Lecturas Matemáticas
Volumen **42** (1) (2021), páginas 27-77
ISSN 0120-1980

From scalars to scales: An overview of local existence and local uniqueness theorems for the initial value problem in one variable

De escalares a escalas: un recorrido por los teoremas de existencia y unicidad local para el problema de valor inicial en una variable

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Abstract. We discuss a collection of local existence and local uniqueness theorems, for the initial value problem in one variable. We consider real-valued equations and the obvious extensions to $\mathbb{R}^n$, as well as equations taking values in a Banach space, and in a scale of Banach spaces. The emphasis is not on giving an exhaustive exposition but, rather, on analyzing hypotheses, contrasting techniques of proof, and developing examples. We also make numerous remarks of a historic nature, and suggest directions for further study. Our exposition ends with a discussion of the existence and uniqueness of local solutions, as generic properties.

***Key words*:** Initial value problem in one variable, local solutions, existence, uniqueness.

Resumen. En este artículo expositorio discutimos varios teoremas de existencia local y unicidad local de soluciones, para el problema de valores iniciales en una variable. Consideramos el caso de ecuaciones con valores reales y su extensión a $\mathbb{R}^n$, así como también ecuaciones con valores en un espacio de Banach, y en una scala de espacios de Banach. Nuestro interés no es el producir una exposicion exhaustiva, sino el analizar hipótesis, comparar métodos de prueba y desarrollar ejemplos. También incluímos muchos comentarios de tipo histórico, y sugerimos direcciones para ampliar el estudio. Nuestra presentación termina con una discusión de la existencia y la unicidad de soluciones locales como propiedades genéricas.

***Palabras clave*:** Problema de valores iniciales en una variable, soluciones locales, existencia, unicidad.

2020 Mathematics Subject Classification: 34A12, 34A30, 34A34, 34A36.

1 Introduction

We discuss local existence and local uniqueness theorems, for the initial value problem in one variable. In doing so, we do not make any claims of completeness, that is not our purpose. Rather, the emphasis is on analyzing hypotheses, contrasting techniques of proof, and developing examples, around a collection of fundamental results. We include many remarks of a historical nature and we often propose directions for further study.

Our exposition intertwines the real case, the obvious extensions to equations with values in $\mathbb{R}^n$, the Banach space case, and the case of scales of Banach spaces. We end with a discussion of local existence and local uniqueness results, as generic properties.

To be sure, the study of problems involving ordinary differential equations, is a huge subject. We acknowledge that our chosen path visits only briefly, or does not visit at all, many areas of great interest, such as continuous dependence on parameters and the initial condition, interval of definition and extension of solutions, analyticity, particular techniques for solving specific problems, and much more. These topics are well developed in some of the references listed at the end of our exposition. We also call attention to the excellent lecture notes written by Peter J. Olver [41], which emphasize the role of differential equations in modeling events of the physical world. Many other sources, a good number of them original, are called upon at the appropriate times.

Let us point out that we use the standard notation in the subject and that all the linear spaces we consider are real.

2 Cauchy-Lipschitz's theorem and Peano's theorem, in $\mathbb{R}$

By a general ordinary differential equation of the first order, we mean the equation

$$F\left(t, y, y'\right) = 0, \tag{1}$$

where F is a function defined in a subdomain of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, that is an open and connected subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. If we can solve for y', the equation is then written in the normal form

$$y' = f\left(t, y\right), \tag{2}$$

for a function f defined in a subdomain D of $\mathbb{R} \times \mathbb{R}$. In what follows, we consider only equations written in such a normal form.

Let us mention that the expression (1) is not quite the most general form. For an interesting discussion on this matter see, for instance, ([29], p. 2).

Definition 1. *Let us fix a subdomain D of $\mathbb{R} \times \mathbb{R}$ and a function $f : D \to \mathbb{R}$.*

1. *A function $y\left(t\right)$ defined on some interval I is admissible if $\left(t, y\left(t\right)\right) \in D$ for all $t \in I$.*
2. *A solution of (2) is an admissible and differentiable function $y : I \to \mathbb{R}$ for some non-trivial interval I, so that $y'\left(t\right) = f\left(t, y\left(t\right)\right)$ for all $t \in I$.*

An interval is non-trivial if it has a non-empty interior. To avoid repetition, let us say that all the intervals to be considered will be non-trivial.

Augustin-Louis Cauchy proved, in 1821, the existence of local solutions for (2), assuming that f is continuous and $\partial_y f$ is bounded (see [6], p. 340). Cauchy's method involved what is now called the Cauchy-Euler's approximation method ([29], p. 1). It can be described as "... proceeding along tangents with slopes given by f, as suggested by the equation (2)" ([6], p. 341).

Although Cauchy included this result in his lectures at the École Polytechnique, it remained practically unknown until 1844, when it appeared in François Moigno's *Leçons de Calcul Différentiel et de Calcul Intégral*, which were based on Cauchy's lectures notes. In 1981, a fragment of Cauchy's notes, including the result, was discovered by Christian Gilain.

In 1870, Rudolf Lipschitz improved upon Cauchy's result, replacing the assumption on the derivative by the following condition on the variable y: There is a constant $k > 0$ so that

$$|f(t, y_1) - f(t, y_2)| \leq k |y_1 - y_2|$$

for every t, y_1, y_2 such that $(t, y_1), (t, y_2) \in D$.

If a function f satisfies this condition, we say that it satisfies a Lipschitz's condition in y with constant k or that is Lipschitz in y with constant k. Let us observe that the condition is assumed to hold uniformly on t.

The function $f(y) = |y|$ shows that the Lipschitz's condition is weaker than the existence of the derivative in y.

Continuity and the Lipschitz's condition imply, not only a local existence theorem, but also a uniqueness theorem as well, for the initial value problem, also called Cauchy problem,

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0, \end{cases} \quad , \tag{3}$$

when $(t_0, y_0) \in D$ is fixed. More precisely,

Theorem 2. *(Cauchy-Lipschitz's theorem) Let $f : D \to \mathbb{R}$ be continuous, and Lipschitz in y with constant k. Let us fix (t_0, y_0) in D and let us consider a closed rectangle R: $|t - t_0| \leq a$, $|y - y_0| \leq b$, in D. Then, if $|f(t, y)| \leq M$ for $(t, y) \in R$, there is one, and only one, solution $y(t)$ of (3) defined on $|t - t_0| \leq h$, where $h = \min\{a, b/M\}$.*

The local existence of solutions for the initial value problem means that given $(t_0, y_0) \in D$ there is a solution $y(t)$ whose graph, in D, passes through (t_0, y_0), that is, $y(t_0) = y_0$. The uniqueness can be stated as saying that the graphs of two distinct solutions cannot have any point of D in common.

Let us point out that the domain of the function $y(t)$ can be extended beyond the bounds set by Theorem 2. In this respect, we observe that if $y(t)$ is the solution guaranteed by Theorem 2, there is a rectangle $R' \subset D$ such that the curve $(t, y(t))$ goes all the way to the left and right sides of R'. If (t_1, y_1) is the point where the curve intersects the right side of R', we can use Theorem 2 with initial value (t_1, y_1), to extend the curve beyond R' to the right. The same argument works on the left.

The following result shows that the curve $(t, y(t))$ can be extended to the boundary of D, when D is bounded.

Theorem 3. *([29], p. 15, Theorem 11) Let D be a bounded domain and let $f : D \to \mathbb{R}$ be continuous, and Lipschitz in y with constant k in a rectangle containing (t_0, y_0). Since the graph of the solution y has to be in D, $y(t)$ is defined only for $t_0 + a < t < t_0 + b$, for suitable a and b.*

Then, if $d(t)$ denotes the distance from the point $(t, y(t))$ to the boundary ∂D, there is

$$\lim_{t \to (t_0+a)^+} d(t) = \lim_{t \to (t_0+b)^-} d(t) = 0.$$

If the domain D is unbounded, we have

Corollary 4. *([29], p. 17, Corollary) When D is unbounded, as $t \to (t_0 + a)^+$ or $t \to (t_0 + b)^-$, either*

1. *$y(t)$ becomes unbounded, or*
2. *$y(t)$ approaches the boundary of D.*

As illustrations of these results, let us consider first the problem

$$\begin{cases} y' = \sqrt[3]{y} \\ y(t_0) = y_0 \end{cases} \tag{4}$$

in the bounded domain

$$D = \{(t, y) : 0 < t < 1, 0 < y < 1\}. \tag{5}$$

If we assume $y \neq 0$, calculating the integral

$$\int_{y_0}^{y} \frac{dy}{\sqrt[3]{y}} = \int_{t_0}^{t} dt$$

and then solving for y, we obtain the solution

$$y(t) = \left(\frac{2}{3}(t - t_0) + y_0^{2/3}\right)^{3/2}.$$

We claim that the curve $(t, y(t))$ will always reach the boundary of D. Indeed, since we only consider the curve $(t, y(t))$ within D, we impose the condition

$$0 < y(t) < 1,$$

so

$$t_0 - \frac{3}{2}y_0^{2/3} < t < t_0 - \frac{3}{2}y_0^{2/3} + \frac{3}{2}.$$

For instance, if $t_0 - \frac{3}{2}y_0^{2/3} = 0$, there is

$$\lim_{t\to 0^+} y(t) = 0$$

and there is

$$\lim_{t\to 1^-} y(t) = \left(\frac{2}{3} + y_0^{2/3} - \frac{2}{3}t_0\right)^{3/2} = \left(\frac{2}{3}\right)^{3/2} < 1.$$

Therefore, the solution reaches the boundary of D, on the left at $(0,0)$ and on the right at $\left(1, \left(\frac{2}{3}\right)^{3/2}\right)$. Any other case will work similarly.

When the domain D is unbounded, for instance

$$D = \{(t,y) : t > 0, y > 0\},$$

$$\lim_{t\to\infty} y(t) = \infty$$

for any initial value (t_0, y_0) and, again, there is

$$\lim_{t\to 0^+} y(t) = 0.$$

As a second example, we consider the problem

$$\begin{cases} y' = y^2 \\ y(t_0) = y_0 \end{cases} \tag{6}$$

for $y_0 \neq 0$, on the domain $\mathbb{R} \times \mathbb{R}$. By integration, we find the solution

$$y(t) = \frac{y_0}{1 - y_0(t - t_0)},$$

which becomes unbounded as t approaches the value $\frac{1}{y_0} + t_0$ from the left and from the right.

When we consider the problem (6) in a bounded domain, say (5), a similar analysis will show that the solution will reach the boundary on the left and also on the right.

