. Notice that $\|\cdot\|_l \le \|\cdot\|_m$ for $l \le m$. We denote by $\mathcal{H}_l(\mathbb{C}) := \mathcal{H}_l(\mathbb{Q}_p^N, \mathbb{C})$ the complex Hilbert space obtained by completing $\mathcal{D}(\mathbb{Q}_p^N)$ with respect to $\langle \cdot, \cdot \rangle_l$.

***Remark* 3.1.** The spaces $\mathcal{H}_l(\mathbb{C})$, for any $l \in \mathbb{N}$, are nuclear and consequently they are separable. The spaces $\mathcal{H}_l(\mathbb{Q}_p^N, \mathbb{C})$ were introduced in [16], see also [3].

3.2. A Wavelet basis for the spaces $\mathcal{H}_l(\mathbb{C})$

In this section we introduce orthonormal bases for the spaces $\mathcal{H}_l(\mathbb{C})$, where l is a non-negative integer.

Let us consider the following set of functions

$$\psi_{\gamma,\eta,\zeta}^{(l)}(x) = \frac{p^{\frac{-N\gamma}{2}} \chi(p^{-1}\zeta \cdot (p^\gamma x - \eta)) \Omega(\|p^\gamma x - \eta\|_p)}{[\max(1, p^{1-\gamma})]^l}, \text{with } x \in \mathbb{Q}_p^N, \quad \gamma \in \mathbb{Z}, \tag{10}$$

$$\eta \in \mathbb{Q}_p^N/\mathbb{Z}_p^N, \quad \eta = (\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(N)}), \quad \eta^{(l)} = \sum_{i=\beta_l}^{-1} \eta_{(i)}^{l} p^i, \quad \beta_l \in \mathbb{Z}^-,$$

$$\eta_i^{(l)} = 0, 1, \ldots, p-1, \quad \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N), \quad \zeta_l = 0, 1, \ldots, p-1,$$

where at least one of ζ_l is not equal to zero.

***Remark* 3.2.** In the case $l = 0$, the set of functions $\psi^{(0)}_{\gamma,\eta,\zeta}$ coincides with the N-dimensional basis of p-adic wavelets of $\mathbb{Q}^N$, introduced by Albeverio and Kozyrev in [2].

Proposition 3.3. The Fourier transform of $\psi^{(l)}_{\gamma,\eta,\zeta}$ is given by

$$\widehat{\psi^{(l)}}_{\gamma\eta\zeta}(\xi) = \frac{p^{\frac{N\gamma}{2}}}{[\max(1, p^{1-\gamma})]^l} \chi(p^{-\gamma}\xi \cdot \eta)\Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p). \tag{11}$$

***Proof*.** It is sufficient to compute the Fourier transform of function of type $\varphi_\zeta(x) = \chi(p^{-1}\zeta \cdot x)\Omega(\|x\|_p)$. Now for the calculation of the formula (11) we use the above function and the result presented in ([12], VII, 2.17). ☑

***Remark* 3.4.** Let us $l, k \in \mathbb{Z}$ with $l \leq k$. We remember that the product of indicators is either an indicator or zero see e.g. [1] and [12]:

$$\Omega(\|p^l x - a\|_p)\Omega(\|p^k x - b\|_p) = \Omega(\|p^l x - a\|_p)\Omega(\|p^{k-l}a - b\|_p), \tag{12}$$

with $x, a, b \in \mathbb{Q}_p^N$.

Lemma 3.5.

(1) The support of the function $\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}$ is

$$supp\left(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}\right) = -p^{\gamma-1}\zeta + p^\gamma \mathbb{Z}_p^N = B^N_{-\gamma}\left(-p^{\gamma-1}\zeta\right).$$

(2) The product $\widehat{\psi^{(l)}}_{\gamma,\eta,\zeta}(\xi)\overline{\widehat{\psi^{(l)}}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}(\xi)}$ is non-zero if $\gamma = \gamma^\dagger$ and $\zeta = \zeta^\dagger$.