Let us observe that Theorem 3 and Corollary 4 only assume that the right-hand side of the equation is locally Lipschitz, that is, it is Lipschitz on a closed rectangle containing the initial value.

When the function $f(t,y)$ is Lipschitz on D, we have the following result:

Theorem 5. *([29], p. 17, Theorem 12) Let D be the domain $(a,b)\times\mathbb{R}$, for $a<b$ fixed. Let $f : D \to \mathbb{R}$ be a function that is continuous, and it is Lipschitz on y uniformly on t. Then, a solution $y(t)$ with graph passing through any point $(t_0, y_0) \in D$, will be defined on the whole interval (a,b).*

This theorem tells us, for instance, that every solution of the equation

$$y' = P(t)\cos y + Q(t)\sin y$$

for $(t, y) \in (a,b)\times\mathbb{R}$, where P and Q are polynomials, has to be defined for every $t \in (a,b)$.

Remark 6. *The Cauchy-Lipschitz's theorem can be proved by constructing approximate solutions using the Cauchy-Euler's approximation method (see, for instance, [29], pp. 3-4, Theorems 3 and 4).*

Remark 7. *The existence and uniqueness of local solutions for (3) can also be proved by the Picard's method, which uses some kind of fixed-point theorem or, more constructively, uses the method of successive approximations, also known as Picard's successive iterations (see, for instance, [29], Chapter 1, Section 7). In both cases, the equation in (3) is first formally integrated, to obtain*

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds, \tag{7}$$

for $|t - t_0| \le h$.

Under the assumptions in Theorem 2, this integral equation is equivalent to, that is has the same solutions as, the initial value problem (3). In the Picard's method, a solution for (7) will be a fixed point for the map

$$y \to y_0 + \int_{t_0}^{t} f(s, y(s))\, ds,$$

defined on an appropriate space of continuous functions. In the method of successive approximations, we define

$$\begin{aligned} y_1(t) &= y_0 + \int_{t_0}^{t} f(s, y_0)\, ds, \\ y_n(t) &= y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\, ds \end{aligned}$$

for $n \ge 1$, and we prove by induction that the function y_n is admissible for $|t - t_0| \le h$ and for all $n \ge 1$. Furthermore, the hypotheses of Theorem 2 imply that the sequence $\{y_j\}_{j\ge1}$ converges uniformly with respect to t for $|t - t_0| \le h$. The limit function will then be a solution of (7). Both methods are actually closely related, as shown by the so *called contraction mapping principle of Banach and Cacciopoli ([25], p. 5, Theorem 3.1), named for the mathematicians Stefan Banach and Renato Cacciopoli.*

As for the uniqueness, if $y_1(t)$ and $y_2(t)$ satisfy (7) for t in an interval I containing t_0,

$$\sup_{t\in I}|y_1(t)-y_2(t)|\leq k\left(\sup_{t\in I}|t-t_0|\right)\sup_{t\in I}|y_1(t)-y_2(t)|.$$

Hence, $y_1=y_2$ if $k\left(\sup_{t\in I}|t-t_0|\right)<1$.

Let us observe that the method of successive approximations generally finds a unique solution in an interval possibly smaller than the interval guaranteed in the statement of Theorem 2 by the Cauchy-Euler's method ([29], p. 21).

Due to the various methods of proof, the Cauchy-Lipschitz's theorem is also known as the Picard-Lindelöf's theorem, for the mathematicians Émile Picard and Ernst Lindelöf.

To assure the existence of local solutions for (3), it suffices to assume that the function f is continuous. This result, due to Giussepe Peano, first appeared, with an incorrect proof, in the *Atti della Accademia delle Scienze di Torino*, in 1886. Peano published a corrected proof in *Mathematische Annalen* in 1890.

Theorem 8. *(Peano's theorem) Let $f:D\to\mathbb{R}$ be continuous. Let us fix (t_0,y_0) in D and let us consider a closed rectangle R: $|t-t_0|\leq a$, $|y-y_0|\leq b$, in D. Then, there exist $0<\delta\leq a$ and a function $y(t)$ defined in $|t-t_0|\leq\delta$ that is a solution of (3).*

Remark 9. *Let us mention that, in the case of Peano's theorem, it is not possible to use directly the method of successive approximations as described in Remark 7. In fact, the approximations will not generally converge, as shown in ([13], p. 42). Actually, the first example was constructed by Martin Müller in 1927 (see reference 2 in [59]). Thus, to prove Peano's theorem it is necessary to incorporate other techniques. For instance, the theorems by Cesare Arzelà and by Guido Ascoli (see, for instance, [45], Chapter III, Section 1.1) are used in the proof given in ([45], p. 144), which follows the Cauchy-Euler's approximation method, while the proof in ([29], p. 10, Theorem 6) relies on the Weierstrass's polynomial approximation theorem (see, for instance, [29], p. 10) and uses a modified version of the successive approximations method. A third proof, based on Juliusz Schauder's fixed point theorem ([25], p. 10), is presented in ([25], p. 14, Theorem 1.1).*

Example 10. *When it is only assumed that the function f is continuous, there is not a general uniqueness theorem. In fact, let us consider the autonomous initial value problem*

$$\begin{cases} y'=\sqrt{|y|} \\ y(t_0)=0, \end{cases} \quad , \tag{8}$$

where autonomous means that the function f does not depend on t.

The function $f(y)=\sqrt{|y|}$ is continuous, but not Lipschitz, in any neighborhood of zero. Indeed, for $\delta>0$,

$$\frac{|f(\delta)-f(0)|}{\delta}=\frac{1}{\sqrt{\delta}},$$

which remains unbounded as $\delta\to 0^+$.

It should be clear that the zero function is a solution of (8). Moreover, the function

$$y_0(t) = \begin{cases} -\frac{(t-t_0)^2}{4} & \textit{for} \quad t \le t_0 \\ \frac{(t-t_0)^2}{4} & \textit{for} \quad t \ge t_0 \end{cases}$$

is also a solution. Furthermore, if $C_1 \le t_0 \le C_2$, the function

$$y_{C_1,C_2}(t) = \begin{cases} -\frac{(t-C_1)^2}{4} & \textit{for} \quad t \le C_1 \\ 0 & \textit{for} \quad C_1 \le t \le C_2 \\ \frac{(t-C_2)^2}{4} & \textit{for} \quad t \ge C_2 \end{cases}$$

is a solution as well. That is to say, (8) has infinitely many solutions.

Let us observe that the functions

$$y_{C_1}(t) = \begin{cases} -\frac{(t-C_1)^2}{4} & \textit{for} \quad t \le C_1 \\ 0 & \textit{for} \quad t \ge C_1, \end{cases}$$

$$y_{C_2}(t) = \begin{cases} 0 & \textit{for} \quad t \le C_2 \\ \frac{(t-C_2)^2}{4} & \textit{for} \quad t \ge C_2, \end{cases}$$

there are also solutions.

Remark 11. *The behavior observed in Example 10 is typical, in the sense that the initial value problem (3) has either one solution, or infinitely many (see, for instance, [45], Chapter III, Section 3). Actually, when there infinitely many solutions, they form a continuum, or closed and connected set, in the space of continuous functions with the topology of the uniform convergence on compact sets (for the proof see, for instance, [28], p. 15). In point of fact, this is related to a result proved by Hellmuth Kneser in 1925.*

In an article published in Mathematische Zeitschrift *in 1925 (see reference in [27]), Mikhail Lavrentieff constructed a function $f(t, y)$, continuous on a closed rectangle $R \subset \mathbb{R}^2$, so that, for each (t_0, y_0) in the interior of R, the problem (3) has more than one solution on every interval $[t_0, t_0 + \varepsilon]$ and $[t_0 - \varepsilon, t_0]$, for $\varepsilon > 0$ small enough. Therefore, the initial value problem has a continuum of solutions. A somewhat less involved example is given in [27].*

Going back to Example 10, for $C_1 = 0$, the graph of the solution y_{C_1} lies beneath the graphs of all the others, while the graph of the solution y_{C_2}, when $C_2 = 0$, lies above. This is a manifestation of the notion of minimum and maximum solutions of an initial value problem, and leads to Peano's phenomenon (see, for instance, [45], Chapter III, Section 2), which we will not discuss here. We will just say that, by definition, the maximum solution and the minimum solution are unique, and that, when there is more than one solution, each point in the band between the graphs of the minimum and maximum solutions, is on the graph of another solution. This is what the Peano's phenomenon is about. Thus, the problem in the non-uniqueness case will have infinitely many solutions that, actually, constitute a continuum. The initial value problem (8) exhibits such behavior.

Remark 12. *Let us mention that Example 10 is special in that* $f(y_0) = 0$. *When* $f(y_0) \neq 0$, *the autonomous initial value problem*

$$\begin{cases} y' = f(y) \\ y(t_0) = y_0 \end{cases} \tag{9}$$

has a unique local solution under the sole condition that the right-hand side $f(y)$ *be continuous. Indeed, we can write, for* y *in a suitable neighborhood of* y_0,

$$F(y) \stackrel{def}{=} t_0 + \int_{y_0}^{y} \frac{du}{f(u)}.$$

The function F *is, near* y_0, *continuously differentiable and either strictly increasing or strictly decreasing, which implies that* F *is injective. Therefore, it has an inverse,* F^{-1}, *defined near* t_0, *that is also continuously differentiable. Since* $F(y_0) = t_0$, *we have*

$$F^{-1}(t_0) = y_0.$$

Moreover, from $F\left(F^{-1}(t)\right) = t$,

$$\left(F^{-1}(t)\right)' = \frac{1}{F'\left(F^{-1}(t)\right)} = f\left(F^{-1}(t)\right).$$

So, $F^{-1}(t)$ *is a solution of the problem (9).*

Let us assume that another function $z(t)$ *is also a solution near* t_0. *Then,*

$$1 = \frac{z'(t)}{f(z(t))} = (F(z(t)))'$$

near t_0, *or*

$$F(z(t)) = t + C,$$

for some constant C. *Since*

$$t_0 = F(y_0) = F(z(t_0)) = t_0 + C,$$

we conclude that $C = 0$. *Hence,*

$$F(z(t)) = t = F\left(F^{-1}(t)\right),$$

which implies that $z(t) = F^{-1}(t)$, *for* t *near* t_0.

We end this section with another example, which, we will see, is closely related to Example 10.

Example 13. *Let us consider the non-autonomous initial value problem*

$$\left\{ \begin{array}{l} z' = t\sqrt{|z|} \\ z(0) = 0 \end{array} \right. . \tag{10}$$

It should be clear that the zero function is a solution of (10) for $t \in \mathbb{R}$*. Moreover, the function*

$$z_0(t) = \frac{t^4}{16}$$

is also a solution for $t \in \mathbb{R}$*. Indeed,*

$$z_0'(t) = \frac{t^3}{4} = t\sqrt{\frac{t^4}{16}} = t\sqrt{z_0(t)} = t\sqrt{|z_0(t)|}.$$

Hence, according to Remark 11, the problem (10) has infinitely many solutions. For example, if $C > 0$*, the function*

$$z_C(t) = \left\{ \begin{array}{ccc} 0 & for & 0 \le t \le \sqrt{C} \\ \frac{(t^2 - C)^2}{16} & for & t \ge \sqrt{C} \end{array} \right. \tag{11}$$

is a solution as well, for $t \ge 0$*. This is clear when* $0 \le t \le \sqrt{C}$*. If* $t \ge \sqrt{C}$*,*

$$z_C'(t) = t\frac{t^2 - C}{4} = t\sqrt{\frac{(t^2 - C)^2}{16}} = t\sqrt{z_C(t)} = t\sqrt{|z_C(t)|}.$$

Formula (11) comes about in a rather natural way. Indeed, we make the following claim: If $y(t)$ *is a solution of*

$$\left\{ \begin{array}{l} y' = \sqrt{|y|} \\ y(0) = 0 \end{array} \right.$$

for $t \ge 0$*, then, the function*

$$z(t) = \frac{y(t^2)}{4}$$

is a solution of (10), for $t \ge 0$*. Indeed,*

$$z'(t) = t\frac{y'(t^2)}{2} = t\sqrt{\frac{|y(t^2)|}{4}} = t\sqrt{|z(t)|}.$$

Now, if we pick, in the notation of Example 10,

$$y_{C_2}(t) = \left\{ \begin{array}{ccc} 0 & for & 0 \le t \le C_2 \\ \frac{(t - C_2)^2}{4} & for & t \ge C_2 \end{array} \right.$$

with $C_2 > 0$*,*

$$\frac{y_{C_2}(t^2)}{4} = \left\{ \begin{array}{ccc} 0 & for & 0 \le t^2 \le C_2 \\ \frac{(t^2 - C_2)^2}{4} & for & t^2 \ge C_2 \end{array} \right.$$

or

$$\frac{y_{C_2}\left(t^2\right)}{4} = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq \sqrt{C_2} \\ \frac{\left(t^2 - C_2\right)^2}{4} & \text{for} \quad t \geq \sqrt{C_2}\,, \end{cases}$$

which gives us (11), if we write C instead of C_2.

3 Extensions to $\mathbb{R}^n$ and to a general Banach space

The local existence and local uniqueness theorems discussed in the previous section apply, with the obvious change in notation, to systems of differential equations. The function f is then defined in a subdomain of $\mathbb{R} \times \mathbb{R}^n$ for some n, and takes values in $\mathbb{R}^n$ (see, for instance, [29], Chapter 2). Generally, we will use in $\mathbb{R}^n$ the euclidean norm.

As a consequence, it is possible to treat equations of higher order,

$$y^{(n)} = f\left(t, y, y', \ldots, y^{n-1}\right)$$

where f is defined on a subdomain D of $\mathbb{R} \times \mathbb{R}^n$, by turning the equation into a system of n equations of first order in n unknowns. To this effect we consider y and its first $(n-1)$ derivatives as the unknowns $x_1, x_2, \ldots, x_n$. These new variables have to satisfy the conditions

$$x_1' = x_2, x_2' = x_3, \ldots, x_{n-1}' = x_n,$$

while

$$x_n' = f\left(t, x_1, x_2, \ldots, x_n\right).$$

For the details, we refer to ([29], Chapter 2, Section 6).

Likewise, the notion of minimum and maximum solutions, as well as Peano's phenomenon, can be formulated for systems (see, for instance, [45], Chapter III, Section 2.3).

The method of successive approximations discussed in Remark 7 still works when $\mathbb{R}^n$ is replaced by a real Banach space. We only need to interpret the integral in an appropriate way. Since f is at least continuous, it is not necessary to resort to the Bochner integral. In fact, it suffices to use a suitable version of the Riemann integral (see, for instance, [2], Section 2). Therefore, there is local existence and local uniqueness in this context.

As an illustration, we state and prove the following result:

Theorem 14. *Let X be a Banach space with norm $\|\cdot\|$ and let I be an interval; let $A : I \to L(X)$ be a continuous function, where $L(X)$ is the Banach space of linear and continuous operators from X into X with the operator norm $\|\cdot\|_{L(X)}$; finally, let $f : I \to X$ be continuous.*

Then, given $t_0 \in I$ and $y_0 \in X$, the initial value problem

$$\begin{cases} y' = A(t)(y) + f(t) \\ y(t_0) = y_0 \end{cases} \tag{12}$$

has one, and only one, solution defined on some interval containing t_0.

Proof. Let us begin by saying that, in this context, y' means the Fréchet derivative of the function $t \to y(t)$ from $\mathbb{R}$ into X, which is defined, for $t \in I$ fixed, as

$$\lim_{h \to 0} \frac{y(t+h) - y(t)}{h}, \tag{13}$$

whenever this limit exists in X.

To prove local uniqueness, let us suppose that $y : J \to X$ is a solution of the problem

$$\begin{cases} y' = A(t)(y) \\ y(t_0) = 0 \end{cases} \tag{14}$$

on some interval J containing t_0. As in the scalar case, we can write, equivalently (see, for instance, [2], p. 7, Proposition 2.6)

$$y(t) = \int_{t_0}^{t} A(s)\, y(s)\, ds.$$

If we assume that J is compact, there is $M > 0$ so that

$$\sup_{t \in J} \|A(t)\|_{L(X)} = M.$$

Therefore,

$$\sup_{t \in J} \|y(t)\| \leq Ml(J) \left(\sup_{t \in J} \|y(t)\| \right),$$

where $l(J)$ denotes the length of the interval J.

If we choose J so that $Ml(J) < 1$, we conclude that the solution y must be identically zero on J.

As for the local existence, we observe that (12) is equivalent (see, for instance, [2], p. 7, Proposition 2.6) to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} A(s)\, y(s)\, ds + \int_{t_0}^{t} f(s)\, ds.$$

We define a sequence $\{y_j\}_{j \geq 0}$ of functions $y_j : I \to X$ as follows:

$$\begin{aligned} y_0(t) &= y_0 + \int_{t_0}^{t} f(s)\, ds, \\ y_1(t) &= y_0 + \int_{t_0}^{t} A(s)\, y_0(s)\, ds + \int_{t_0}^{t} f(s)\, ds, \end{aligned}$$

and

$$y_k(t) = y_0 + \int_{t_0}^{t} A(s)\, y_{k-1}(s)\, ds + \int_{t_0}^{t} f(s)\, ds, \tag{15}$$

for $k \geq 2$.

Let J be a compact subinterval of I that contains t_0 and let $M = \sup_{t\in J} \|A(t)\|_{L(X)}$. We will prove by induction that

$$\|y_{k+1}(t) - y_k(t)\| \leq \left(\|y_0\| + l(J) \sup_{s\in J} \|f(s)\|\right) \frac{(M|t-t_0|)^{k+1}}{(k+1)!} \tag{16}$$

for $t \in J$ and $k \geq 0$.

Indeed, when $k = 0$,

$$\|y_1(t) - y_0(t)\| = \left\|\int_{t_0}^{t} A(s) y_0(s)\, ds\right\| \leq \left(\|y_0\| + l(J) \sup_{s\in J} \|f(s)\|\right) M|t-t_0|.$$

If we assume that (16) holds for $k \leq m$,

$$\begin{aligned} \|y_{m+2}(t) - y_{m+1}(t)\| &\leq \left\|\int_{t_0}^{t} A(s)(y_{m+1}(s) - y_m(s))\, ds\right\| \\ &\leq M\left|\int_{t_0}^{t} \|y_{m+1}(s) - y_m(s)\|\, ds\right|. \end{aligned}$$

We can see that the assumptions made on the right-hand side of the equation in (12), automatically imply a Lipschitz's condition in y, at least locally.

If $t > t_0$,

$$\begin{aligned} M\int_{t_0}^{t} \|y_{m+1}(s) - y_m(s)\|\, ds &\leq M^{m+2}\left(\|y_0\| + l(J) \sup_{s\in J} \|f(s)\|\right) \int_{t_0}^{t} \frac{(s-t_0)^{m+1}}{(m+1)!} ds \\ &= \left(\|y_0\| + l(J) \sup_{s\in J} \|f(s)\|\right) \frac{(M|t-t_0|)^{m+2}}{(m+2)!}, \end{aligned}$$

where $l(J)$ denotes the length of the interval J.

If $t < t_0$,

$$M\left|\int_{t_0}^{t} \|y_{m+1}(s) - y_m(s)\|\, ds\right| = M\int_{t}^{t_0} \|y_{m+1}(s) - y_m(s)\|\, ds$$

$$\begin{aligned} &\leq M^{m+2}\left(\|y_0\| + l(J) \sup_{s\in J} \|f(s)\|\right) \int_{t}^{t_0} \frac{(t_0-s)^{m+1}}{(m+1)!} ds \\ &= \left(\|y_0\| + l(J) \sup_{s\in J} \|f(s)\|\right) \frac{(M|t-t_0|)^{m+2}}{(m+2)!}. \end{aligned}$$

The identity

$$y_{k+m}(t) - y_k(t) = \sum_{l=k}^{k+m-1} (y_{l+1}(t) - y_l(t))$$

and the estimate (16) imply that

$$\begin{aligned} \|y_{k+m}(t) - y_k(t)\| &\leq \left(\|y_0\| + l(J) \sup_{s \in J} \|f(s)\|\right) \sum_{l=k}^{k+m-1} \frac{(Ml(J))^{l+1}}{(l+1)!} \\ &\leq \left(\|y_0\| + l(J) \sup_{s \in J} \|f(s)\|\right) \sum_{l \geq k} \frac{(Ml(J))^{l+1}}{(l+1)!}. \end{aligned}$$

Hence,

$$\|y_{k+m}(t) - y_k(t)\| \underset{k \to \infty}{\to} 0$$

uniformly in $m \geq 1$ and $t \in J$.

That is to say, the sequence $\{y_k\}_{k \geq 0}$ is Cauchy, uniformly in $t \in J$ and, therefore, it converges uniformly in $t \in J$, to a function $y : J \to X$. So, taking the limit in (15) as $k \to \infty$, we get

$$y(t) = y_0 + \int_{t_0}^{t} A(s) y(s) \, ds + \int_{t_0}^{t} f(s) \, ds$$

for $t \in J$. This means that the function y is a solution of the problem (12) in the interval J.