***Proof*.** (1) It follows from observation:

$$\xi \in supp(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}) \Leftrightarrow \|p^{-\gamma}\xi + p^{-1}\zeta\|_p \leq 1 \Leftrightarrow \text{ There exists } w \in \mathbb{Z}_p^N, \text{ and}$$
$$p^{-\gamma}\xi + p^{-1}\zeta = w \Leftrightarrow \xi \in -p^{\gamma-1}\zeta + p^\gamma \mathbb{Z}_p^N.$$

(2) Consider $\xi \in supp\left(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}\right) \cap supp\left(\widehat{\psi^{(l)}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}}\right)$, then $\xi \in supp\left(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}\right)$ and $\xi = -p^{\gamma-1}\zeta + p^{\gamma}w$ for some $w \in \mathbb{Z}_p^N$. Using the formula (11) we have

$$\begin{aligned}
&\widehat{\psi^{(l)}}_{\gamma,\eta,\zeta}(\xi)\overline{\widehat{\psi^{(l)}}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}(\xi)}\\
&= \frac{p^{\frac{N(\gamma+\gamma^\dagger)}{2}}}{[\max(1,p^{1-\gamma})]^l[\max(1,p^{1-\gamma^\dagger})]^l}\chi(-p^{-1}\zeta\cdot\eta)\chi(w\cdot\eta)\chi(-p^{\gamma-\gamma^\dagger-1}\zeta\cdot\eta^\dagger)\times\\
&\qquad\chi(p^{\gamma-\gamma^\dagger}w\cdot\eta^\dagger)\Omega(\|w\|_p)\Omega(\|p^{\gamma-\gamma^\dagger}w + p^{-1}\zeta^\dagger - p^{\gamma-\gamma^\dagger-1}\zeta)\|_p).
\end{aligned}$$

Since $\|p^{\gamma-\gamma^\dagger}w\|_p \leq p^{\gamma^\dagger-\gamma}$, $\|p^{-1}\zeta^\dagger\|_p = p$ and $\|p^{\gamma-\gamma^\dagger-1}\zeta)\|_p = p^{\gamma^\dagger-\gamma+1}$, then if $\gamma \neq \gamma^\dagger$ we have $\Omega(\|p^{\gamma-\gamma^\dagger}w + p^{-1}\zeta^\dagger - p^{\gamma-\gamma^\dagger-1}\zeta)\|_p) = 0$.

If $\gamma = \gamma^\dagger$ and $\zeta \neq \zeta^\dagger$, for the above $\Omega(\|w + p^{-1}\zeta^\dagger - p^{-1}\zeta)\|_p) = 0$.

We conclude that for ξ in $supp\left(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}\right)$, the product $\widehat{\psi^{(l)}}_{\gamma,\eta,\zeta}(\xi)\overline{\widehat{\psi^{(l)}}}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}(\xi)$ is non-zero if $\gamma = \gamma^\dagger$ and $\zeta = \zeta^\dagger$. A similar result is obtained when considering $\xi \in supp\left(\widehat{\psi^{(l)}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}}\right)$. ☑

Theorem 3.6. *The set of functions*

$$\psi^{(l)}_{\gamma\eta\zeta}(x) = \frac{p^{\frac{-N\gamma}{2}}}{[\max(1,p^{1-\gamma})]^l}\chi(p^{-1}\zeta\cdot(p^{\gamma}x-\eta))\Omega(\|p^{\gamma}x-\eta\|_p), \tag{13}$$

with $\gamma, \zeta,\ \eta$ as before, is an orthonormal basis of $\mathcal{H}_l(\mathbb{C})$.

Proof. We first show that the functions (13) are orthonormal, with respect to the scalar product $\langle\cdot,\cdot\rangle_l$ given above:

$$\begin{aligned}
\langle\psi^{(l)}_{\gamma,\eta,\zeta},\psi^{(l)}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}\rangle_l &= \int_{\mathbb{Q}_p^N}[\xi]_p^{2l}\,\widehat{\psi^{(l)}}_{\gamma,\eta,\zeta}(\xi)\overline{\widehat{\psi^{(l)}}}_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger,}(\xi)d^N\xi\\
&= \frac{p^{\frac{N(\gamma+\gamma^\dagger)}{2}}}{[\max(1,p^{1-\gamma})]^l[\max(1,p^{1-\gamma^\dagger})]^l}\int_{\mathbb{Q}_p^N}[\xi]_p^{2l}\,\chi(p^{-\gamma}\xi\cdot\eta)\chi(-p^{-\gamma^\dagger}\xi\cdot\eta^\dagger)\\
&\qquad\times\Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p)\Omega(\|p^{-\gamma^\dagger}\xi + p^{-1}\zeta^\dagger\|_p)d^N\xi.
\end{aligned}$$

By part 2 of Lemma 3.5, the scalar product can be non-zero only when $\gamma = \gamma^\dagger$, and $\zeta = \zeta^\dagger$. Then the previous integral equals

$$\begin{aligned}
&\langle\psi_{\gamma,\eta,\zeta},\psi_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}\rangle_l = \delta_{\gamma,\gamma^\dagger}\delta_{\zeta,\zeta^\dagger}\frac{p^{N\gamma}}{[\max(1,p^{1-\gamma})]^{2l}}\\
&\times\int_{\mathbb{Q}_p^N}[\xi]_p^{2l}\,\chi(p^{-\gamma}\xi\cdot(\eta-\eta^\dagger))\Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p)d^N\xi.
\end{aligned}$$

Suppose that $\xi \in supp\left(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}\right)$. Then $\xi \in -p^{\gamma-1}\zeta + p^{\gamma}\mathbb{Z}_p^N$, and $\|\xi\|_p = p^{1-\gamma}$.

$$\begin{aligned}\langle \psi_{\gamma,\eta,\zeta}, \psi_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}\rangle_l = \delta_{\gamma,\gamma^\dagger}\delta_{\zeta,\zeta^\dagger}\frac{p^{N\gamma}}{[\max(1,p^{1-\gamma})]^{2l}}[\max(1,p^{1-\gamma})]^{2l} \\ \times \int_{\mathbb{Q}_p^N} \chi(p^{-\gamma}\xi\cdot(\eta-\eta^\dagger))\Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p)d^N\xi.\end{aligned}$$

By changing variables as $z = p^{-\gamma}\xi + p^{-1}\zeta, \quad d^N\xi = p^{-N\gamma}d^N z$, in the previous integral, we obtain

$$\begin{aligned}&\langle \psi_{\gamma,\eta,\zeta}, \psi_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}\rangle_l \\ &\quad = \delta_{\gamma,\gamma^\dagger}\delta_{\zeta,\zeta^\dagger}p^{N\gamma}\int_{\mathbb{Q}_p^N}\chi(p^{-\gamma}(\eta-\eta^\dagger)\cdot(z-p^{-1}\zeta)p^{\gamma})\Omega(\|z\|_p)p^{-N\gamma}d^N z \\ &\quad = \delta_{\gamma,\gamma^\dagger}\delta_{\zeta,\zeta^\dagger}\chi(-p^{-1}(\eta-\eta^\dagger)\cdot\zeta)\int_{\mathbb{Q}_p^N}\chi[(\eta-\eta^\dagger)\cdot z]\Omega(\|z\|_p)d^N z \\ &\quad = \delta_{\gamma,\gamma^\dagger}\delta_{\zeta,\zeta^\dagger}\chi(-p^{-1}(\eta-\eta^\dagger)\cdot\zeta)\Omega(\|\eta-\eta^\dagger\|_p). \qquad (14)\end{aligned}$$

If $\eta \neq \eta^\dagger$, then $\|\eta-\eta^\dagger\|_p \geq p > 1$ and so the previous integral is zero. Consequently we have

$$\langle \psi_{\gamma,\eta,\zeta}, \psi_{\gamma^\dagger,\eta^\dagger,\zeta^\dagger}\rangle_l = \delta_{\gamma,\gamma^\dagger}\delta_{\zeta,\zeta^\dagger}\delta_{\eta,\eta^\dagger}.$$

We can conclude that the system of functions (13) is orthonormal.

To prove the completeness of the system of functions (13), we use fact that the space $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$ is dense in $\mathcal{H}_l(\mathbb{C})$ and that the set of functions $\psi^{(l)}_{\gamma,\zeta,n}$ is invariant under dilations and translations, therefore, it is sufficient to verify the Parseval identity for the characteristic function $\Omega(\|x\|_p)$:

$$\begin{aligned}\langle \Omega(\|x\|_p), \psi^{(l)}_{\gamma,\eta,\zeta}\rangle_l = \frac{1}{[\max(1,p^{1-\gamma})]^{l}}\int_{\mathbb{Q}_p^N}\int_{\mathbb{Q}_p^N}[\xi]_p^{2l}\,\widehat{\Omega}(\|\xi\|_p)\overline{\widehat{\psi}}_{\gamma,\eta,\zeta}(\xi)d^N\xi \\ = \frac{1}{[\max(1,p^{1-\gamma})]^{2l}}\int_{\mathbb{Q}_p^N}\int_{\mathbb{Q}_p^N}[\xi]_p^{2l}\,\Omega(\|\xi\|_p)p^{\frac{N\gamma}{2}}\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|p^{-\gamma}\xi+p^{-1}\zeta\|_p)d\xi. \qquad (15)\end{aligned}$$