This completes the proof of the theorem. □

Remark 15. *If we take the limit, as $m \to \infty$, on both sides of the inequality*

$$\|y_{k+m}(t) - y_k(t)\| \leq \left(\|y_0\| + l(J) \sup_{s \in J} \|f(s)\|\right) \sum_{l=k}^{k+m-1} \frac{(M|t-t_0|)^{l+1}}{(l+1)!},$$

we can write

$$\begin{aligned} \|y_k(t) - y(t)\| &\leq \left(\|y_0\| + l(J) \sup_{s \in J} \|f(s)\|\right) \sum_{l \geq k} \frac{(M|t-t_0|)^{l+1}}{(l+1)!} \\ &\leq \left(\|y_0\| + l(J) \sup_{s \in J} \|f(s)\|\right) (M|t-t_0|)^{k+1} e^{M|t-t_0|} \end{aligned}$$

for $t \in J$.

Remark 16. *When $X = \mathbb{R}^n$, Theorem 14 gives a local existence and local uniqueness result for systems of n linear equations. In particular, for $n = 1$, it covers the case of one linear scalar equation.*

Remark 17. *If*

$$\sup_{t \in I} \|A(t)\|_{L(X)} < \infty,$$

the proof of Theorem 14 implies that the sequence $\{y_k\}_{k \geq 0}$ of the successive approximations (15), converges uniformly on compact subintervals of I containing t_0, to a solution, defined on I, of the problem (12). Moreover, this solution is unique. To show uniqueness, we proceed as follows:

Let y be a solution on I of the problem (14). Since all the intervals we consider have non-empty interior, either $[a, t_0] \subseteq I$ for some $a < t_0$ or $[t_0, b] \subseteq I$ for some $b > t_0$. Let us first assume that $[a, t_0] \subseteq I$ for some $a < t_0$. According to the uniqueness part of Theorem 14, $y(t)$ must be zero for t in some interval $[a', t_0]$ for $a' \geq a$. If $a' = a$, then y is zero on $[a, t_0]$. If $a' > a$ and y is not identically zero on $[a, t_0]$, let us consider

$$t_1 = \max\{s \in [a, t_0] : \|y(s)\| = 0\},$$

where t_1 must be strictly larger than a.

We define a function $Y(t)$, for $t \in [a, t_0]$, as

$$Y(t) = \max\{s \in [t, t_0] : \|y(s)\|\}.$$

By definition, $Y(t) = 0$ for $t \in [t_1, t_0]$ and $Y(t) > 0$ for $t \in [a, t_1)$. Therefore,

$$\begin{aligned} Y(t) &= \max_{t \leq s \leq t_1} \left\| \int_{t_1}^{s} A(r)\, y(r)\, dr \right\| \leq \int_{s}^{t_1} \|A(r)\|_{L(X)} \|y(r)\|\, dr \\ &\leq \int_{t}^{t_1} \|A(r)\|_{L(X)} \|y(r)\|\, dr \leq M\,|t - t_1| \max_{t \leq s \leq t_1} \|y(s)\| \end{aligned}$$

or

$$Y(t) \leq M\,|t - t_1|\, Y(t)$$

for $t \in |a, t_1|$.

Hence, for $t \in [a, t_1)$, we have $1 \leq M\,|t - t_1|$, which is not possible.

So, we have shown that the solution y must be identically zero on every interval $[a, t_0]$ contained in I.

Obvious modifications of the proof we just presented, will show that the solution y must be identically zero on every interval $[t_0, b]$ contained in I.

Finally, we observe that

$$I = \left(\bigcup_{a < t_0} \{[a, t_0] : [a, t_0] \subseteq I\} \right) \bigcup \left(\bigcup_{b > t_0} \{[t_0, b] : [t_0, b] \subseteq I\} \right),$$

so, this proof of global uniqueness is complete.

Remark 18. *If the function $t \to A(t)$ is constant, that is $A(t) = A \in L(X)$ for every $t \in I$, the solution y has an explicit formula. One way of seeing this, is to use the integrating factor method in the context of a Banach space, as developed in [2].*

Another way, which of course results in the same explicit formulation, is to observe that y can be written as $v + w$, where v and w solve the problems

$$\begin{cases} v' = A(v) \\ v(t_0) = y_0 \end{cases} \tag{17}$$

and

$$\begin{cases} w' = A(w) + f \\ w(t_0) = 0 \end{cases} \tag{18}$$

respectively.

The function $v = e^{(t-t_0)A}(y_0)$*, defined for every* $t \in \mathbb{R}$*, is the solution of (17). We refer to [2], for the definition and the properties of exponential functions of operators.*

To solve (18), we first observe that $v(t) = e^{tA}(c)$ *is a solution of the homogeneous equation, for any* $c \in X$*. Following the idea of the method called variation of parameters, we propose a particular solution of the form*

$$w(t) = e^{tA}(c(t)),$$

where $c(t)$ *is a function to be determined. To find* $c(t)$*, we substitute* $w(t)$ *in the differential equation,*

$$e^{tA}(c'(t)) + Ae^{tA}(c(t)) = Ae^{tA}(c(t)) + f(t).$$

Applying the inverse e^{-tA} *to both sides of this equation,*

$$c'(t) = e^{-tA},$$

or

$$c(t) = \int_{t_0}^{t} e^{-sA} f(s)\, ds.$$

Hence,

$$w(t) = \int_{t_0}^{t} e^{(t-s)A} f(s)\, ds,$$

for $t \in I$*.*

Finally,

$$y(t) = v(t) + w(t) = e^{(t-t_0)A}(y_0) + \int_{t_0}^{t} e^{(t-s)A} f(s)\, ds.$$

Unlike the Cauchy-Lipschitz's theorem, Peano's theorem does not hold generally in the case of a Banach space. We present the following example, suggested by Jean Dieudonné (see [16], p. 287, Problem 5):

Example 19. *Let* c_0 *be the Banach space consisting of real sequences* $y = \{y_j\}_{j\geq 1}$ *that converge to zero, with the norm* $\|y\|_{c_0} = \sup_{j\geq 1} |y_j|$*.*

We define a function $f : c_0 \to c_0$ *as*

$$f(\{y_j\}) = \left\{ \sqrt{|y_j|} + \frac{1}{j+1} \right\}$$

and we consider the autonomous initial value problem

$$\begin{cases} y' = f(\{y_j\}) \\ y(0) = 0, \end{cases} \tag{19}$$

where y' *is the Fréchet derivative of the function* $t \to y(t)$ *from* $\mathbb{R}$ *into* c_0*, defined by (13). Equivalently, it can be defined, in this case, as the sequence*

$$\left\{ \lim_{h \to 0} \frac{y_j(t+h) - y_j(t)}{h} \right\}_j,$$

where the limit is taken in $\mathbb{R}$*, uniformly with respect to* $j \geq 1$.

Our first claim is that the function f *is continuous from* c_0 *into itself. In fact, since the function* $y \to \sqrt{|y|}$ *is uniformly continuous from* $\mathbb{R}$ *into* $\mathbb{R}$*,*

$$\|f(\{y_j\} - f(\{x_j\}))\|_{c_0} = \sup_{j \geq 1} \left| \sqrt{|y_j|} - \sqrt{|x_j|} \right| < \varepsilon,$$

provided that $\sup_{j \geq 1} |y_j - x_j| < \delta$*, for some* $\delta(\varepsilon) > 0$.

Peano's theorem will not hold generally, if we show that (19) does not have a solution.

Let us assume, on the contrary, that there is a Fréchet differentiable function $y : I \to c_0$ *for some real interval* $I = (a, b)$ *containing zero, that solves (19). Then, for each* $j \geq 1$ *we have*

$$y_j' = \sqrt{|y_j|} + \frac{1}{j+1}$$

in I.

For simplicity, we write this equation as

$$x' = \sqrt{|x|} + N, \tag{20}$$

with $N > 0$.

The right-hand side of (20) is never zero, so Example 10 implies that $x(t)$ *is the unique solution of the problem*

$$\begin{cases} x' = \sqrt{|x|} + N \\ x(0) = 0\,. \end{cases} \tag{21}$$

Since x' *is positive on* $(0, b)$ *and* $x(0) = 0$*, we can write*

$$x' = \sqrt{x} + N$$

for $0 \leq t < b$. *Using the change of variable* $u = \sqrt{x}$,

$$\begin{aligned} \int \frac{dx}{\sqrt{x}+N} &= \int \frac{2u}{u+N} du \\ &= 2\left[u - N\ln\left(u+N\right)\right] + C = 2\left[\sqrt{x} - N\ln\left(\sqrt{x}+N\right)\right] + C, \end{aligned}$$

for any $C \in \mathbb{R}$.

Hence, integrating both sides of (21) between 0 *and* $t < b$, *we have*

$$2\left[\sqrt{x(t)} - N\ln\left(\sqrt{x(t)}+N\right)\right] + 2N\ln N = t$$

or

$$\sqrt{x(t)} = \frac{t}{2} + 2N\ln\frac{\sqrt{x(t)}+N}{N} \geq \frac{t}{2}.$$

Therefore,

$$y_j(t) \geq \frac{t^2}{4}$$

for all $j \geq 1$ *and for* $0 \leq t < b$.

That is to say, the sequence $\{y_j(t)\}_{j\geq 1}$ *does not belong to* c_0 *for any* $t \in (0, b)$, *which contradicts our initial assumption.*

Remark 20. *In Example 19, the right hand side of (19) is a function mapping a certain infinite dimensional Banach space into itself. It is natural to ask whether Peano's theorem would hold when the space is an infinite dimensional Hilbert space. The example due to Alexander N. Godunov [21], shows that the answer is no. Indeed, Godunov constructed a continuous function* $f : \mathbb{R}\times l^2 \to l^2$, *for which the problem with initial condition* $y(0) = 0$, *does not have a solution (see* Mathematical Reviews *MR0328253). In an article published in 1974 in the* Mathematical Notes of the Academy of Sciences of the USSR *(Vol. 15, 273-279), Godunov extended the result to an infinite dimensional separable Hilbert space and to an arbitrary condition* $y(t_0) = y_0$.

Godunov proved in [22] (see Mathematical Reviews *MR0470394), that a local existence and local uniqueness theorem holds for the abstract initial value problem*

$$\begin{cases} x' = f(t, x(t)) \\ x(t_0) = x_0, \end{cases}$$

when the function f *is*

1. *continuous and bounded on* $[t_0, t_0 + a] \times B_r(x_0)$ *with values in a Hilbert space* H, *where* $B_r(x_0)$ *denotes the closed ball centered at* x_0 *with radius* $r > 0$, *and*
2. *it satisfies two other rather technical conditions, which amount to an associated scalar initial value problem having zero as the only solution.*

4 Osgood's condition and Osgood's theorem

William Fogg Osgood, in an article published in 1898 in *Monatshefte der Mathematik und Physik* (see reference 3 in [59]), proved a local existence and uniqueness result, using the following condition:

Definition 21. *Let $f : D \to \mathbb{R}$ be a function defined on a subdomain D of $\mathbb{R} \times \mathbb{R}$. We say that f satisfies Osgood's condition if*

$$|f(t, y_1) - f(t, y_2)| \leq \varphi(|y_1 - y_2|) \tag{22}$$

for $(t, y_1), (t, y_2)$ in D, where the function $\varphi : [0, \infty) \to [0, \infty)$ is continuous, strictly positive on $(0, \infty)$, and

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{du}{\varphi(u)} = \infty. \tag{23}$$

The assumptions on φ stated in Definition 21, imply that the function φ must be zero at $t = 0$.

Remark 22. *Let us observe that (22) describes, quantitatively, how the function f is uniformly continuous in y, uniformly with respect to t.*

That a function $f(t, y)$ is Lipschitz in y with constant k, exactly means that it satisfies Osgood's condition for

$$\varphi(u) = ku.$$

As other examples of the function φ in Osgood's condition, we mention

$$\varphi_n(u) = \begin{cases} ku(|\ln u| + 1)^n & \text{for} \quad u > 0 \\ 0 & \text{for} \quad u = 0, \end{cases}$$

for $n = 1, 2, \ldots$, and

$$\varphi(u) = \begin{cases} ku(|\ln u| + 1)\ln(|\ln u| + 1) & \text{for} \quad u > 0 \\ 0 & \text{for} \quad u = 0, \end{cases}$$

for some $k > 0$.

Theorem 23. *(Osgood's theorem) Let $f : D \to \mathbb{R}$ be a function defined on a subdomain D of $\mathbb{R} \times \mathbb{R}$ that is continuous and satisfies Osgood's condition. Then, for each $(t_0, y_0) \in D$ the problem*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

has one, and only one, solution defined on some interval $I = (a, b)$ containing t_0.

Remark 24. *To be sure, the existence part of Osgood's theorem is guaranteed by Peano's theorem.*

As for the uniqueness part, it can be proved by contradiction, assuming that there are two different solutions, y_1 and y_2. Then, the problem

$$\begin{cases} z' = f(t, y_1) - f(t, y_2) \\ z(t_0) = 0, \end{cases}$$

is solved by $z = y_1 - y_2$. By our assumption on y_1 and y_2, there is $s \in I$ such that $y_1(s) \neq y_2(s)$. If, for instance, $s > t_0$ and $y_1(s) - y_2(s) = z(s) = z_1$ is positive, we consider the problem

$$\begin{cases} u' = \varphi(|u|) \\ u(t_0) = z_1, \end{cases}$$

which, according to Remark 12, has one, and only one, solution $u(t)$, to be found by separation of variables. A comparison argument between $u(t)$ and $z(t)$ leads to a contradiction. For the details see, for instance, ([46], p. 34). For a different proof, see ([45], p. 180). A third proof, assuming that φ is non-decreasing, can be seen in ([5], p. 13, Theorem 1.4.2) or in [46].

The proof presented in ([45], p. 180), shows that Osgood's theorem works for systems of equations.

There are various extensions and modifications of Osgood's condition (see, for instance, [45], p. 180) under which Osgood's theorem still holds true.

Remark 25. *The conclusion of Osgood's theorem does not generally apply to continuous functions $f(t, y)$ that satisfy (22) for $\varphi(u) = ku^{\alpha}$, with $0 < \alpha < 1$ and $k > 0$. That is to say, the conclusion of Osgood's theorem does not generally apply to continuous functions $f(t, y)$ that are α-Hölder in y for $0 < \alpha < 1$. Example 10 will illustrate this point with $\alpha = \frac{1}{2}$, if we show that $f(y) = |y|^{\beta}$ satisfies (22) exactly when $\beta = \alpha$. Indeed, if $0 < \beta < \alpha$,*

$$\frac{\left|f\left(\frac{1}{n}\right) - f(0)\right|}{\left|\frac{1}{n} - 0\right|^{\alpha}} = \frac{\frac{1}{n^{\beta}}}{\frac{1}{n^{\alpha}}} = n^{\alpha-\beta}$$

for $n = 1, 2, \ldots$, which goes to infinity as $n \to \infty$. If $\beta > \alpha$,

$$\frac{|f(n) - f(0)|}{|n - 0|^{\alpha}} = \frac{n^{\beta}}{n^{\alpha}} = n^{\beta-\alpha}$$

for $n = 1, 2, \ldots$, which also goes to infinity as $n \to \infty$. If $\beta = \alpha$, we fix $y_1, y_2 \in \mathbb{R}$ different from zero. Then,

$$\frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|^{\alpha}} = \frac{\left||y_1|^{\alpha} - |y_2|^{\alpha}\right|}{|y_1 - y_2|^{\alpha}}.$$

Since $||y_1| - |y_2|| \leq |y_1 - y_2|$,

$$\frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|^{\alpha}} \leq \frac{\left||y_1|^{\alpha} - |y_2|^{\alpha}\right|}{||y_1| - |y_2||^{\alpha}}.$$

If $|y_1| > |y_2|$*, let* $a = \frac{|y_2|}{|y_1|}$*. Therefore,*

$$\frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|^{\alpha}} \leq \frac{(1 - a^{\alpha})}{(1 - a)^{\alpha}}$$

and

$$(1 - a)^{\alpha} \geq 1 - a \geq 1 - a^{\alpha}.$$

If $|y_1| < |y_2|$ *we write*

$$\frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|^{\alpha}} = \frac{||y_2|^{\alpha} - |y_1|^{\alpha}|}{|y_2 - y_1|^{\alpha}}$$

and we repeat the same calculation with $a = \frac{|y_1|}{|y_2|}$.

If we were to limit the domain of the function $f(y) = |y|^{\beta}$ *to* $[0, 1]$*, similar calculations will show that* f *is* α*-Hölder for* $0 < \alpha \leq \beta$ *but not for* $0 < \beta < \alpha$*.*

Finally, if a function $f(t, y)$ *is* α*-Hölder in* y *for* $\alpha > 1$*, then the function does not depend on* y*:*

$$|f'(y_2)| = \lim_{y_1 \to y_2} \left| \frac{f(y_1) - f(y_2)}{y_1 - y_2} \right| \leq k \lim_{y_1 \to y_2} |y_1 - y_2|^{\alpha - 1} = 0.$$

Osgood's theorem, like Cauchy-Lipschitz's theorem, is, simultaneously, a local existence and a local uniqueness result. However, it is usually referred to as Osgood's uniqueness theorem, to highlight it as an extension of the uniqueness part in Cauchy-Lipschitz's theorem.

Let us point out that there are other criteria for local existence as well as for local uniqueness (see, for instance, [5]; [12]; [37]; [45], and the references therein), that we will not discuss. Instead, we will end this section with two applications of the Osgood's condition. The first one is due to Aurel Wintner [59] and the second one to Lawrence Markus [39].

Theorem 26. *Let* $D \subseteq \mathbb{R} \times \mathbb{R}$ *be a subdomain. If* $f : D \to \mathbb{R}$ *is continuous and it satisfies Osgood's condition, then, for each* $(t_0, y_0) \in D$*, the sequence* $\{y_m\}_{m \geq 0}$ *of successive approximations, defined as*

$$\begin{aligned} y_1(t) &= y_0 + \int_{t_0}^{t} f(s, y_0)\, ds, \\ y_m(t) &= y_0 + \int_{t_0}^{t} f(s, y_{m-1}(s))\, ds \end{aligned}$$

for $m \geq 1$*, is well defined and converges, uniformly with respect to* t*, for* t *in some interval* (c', c) *containing* t_0*, to a solution* $y(t)$ *of the problem*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

Theorem 27. *Let $f : D \to \mathbb{R}$ be continuous, where D is a subdomain of $\mathbb{R}\times\mathbb{R}$. Further, we assume that for each $(t_0, y_0) \in D$, there is a closed rectangle $R \subset D$, $|t - t_0| \leq a, |y - y_0| \leq b$, and constants $C > 0, \gamma > 0$, such that*

$$|f(t, y + h) + f(t, y - h) - 2f(t, y)| \leq Ch$$

for $(t, y) \in R$ and $0 < h < \gamma$. Then, the problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

has one, and only one, solution defined on some open interval containing t_0.

Remark 28. *According to Remark 22, the statement of Theorem 26 is an extension of Cauchy-Lipschitz's theorem. Wintner refers to Osgood's condition as "something like the best restriction not containing t" ([59], p. 14). As for the proof, Wintner states that it is done using "arguments which, in every respect, are more sophisticated that the 'linear' inequalities of Cauchy-Lipschitz-Picard-Lindelöf." ([59], p. 14). Unfortunately, the nature of the proof does not yield an "explicit estimate of the deviation of the solution $y(t)$ from the approximations $y_n(t)$" ([59], p. 18). Since Wintner presents a detailed explanation of his proof, we will not repeat it here. Let us mention that Wintner considers a system of equations, in which case, the absolute values in the Osgood's condition (22) are replaced by the euclidean norm in $\mathbb{R}^n$.*

The assumption, in the original statement of Wintner's result, that the function $f(t, y)$ is defined in a closed rectangle $t_0 \leq t \leq a$, $\|y - y_0\| \leq b$, renders a solution living "in the future" from the initial "time" t_0. That is, a solution defined on some interval $[t_0, c)$. However, under the assumptions of Theorem 26, the exact same argument proves also the existence of a solution defined "in the past", that is to say, defined in an interval $(c', t_0]$. It shows also the existence of a solution defined on some interval (c', c) containing t_0.

As for Theorem 27, it is a direct consequence of Osgood's theorem, once we prove the following result, which has an interest of its own:

Theorem 29. *Let $f : D \to \mathbb{R}$ be continuous, where D is a subdomain of $\mathbb{R}\times\mathbb{R}$. Given $(t_0, y_0) \in D$ fixed, we assume that there is a closed rectangle $R \subset D$, $|t - t_0| \leq a, |y - y_0| \leq b$, and constants $C > 0, \gamma > 0$, such that*

$$|f(t, y + h) + f(t, y - h) - 2f(t, y)| \leq Ch \tag{24}$$

for all $(t, y) \in R$ and $0 < h \leq \gamma$.