Suppose that $0 \leq -\gamma$ $(\gamma \leq 0)$. By using (12), we obtain that the product of indicators is zero:

$$\Omega(\|\xi\|_p)\Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p) = \Omega(\|\xi\|_p)\Omega(\|p^{-1}\zeta\|_p).$$

Suppose that $-\gamma < 0$ $(0 < \gamma)$. By using (12), we obtain that the product of indicators is non-zero:

$$\Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p)\Omega(\|\xi\|_p) = \Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p)\Omega(\|-p^{\gamma-1}\zeta\|_p), \text{ if } \gamma \geq 1.$$

Therefore we have that

$$\begin{aligned}
\langle \Omega(\|x\|_p), \psi^{(l)}_{\gamma,\eta,\zeta}\rangle_l &= \frac{p^{\frac{N\gamma}{2}}}{[\max(1,p^{1-\gamma})]^{2l}} \int_{\mathbb{Q}_p^N}\int_{\mathbb{Q}_p^N} [\xi]_p^{2l}\,\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|\xi\|_p) \\
&\quad \times \Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p) d^N\xi \\
&= \frac{p^{\frac{N\gamma}{2}}[\max(1,p^{1-\gamma})]^{2l}}{[\max(1,p^{1-\gamma})]^{2l}} \Omega(\|-p^{\gamma-1}\zeta\|_p) \int_{\mathbb{Q}_p^N}\int_{\mathbb{Q}_p^N} [\xi]_p^{2l}\,\chi(p^{-\gamma}\xi\cdot\eta) \\
&\quad \times \Omega(\|p^{-\gamma}\xi + p^{-1}\zeta\|_p) d^N\xi \\
&= p^{\frac{N\gamma}{2}}\Omega(\|-p^{\gamma-1}\zeta\|_p)\int_{\mathbb{Q}_p^N} \chi[p^{-\gamma}(p^{\gamma}(z-p^{-1}\zeta))\cdot\eta]\Omega(\|z\|_p)p^{-N\gamma}d^Nz \\
&= p^{-\frac{N\gamma}{2}}\Omega(\|-p^{\gamma-1}\zeta\|_p)\chi(-p^{-1}\zeta\cdot\eta)\int_{\mathbb{Q}_p^N}\chi(z\cdot\eta)\Omega(\|z\|_p)d^Nz \\
&= p^{-\frac{N\gamma}{2}}\chi(-p^{-1}\zeta\cdot\eta)\Omega(\|\eta\|_p), \text{ for } \gamma\geq 1. \qquad (16)
\end{aligned}$$

If $\eta \neq 0$, then the previous product is zero. Therefore if $\eta = 0$, and $\gamma \geq 1$ we have

$$\langle \Omega(\|x\|_p), \psi^{(l)}_{\gamma,\eta,\zeta}\rangle_l = p^{-\frac{N\gamma}{2}}.$$

We remember that the number of vectors ζ is $|\zeta| = p^N - 1$.

Finally,

$$\begin{aligned}
\sum_{\gamma\in\mathbb{Z},\ \eta\in\mathbb{Q}_p^N/\mathbb{Z}_p^N,\ |\zeta|=1}^{p^N-1} |\langle \Omega(\|x\|_p), \psi^{(l)}_{\gamma,\eta,\zeta}\rangle_l|^2 &= \sum_{\gamma=1}^{\infty}\sum_{|\zeta|=1}^{p^N-1}(p^{-\frac{N\gamma}{2}})^2 \\
&= \sum_{\gamma=1}^{\infty}(p^N-1)p^{-N\gamma} = 1 = \|\Omega(\|x\|_p)\|_l^2. \qquad (17)
\end{aligned}$$

☑

Definition 3.7. Let $a : \mathbb{R}^+ \to \mathbb{C}$ be a fixed function. We define the pseudodifferential operators A with symbol $a\left(\|\xi\|\right)$ as follows:

$$\begin{array}{ccc}
\mathcal{D}(\mathbb{Q}_p^N) & \to & L^2 \\
\varphi & \to & (A\varphi)(x),
\end{array}$$

where

$$(A\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}\{a(\|\xi\|_p)\mathcal{F}_{x\to\xi}\varphi\}. \qquad (18)$$

Theorem 3.8. *The set of functions*

$$\psi^{(l)}_{\gamma\eta\zeta}(x) = \frac{p^{\frac{-N\gamma}{2}}}{[\max(1,p^{1-\gamma})]^{l}}\chi(p^{-1}\zeta\cdot(p^{\gamma}x-\eta))\Omega(\|p^{\gamma}x-\eta\|_p), \qquad (19)$$

with $\gamma, \zeta,\ \eta$ as before, are eigenfunctions of the pseudodifferential operator A defined in Definition 3.7. The corresponding eigenvalues are $\lambda = a(p^{1-\gamma})$.