Then, there exists a constant $k > 0$ so that

$$|f(t, y_1) - f(t, y_2)| \leq k |y_1 - y_2| (|\ln |y_1 - y_2|| + 1)$$

for (t, y_1) and $(t, y_2) \in R$ with $0 < |y_1 - y_2| \leq \gamma/2$.

Remark 30. *Given an open interval I, the class of continuous functions $f : I \to \mathbb{R}$ that satisfy the condition*

$$|f(y+h) + f(y-h) - 2f(y)| \leq Ch$$

for some $C > 0$, for all $y \in [a,b] \subset I$ and for all positive h small enough, was denoted Λ_* by Antoni Zygmund ([60], p. 47).

If $f : I \to \mathbb{R}$ satisfies the Lipschitz's condition with constant k, the estimate

$$\begin{aligned} |f(y+h) + f(y-h) - 2f(y)| &\leq |f(y+h) - f(y)| + |f(y-h) - f(y)| \\ &\leq 2kh \end{aligned}$$

shows that $f \in \Lambda_*$. However, not every function in Λ_* is Lipschitz. For example, the function

$$w(y) = \sum_{j \geq 1} b^{-j} \cos b^j y \qquad (25)$$

with $b > 1$ an integer, belongs to Λ_* ([61], p. 47, Theorem 4.9) but it is not Lipschitz. Indeed, the estimate

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| \leq k \sum_{j=1}^{n} (b_j - a_j) < \varepsilon$$

for

$$\sum_{j=1}^{n} (b_j - a_j) < \delta = \frac{\varepsilon}{k},$$

shows that a Lipschitz function f is absolutely continuous. This means that, for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ so

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon$$

for any $n = 1, 2, \ldots$ and any collection $\{[a_j, b_j]\}_{1 \leq j \leq n}$ of subintervals of I with disjoint interiors, that satisfy

$$\sum_{j=1}^{n} (b_j - a_j) < \delta.$$

If the function w in (25) were Lipschitz and, therefore, absolutely continuous, it would be differentiable almost everywhere (see, for instance, [48], p. 148, Theorem 7.20). That is, it would be differentiable except on a set of Lebesgue measure zero. Therefore ([61], p. 40), the result of differentiating term by term the Fourier series associated with w would *be the Fourier series associated with w' and so ([61], p. 45, Theorem 4.4), the sequence of its coefficients should converge to zero. However, this is not the case, since*

$$w' \sim -\sum_{j\geq 1} \sin b^j y.$$

Although not every function in Λ_* is Lipschitz, as a consequence of Theorem 29 and the estimate

$$k\,|u|\,(|\ln |u|| + 1) \leq k \sup_{u\ small} \left(|u|^{1-\alpha}\,(|\ln |u|| + 1)\right) |u|^{\alpha},$$

$0 < \alpha < 1$ *fixed, the functions in Λ_* are α-Hölder for $|y_1 - y_2|$ small enough.*

It might be seen that the class Λ_* goes too far as an extension of the class of Lipschitz functions, since it does not retain certain basic properties, such as differentiability almost everywhere. However, as Zygmund argues in [60] using a variety of examples, "from the point of view of trigonometric series, Λ_* is more natural than the class of Lipschitz functions" ([60], p. 49).

For a proof of Theorem 29, Markus refers to page 52 in [60] . However, the result appears to be mentioned in [60] without proof. Therefore, we present here a proof, which is a minor modification of the proof of Theorem 3.4 in ([61], p. 44).

Proof. We fix $(t, y) \in R$ and define a function $g : [0, \gamma] \to \mathbb{R}$ as

$$g(\tau) = f(t, y + \tau) - f(t, y).$$

Then,

$$\begin{aligned} g(\tau) - 2g\left(\frac{\tau}{2}\right) &= f(t, y+\tau) - f(t,y) - 2f\left(t, y + \frac{\tau}{2}\right) + 2f(t,y) \\ &= f\left(t, y + \frac{\tau}{2} + \frac{\tau}{2}\right) + f\left(t, y + \frac{\tau}{2} - \frac{\tau}{2}\right) - 2f\left(t, y + \frac{\tau}{2}\right). \end{aligned}$$

Using (24) with $y + \frac{\tau}{2}$ instead of y and $h = \frac{\tau}{2}$,

$$\left| g(\tau) - 2g\left(\frac{\tau}{2}\right) \right| \leq C\frac{\tau}{2}.$$

Let us observe that the same estimate yields

$$\left| 2g\left(\frac{\tau}{2}\right) - 2^2 g\left(\frac{\tau}{2^2}\right) \right| \leq 2C\frac{\tau}{2^2}.$$

If we assume

$$\left| 2^{n-1} g\left(\frac{\tau}{2^{n-1}}\right) - 2^n g\left(\frac{\tau}{2^n}\right) \right| \leq 2^{n-1} C \frac{\tau}{2^n}$$

for any $n = 1, 2, \ldots$ fixed, then

$$\begin{aligned} \left|2^n g\left(\frac{\tau}{2^n}\right) - 2^{n+1} g\left(\frac{\tau}{2^{n+1}}\right)\right| &= 2\left|2^{n-1} g\left(\frac{\tau/2}{2^{n-1}}\right) - 2^n g\left(\frac{\tau/2}{2^n}\right)\right| \\ &\leq 2\, 2^{n-1} C \frac{\tau/2}{2^n} = 2^n C \frac{\tau}{2^{n+1}}. \end{aligned}$$

Therefore,

$$\left|2^{n-1} g\left(\frac{\tau}{2^{n-1}}\right) - 2^n g\left(\frac{\tau}{2^n}\right)\right| \leq C\frac{\tau}{2}$$

for all $n = 1, 2, \ldots$ and

$$\begin{aligned} \left|g(\tau) - 2^n g\left(\frac{\tau}{2^n}\right)\right| &\leq \sum_{j=1}^{n} \left|2^{j-1} g\left(\frac{\tau}{2^{j-1}}\right) - 2^j g\left(\frac{\tau}{2^j}\right)\right| \\ &\leq Cn\frac{\tau}{2}. \end{aligned} \tag{26}$$

We fix $h \in (0, \frac{\gamma}{2}]$ and we choose n so that

$$\frac{\gamma}{2} \leq 2^n h \leq \gamma.$$

Then,

$$2^n \leq \frac{\gamma}{h}$$

or

$$\begin{aligned} n \ln 2 &\leq \ln \gamma - \ln h \leq |\ln \gamma| + |\ln h| \\ &\leq (|\ln \gamma| + 1)(|\ln h| + 1). \end{aligned}$$

Thus,

$$n \leq \frac{|\ln \gamma| + 1}{\ln 2} (|\ln h| + 1).$$

According to (26), if we take $\tau = 2^n h$,

$$\frac{2^n C n}{2} h \geq |g(2^n h) - 2^n g(h)| \geq |2^n g(h)| - |g(2^n h)|$$

or

$$|2^n g(h)| \leq |g(2^n h)| + 2^{n-1} C n h.$$

Let

$$M = \max_{(t,y)\in R} |f(t,y)|.$$

Then,

$$|g(h)| \leq \frac{2M}{2^n} + \frac{Cn}{2} h \leq \frac{4M}{\gamma} h + \frac{C}{2\ln 2} h(|\ln h| + 1)$$

or

$$|f(t, y+h) - f(t,y)| \leq kh(|\ln h| + 1),$$

where $k = \max\left\{\frac{4M}{\gamma}, \frac{C}{2\ln 2}\right\}$.

Finally, for $(t, y_1), (t, y_2) \in R$ with $0 \le |y_1 - y_2| \le \frac{\gamma}{2}$,

$$|f(t, y_1) - f(t, y_2)| \le k |y_1 - y_2| (|\ln |y_1 - y_2|| + 1).$$

This completes the proof of the theorem. □

Theorem 29 and Remark 30 show that Theorem 27 is truly an extension of Cauchy-Lipschitz's theorem.

5 Carathéodory's conditions

So far, we have looked for local solutions of the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0, \end{cases}$$

assuming that the function f is at least continuous near (t_0, y_0). The following two examples illustrate what might happen when this is no longer the case.

Example 31. *Let $f(t, y)$ be the function defined on $\mathbb{R}$ as*

$$f(t, y) = \begin{cases} \frac{2y}{t} & \textit{for} \quad t \neq 0 \\ 0 & \textit{for} \quad t = 0. \end{cases} \tag{27}$$

It should be clear that f is discontinuous at points of the form $(0, y)$ for any $y \in \mathbb{R}$.

We claim that (27) does not have a solution when $t_0 = 0$ and $y_0 \neq 0$, while it has infinitely many solutions when $t_0 = y_0 = 0$.

Indeed, let us begin with the case $t_0 = 0$ and $y_0 \neq 0$, and let us assume that there is a solution $y(t)$ defined near $(0, y_0)$. Since $y(t)$ is continuous and $y(0) \neq 0$, the function $y(t)$ will be different from zero in an interval $[0, b]$, for some $b > 0$. Let us fix $t \in (\varepsilon, b)$, for $0 < \varepsilon < b$ fixed. The function y' is continuous on (ε, b), so

$$\int_\varepsilon^t \frac{y'(s)}{y(s)} ds = \int_\varepsilon^t \frac{2}{s} ds,$$

or

$$\ln \left| \frac{y(t)}{y(\varepsilon)} \right| = \ln \frac{t^2}{\varepsilon^2}.$$

Therefore,

$$\left| \frac{y(t)}{y(\varepsilon)} \right| = \frac{t^2}{\varepsilon^2}.$$

However, there is

$$\lim_{\varepsilon \to 0^+} \left| \frac{y(t)}{y(\varepsilon)} \right| = \left| \frac{y(t)}{y_0} \right|,$$

while $(t/\varepsilon)^2$ does not have limit as $\varepsilon \to 0^+$.

Hence, the solution $y(t)$ cannot exist.

As for the case $t_0 = y_0 = 0$, it should be clear that the function $y(t) = Ct^2$ is a solution defined on $\mathbb{R}$, for every $C \in \mathbb{R}$. We can derive it in the following way:

Formally, we separate variables,

$$\frac{dy}{y} = 2\frac{dt}{t}$$

and we find antiderivatives for each side, which we can write as

$$\ln|y| = \ln t^2 + C_1$$

for any $C_1 \in \mathbb{R}$, or

$$\ln|y| = \ln\left(Ct^2\right),$$

for $C = e^{C_1} > 0$. Therefore,

$$|y| = Ct^2.$$

If $y(t) < 0$ for some t, then $y(t) = -Ct^2$ and then, $y(t) = -Ct^2$ for all t. If $y(t) > 0$ for some t, we have $y(t) = Ct^2$ so, $y(t) = Ct^2$ for every t. All in all, the functions $y(t) = Ct^2$, for any $C \in \mathbb{R}$, are solutions of the problem.

Finally, let us mention that, if $t_0 \neq 0$ and $y_0 \neq 0$, the function $f(t, y)$ fulfills, near (t_0, y_0), the assumptions of Theorem 2, the Cauchy-Lipschitz's theorem. Solving for C in the equation

$$y_0 = Ct_0^2,$$

we arrive at the unique solution

$$y(t) = y_0\frac{t^2}{t_0^2},$$

which is actually defined for all $t \in \mathbb{R}$.

Thus, this first example proves that continuity of the function f is generally necessary for the existence of solutions.

Example 32. *Let us consider the following situation: If $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside's function, defined as*

$$H(t) = \begin{cases} 1 & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0, \end{cases}$$

the function

$$y(t) = \begin{cases} t & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t \leq 0 \end{cases}$$

solves the equation

$$y' = H(t)$$

at every $t \neq 0$. *Since the function* $y(t)$ *is not differentiable at zero, we cannot say that* $y(t)$ *is a solution to the problem*

$$\begin{cases} y' = H(t) \\ y(0) = 0 \end{cases} \tag{28}$$

near zero. Of course, the difficulty appears because the right-hand side of the equation, $H(t)$*, is not continuous at zero. Thus, strictly speaking, problem (28) does not have a solution in the sense of Definition 1, what we may call the classical sense.*

The same situation presents itself if we take, as right-hand side for the equation, an increasing, or decreasing, function $f(t)$. *Such a function has limit, equal to the value of the function, at every point in* $\mathbb{R}\backslash C$*, where* C *is some countable set (see, for instance, [20], p. 103). For a related example, see ([16], p. 281, Remark 10.4.6).*

Example 32 shows that we might encounter difficulties with the very meaning of the word "solution", when we do not assume that the function f is continuous. It also shows that the continuity of the function f is generally necessary, for the existence of solutions in the classical sense. However, Example 32 also suggests how we might be able to avoid these difficulties altogether: We need to interpret the word "solution" differently, allowing for the removal of certain points, and we need to identify alternative assumptions on the function f.

This plan was carried out, with remarkable success, by Constantin Carathéodory, in 1918 [11]. At the center of Carathéodory's results is the following idea:

We know that the continuity of $f(t, y)$ guarantees that the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{29}$$

is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds. \tag{30}$$

However, (30) still has a meaning under more general conditions on f, provided that we do not expect $y(t)$ to have a continuous derivative. The following two definitions clarify this idea.

Definition 33. *([11], p. 665) Given an open set* $U \subseteq \mathbb{R}^2$*, a function* $f : U \to \mathbb{R}$ *satisfies Carathéodory's conditions on* U *if for each* $(t_0, y_0) \in U$ *and each rectangle* $R \subset U$,

$$R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\},$$

1. *the function* $y \to f(t, y)$ *is continuous, for almost all t with* $|t - t_0| \leq a$ *in the sense of the Lebesgue measure, and*
2. *the function* $t \to f(t, y)$ *is Lebesgue measurable for each* y *with* $|y - y_0| \leq b$.

Definition 34. *A function $y : I \to \mathbb{R}$, where I is an interval containing t_0, is a solution of the problem (29) in an extended sense if*

1. *y is absolutely continuous,*
2. *$(t, y(t)) \in U$ for $t \in I$, that is y is admissible,*
3. *$y'(t) = f(t, y(t))$ for almost all $t \in I$, and*
4. *$y(t_0) = y_0$.*

Remark 35. *When the function f is continuous, classical solutions and solutions in an extended sense are the same. Indeed, it should be clear that any classical solution of the problem is trivially a solution in an extended sense. Conversely, if y is a solution in an extended sense, (30) implies that the function*

$$t \to y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

has a continuous derivative. Therefore, y' exists, and it is continuous, throughout the interval where y is defined. That is to say, y is a classical solution.

Frequently, a function satisfying Carathéodory's conditions is called a Carathéodory's function.

Theorem 36. *Let $f : U \to \mathbb{R}$ be a function that satisfies Carathéodory's conditions. Moreover, assume that for each rectangle $R \subset U$ there is a Lebesgue integrable function $g : I \to [0, \infty)$, where I is the interval $|t - t_0| \leq a$, so that*

$$|f(t, y)| \leq g(t)$$

for all $(t, y) \in R$.

Then, the initial value problem (29) has a solution in the extended sense, in some interval containing t_0.

Remark 37. *There are several proofs of Theorem 36. In ([25], p. 28, Theorem 1.1), the proof resembles the third proof of Peano's theorem mentioned in Remark 9. Indeed, it uses the Schauder's fixed point theorem ([25], p. 10) for the operator*

$$T(y) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

defined in a suitable Banach space of continuous functions. Another proof (see [13], p. 43, Theorem 1.1), applies Ascoli's theorem to a sequence of specific approximating functions, defined using the integral equation (30). Actually, Theorem 36 and both proofs, are formulated for a system of differential equations.

In the one-dimensional case, Earl A. Coddington's and Norman Levinson's classical book [13] includes, among other related things, a proof for the existence of the maximum solution and the minimum solution ([13], p. 45, Theorem 1.2) that we defined in Remark 11.

Theorem 38. *Let $f : U \to \mathbb{R}$ be a function that satisfies Carathéodory's conditions. Moreover, we assume that, for each rectangle $R \subset U$, the function f satisfies the following two conditions:*

1. *There is a Lebesgue integrable function $g : I \to [0, \infty)$, where I is the interval $|t - t_0| \leq a$, so that*
$$|f(t, y)| \leq g(t)$$
for all $(t, y) \in R$.
2. *There is a Lebesgue integrable function $k : I \to [0, \infty)$, where I is the interval $|t - t_0| \leq a$, so that*
$$|f(t, y_1) - f(t, y_2)| \leq k(t)\,|y_1 - y_2|$$
for all $(t, y_1), (t, y_2) \in R$.

Then, the initial value problem (29) has one, and only one, solution in the extended sense, in some interval containing t_0.

Remark 39. *Theorem 38 can be proved using the contraction mapping principle applied to the operator*
$$y \to T(y)(t) = \int_{t_0}^{t+t_0} f(s, y(s - t_0) + y_0)\, ds$$
defined on a suitable space of continuous functions (see [25], p. 30, Theorem 5.3, and p. 18, Theorem 3.1).

As with Theorem 36, there is a formulation of Theorem 38 for systems of equations and, in particular, for systems of linear equations ([25], p. 30).

Concerning the continuation of solutions in the extended sense, we mention the following result:

Theorem 40. *([25], p.29, Theorem 5.2) Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be open and let $f : U \to \mathbb{R}$ be a function satisfying Carathéodory's conditions. If y is a solution of (29) defined on some interval I containing t_0, there is a continuation of y to a maximal interval of existence. Furthermore, if (a, b) is a maximal interval of existence of y, then either $y(t) \to \infty$ or $y(t)$ tends to the boundary of U, as $t \to a^+$ and $t \to b^-$.*

Remark 41. *Given a Carathéodory's function f, the application*
$$y(t) \to f(t, y(t))$$
defines formally an operator N_f, called Nemytskiĭ's operator, introduced by Viktor V. Nemytskiĭ [40] and studied by A. Mark Krasnosel'skiĭ [33] and Mordukhaĭ M. Vaĭnberg ([53], [54]), Nemytskiĭ's doctoral student at Moscow State University, among others. Roughly speaking, N_f is a variable coefficient composition operator. It has interesting continuity properties on many functional spaces (see, for instance, [18], Chapters 6 and

7). The operator N_f has been extended to scalar measures as well as to vector measures, and these extensions have been used to formulate and solve initial value problems ([3], [4]). For the role played by the Nemytskiĭ's operator in numerous applications, see, for instance, the references in [3] and [4].

Remark 42. *Functions $f(t, y)$ satisfying Carathéodory's conditions are not generally continuous on t and y simultaneously. Nevertheless, there is a simultaneous continuity property lurking in the background. Indeed, in the spirit of the well known theorem in measure theory proved by Nikolai N. Luzin ([36], p. 65), Vaĭnberg proved ([54], p. 148, Theorem 18.2) the following result:*

Let $f : R \to \mathbb{R}$ be a function, where $R \subset \mathbb{R} \times \mathbb{R}$ is the rectangle $|t - t_0| \leq a, |y - y_0| \leq b$. Then, the following statements are equivalent:

1. *The function f satisfies Carathéodory's conditions.*
2. *For each $\varepsilon > 0$ there is a closed set $E \subseteq R$ so that the Lebesgue measure of $R \backslash E$ is $\leq \varepsilon$ and f restricted to E is continuous.*

Remark 43. *The two articles [24] argue lucidly, using applications to control theory and to game theory, the need to consider various generalized notions of solution for a differential problem. These generalized notions include, among others, the extended sense described in Definition 34. For more on the subject, we mention [8], [9], [10] and the references therein.*

6 Banach scales: Definitions and examples

We begin with the following definition, due to Lyev V. Ovsjannikov [42]:

Definition 44. *A family $\left\{X_\rho, \|\cdot\|_\rho\right\}_{\rho>0}$ of Banach spaces is called a Banach scale if, for $\rho_1 > \rho_2 > 0$,*

1. *X_{ρ_1} is a linear subspace of X_{ρ_2} and*
2. *the embedding $X_{\rho_1} \to X_{\rho_2}$ is non-expansive, that is, $\|x\|_{\rho_2} \leq \|x\|_{\rho_1}$ for every $x \in X_{\rho_1}$.*

Example 45. *If $B_\rho \subset \mathbb{R}^n$ denotes the closed ball centered at zero with radius $\rho > 0$, let*

$$\mathcal{C}_\rho = \{f : B_\rho \to \mathbb{R} : f \text{ is continuous}\}.$$

The linear space $\mathcal{C}_\rho$ becomes a Banach space with the sup norm $\|\cdot\|_{\mathcal{C}_\rho}$. Moreover, for $\rho_1 > \rho_2 > 0$, $\mathcal{C}_{\rho_1}$ is a linear subspace of $\mathcal{C}_{\rho_2}$ and $\|f\|_{\mathcal{C}_{\rho_2}} \leq \|f\|_{\mathcal{C}_{\rho_1}}$ for every $f \in \mathcal{C}_{\rho_1}$.

One of the main examples of a Banach scale concerns real analytic functions, but before getting into that, it will be convenient to review a few notations and some background material.

We denote $\mathbb{N}$ and $\mathbb{N}^n$ the space of non-negative integers and the n-tuples of non-negative integers, respectively.