Proof. Let us prove that the functions in (19) are eigenfunctions of the operator (18), i.e. $(A(\psi^{(l)}_{\gamma\eta\zeta}))(x) = a(p^{1-\gamma})\psi^{(l)}_{\gamma\eta\zeta}$. By using that

$$(A(\psi^{(l)}_{\gamma\eta\zeta}))(x) = \mathcal{F}^{-1}_{\xi\to x}\{a(\|\xi\|_p)\mathcal{F}_{x\to\xi}\psi^{(l)}_{\gamma\eta\zeta}\} = \int_{\mathbb{Q}_p^N}\chi(-\xi\cdot x)a(\|\xi\|_p)\widehat{\psi^{(l)}}_{\gamma\eta\zeta}(\xi)d^N\xi$$

and the proposition 3.3, we obtain

$$\begin{aligned}&(A(\psi^{(l)}_{\gamma\eta\zeta}))(x)\\ &= \frac{p^{\frac{N\gamma}{2}}}{[\max(1,p^{1-\gamma})]^l}\int_{\mathbb{Q}_p^N}\chi(-\xi\cdot x)a(\|\xi\|_p)\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|p^{-\gamma}\xi+p^{-1}\zeta\|_p)d^N\xi\\ &= \frac{p^{\frac{N\gamma}{2}}}{[\max(1,p^{1-\gamma})]^l}\int_{\mathbb{Q}_p^N}\chi((p^{-\gamma}\eta-x)\cdot\xi)a(\|\xi\|_p)\Omega(\|p^{-\gamma}\xi+p^{-1}\zeta\|_p)d^N\xi\end{aligned}$$

Suppose that $\xi \in supp\left(\widehat{\psi^{(l)}_{\gamma,\eta,\zeta}}\right)$. Then $\xi \in -p^{\gamma-1}\zeta + p^{\gamma}\mathbb{Z}_p^N$ and $\|\xi\|_p = p^{1-\gamma}$.

Now we have

$$(A(\psi^{(l)}_{\gamma\eta\zeta}))(x) = \frac{p^{\frac{N\gamma}{2}}a(p^{1-\gamma})}{[\max(1,p^{1-\gamma})]^l}\int_{\mathbb{Q}_p^N}\chi((p^{-\gamma}\eta-x)\cdot\xi)\Omega(\|p^{-\gamma}\xi+p^{-1}\zeta\|_p)d^N\xi.$$

By changing variables as $z = p^{-\gamma}\xi + p^{-1}\zeta,\quad d^N\xi = p^{-N\gamma}d^Nz$, in the previous integral, we obtain

$$\begin{aligned}&(A(\psi^{(l)}_{\gamma\eta\zeta}))(x)\\ &= \frac{p^{\frac{N\gamma}{2}}a(p^{1-\gamma})}{[\max(1,p^{1-\gamma})]^l}\int_{\mathbb{Q}_p^N}\chi[p^{\gamma}(p^{-\gamma}\eta-x)\cdot(z-p^{-1}\zeta)]\Omega(\|z\|_p)p^{-N\gamma}d^Nz\\ &= \frac{p^{\frac{-N\gamma}{2}}a(p^{1-\gamma})}{[\max(1,p^{1-\gamma})]^l}\chi[-p^{\gamma-1}(p^{-\gamma}\eta-x)\cdot\zeta]\int_{\mathbb{Q}_p^N}\chi[p^{\gamma}(p^{-\gamma}\eta-x)\cdot z]\Omega(\|z\|_p)d^Nz\\ &= \frac{p^{\frac{-N\gamma}{2}}a(p^{1-\gamma})}{[\max(1,p^{1-\gamma})]^l}\chi[-p^{\gamma-1}(p^{-\gamma}\eta-x)\cdot\zeta]\Omega(\|p^{\gamma}(p^{-\gamma}\eta-x)\|_p)\\ &= \frac{p^{\frac{-N\gamma}{2}}a(p^{1-\gamma})}{[\max(1,p^{1-\gamma})]^l}\chi[p^{-1}\zeta\cdot(p^{\gamma}x-\eta)]\Omega(\|p^{\gamma}x-\eta\|_p) = a(p^{1-\gamma})\psi^{(l)}_{\gamma\eta\zeta}(x).\end{aligned}$$

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