If α and β belong to $\mathbb{N}^n$, $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$ for all j. The notation $\alpha < \beta$, $\alpha \geq \beta$, etc. should then be clear.

Given α in $\mathbb{N}^n$, we write

$$|\alpha| \stackrel{def}{=} \alpha_1 + \cdots + \alpha_n$$

and

$$\alpha! \stackrel{def}{=} \alpha_1! \cdots \alpha_n!,$$

while for $x \in \mathbb{R}^n$, $\|x\|$ is the euclidean norm,

$$||x|| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}.$$

If $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$,

$$x^\alpha \stackrel{def}{=} x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

When $\alpha \in \mathbb{N}^n$, ∂^α is the partial derivative

$$\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

If $\alpha = 0$, we interpret ∂^0 as no derivative. If $|\alpha| = 1$, $\partial^\alpha = \partial_{x_j}^1 = \partial_{x_j}$ for some $1 \leq j \leq n$. For brevity, we will write ∂_j instead of ∂_{x_j}.

Given $f : U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^n$ is open, we say that $f \in C^\infty(U)$ if $\partial^\alpha f$ exists and is continuous, on U, for every $\alpha \in \mathbb{N}^n$.

Definition 46. *A function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$, is real analytic if for each $x_0 \in U$ fixed, f is representable, near x_0, by a power series. That is,*

$$f(x) = \sum_{\alpha \geq 0} a_\alpha (x - x_0)^\alpha, \tag{31}$$

for x in a ball $|x - x_0| < r$ contained in U for some $r > 0$. Since the convergence is actually absolute, the series converges independently of the order of summation.

Remark 47. *As a consequence of the representation (31), $f \in C^\infty(U)$ and $a_\alpha = \frac{(\partial^\alpha f)(x_0)}{\alpha!}$ (see, for instance, [32]).*

There are several characterizations (see, for instance, [38], [30], [32], and the references therein), some of a more analytic nature, others more geometric, of real analytic functions $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$. We state, without proof, the following characterization ([30], see also [32]), which suits well our purpose of illustrating the notion of Banach scale.

Theorem 48. *Given a function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$, the following statements are equivalent:*

1. *f is real analytic,*
2. *$f \in C^{\infty}(U)$ and for each $K \subset U$ convex and compact, there are constants $M = M_K > 0$ and $C = C_K > 0$ so that*

$$\sup_{x \in K} |(\partial^{\alpha} f)(x)| \leq MC^{|\alpha|}\alpha! \tag{32}$$

for all n-tuples $\alpha \in \mathbb{N}^n$.

Let us mention that the proof of 1) $\Rightarrow$ 2) uses the n-dimensional version of the Cauchy integral formula. The proof of 2) $\Rightarrow$ 1) uses (32) to show that the n-th remainder in the Taylor expansion of f about $x_0 \in U$, goes to zero near x_0, as $n \to \infty$.

Example 49. *Given $U \subseteq \mathbb{R}^n$ open and given $\rho > 0$, let*

$$\mathcal{A}_{\rho} = \left\{ f \in C^{\infty}(U) : \sup_{x \in U, \alpha \in \mathbb{N}^n} \frac{\rho^{|\alpha|}}{\alpha!} |(\partial^{\alpha} f)(x)| < \infty \right\}.$$

According to Theorem 48, the linear space $\mathcal{A}_{\rho}$ consists of functions that are real analytic on U. Furthermore, the uniform estimate on $\partial^{\alpha} f$ readily implies that they can be extended, as analytic functions, to an open subset of the complex space $\mathbb{C}^n$ containing the closure of U.

We consider on $\mathcal{A}_{\rho}$ the norm

$$\|f\|_{\mathcal{A}_{\rho}} = \sup_{x \in U, \alpha \in \mathbb{N}^n} \frac{\rho^{|\alpha|}}{\alpha!} |(\partial^{\alpha} f)(x)|.$$

With this norm, $\mathcal{A}_{\rho}$ becomes a Banach space. Moreover, it should be clear that, for $\rho_1 > \rho_2 > 0$, $\mathcal{A}_{\rho_1}$ is a linear subspace of $\mathcal{A}_{\rho_2}$ and $\|f\|_{\mathcal{A}_{\rho_2}} \leq \|f\|_{\mathcal{A}_{\rho_1}}$ for every $f \in \mathcal{A}_{\rho_1}$. Hence, the family $\{\mathcal{A}_{\rho}\}_{\rho>0}$ thus defined, is a Banach scale.

Definition 50. *[42] Let $\{X_{\rho}\}_{\rho>0}$ be a Banach scale. An operator $A : \bigcup_{\rho>0} X_{\rho} \to \bigcup_{\rho>0} X_{\rho}$ acts on the scale if, for each $0 < \rho_2 < \rho_1$ fixed, A is a linear and continuous operator from X_{ρ_1} into X_{ρ_2}.*

Remark 51. *If an operator A acts on a Banach scale $\{X_{\rho}\}_{\rho>0}$, according to 1) and 2) in Definition 44, the norm $N(\rho_1, \rho_2)$ of the operator $A : X_{\rho_1} \to X_{\rho_2}$ is non-increasing as a function of ρ_1 while ρ_2 remains fixed, and it is non-decreasing as a function of ρ_2 while ρ_1 remains fixed.*

Definition 52. *If an operator A acts on a Banach scale $\{X_{\rho}\}_{\rho>0}$, the operator is bounded on the scale when there is a constant $N > 0$ so that*

$$N(\rho_1, \rho_2) \leq N \tag{33}$$

for every $\rho_1, \rho_2 > 0$ with $\rho_2 < \rho_1$.

Example 53. *Given a continuous function $\theta : \mathbb{R}^n \to \mathbb{R}$, the multiplication operator $M_\theta : \bigcup_{\rho>0} \mathcal{C}_\rho \to \bigcup_{\rho>0} \mathcal{C}_\rho$ defined as $M(f) = \theta f$ acts on the scale $\{\mathcal{C}_\rho\}_{\rho>0}$. Indeed, if $0 < \rho_2 < \rho_1$ fixed, it should be clear that M_θ is a linear operator from $\mathcal{C}_{\rho_1} \to \mathcal{C}_{\rho_2}$. Moreover,*

$$\sup_{x\in B_{\rho_2}} |(\theta f)(x)| \le \sup_{x\in B_{\rho_2}} |\theta(x)| \sup_{x\in B_{\rho_1}} |f(x)|,$$

so $M_\theta : \mathcal{C}_{\rho_1} \to \mathcal{C}_{\rho_2}$ is continuous and

$$N(\rho_1, \rho_2) \le \sup_{x\in B_{\rho_2}} |\theta(x)|.$$

If the function θ is zero outside a compact set K,

$$\sup_{0<\rho_2<\rho_1} N(\rho_1, \rho_2) \le \sup_{x\in K} |\theta(x)|.$$

That is, M_θ is bounded on the scale.

Definition 54. *If an operator A acts on a Banach scale $\{X_\rho\}_{\rho>0}$, the operator is singular on the scale if there is a constant $N > 0$ so that*

$$(\rho_1 - \rho_2) N(\rho_1, \rho_2) \le N \tag{34}$$

for every $\rho_1, \rho_2 > 0$ with $\rho_2 < \rho_1$.

Remark 55. *For an operator that is singular on a Banach scale,*

$$\inf_{\rho_1>\rho_2>0} N(\rho_1, \rho_2) = 0.$$

Indeed, if there is $M > 0$ so that $N(\rho_1, \rho_2) \ge M$ for all $\rho_1, \rho_2 > 0$ with $\rho_2 < \rho_1$,

$$0 < \rho_1 - \rho_2 \le \frac{N}{M}$$

for all $\rho_1, \rho_2 > 0$ with $\rho_2 < \rho_1$, which is not possible.

The following example is mentioned in [42] without proof.

Example 56. *The partial derivative ∂_j is singular on the scale $\{\mathcal{A}_\rho\}_{\rho>0}$.*

Indeed, it should be clear that

$$\partial_j : \bigcup_{\rho>0} \mathcal{A}_\rho \to \bigcup_{\rho>0} \mathcal{A}_\rho.$$

Moreover, for $0 < \rho_2 < \rho_1$ fixed, $f \in \mathcal{A}_{\rho_1}$, $\alpha \in \mathbb{N}^n$, and $x \in U$,

$$\frac{\rho_2^{|\alpha|}}{\alpha!} |(\partial^\alpha (\partial_j f))(x)| = \left(\frac{\rho_2}{\rho_1}\right)^{|\alpha|} \frac{(\alpha_j + 1)}{\rho_1} \frac{\rho_1^{|\alpha|+1}}{(\alpha + e^j)!} \left|\left(\partial^{\alpha+e^j} f\right)(x)\right|,$$

where

$$e_k^j = \begin{cases} 1 & \textit{if} \quad k = j \\ 0 & \textit{if} \quad k \neq j \,. \end{cases}$$

Since

$$\left(\frac{\rho_2}{\rho_1}\right)^{|\alpha|} (\alpha_j + 1) \leq \left(\frac{\rho_2}{\rho_1}\right)^{|\alpha|} (|\alpha| + 1) \underset{|\alpha| \to \infty}{\to} 0,$$

there is a constant $C = C(\rho_1, \rho_2) > 0$ *such that*

$$\begin{aligned} \|\partial_j f\|_{\mathcal{A}_{\rho_2}} &= \sup_{x \in U, \alpha \in \mathbb{N}^n} \frac{\rho_2^{|\alpha|}}{\alpha!} \left|\left(\partial^\alpha \left(\partial_j f\right)\right)(x)\right| \leq C \sup_{x \in U, \beta \in \mathbb{N}^n} \frac{\rho_1^{|\beta|}}{\beta!} \left|\left(\partial^\beta f\right)(x)\right| \\ &= C \|f\|_{\mathcal{A}_{\rho_1}} . \end{aligned}$$

Therefore, the operator ∂_j *acts on the scale* $\{\mathcal{A}_\rho\}_{\rho > 0}$.

Let us prove that the partial derivative operator ∂_j *satisfies (34), by finding an appropriate bound for* $\left(\frac{\rho_2}{\rho_1}\right)^{|\alpha|} \frac{(\alpha_j+1)}{\rho_1}$.

$$\begin{aligned} \sup_{\alpha \in \mathbb{N}^n} \left[\left(\frac{\rho_2}{\rho_1}\right)^{|\alpha|} \frac{(\alpha_j + 1)}{\rho_1}\right] &= \sup_{\alpha \in \mathbb{N}^n} \left[\left(\frac{\rho_2}{\rho_1}\right)^{|\alpha|+1} \frac{(\alpha_j + 1)}{\rho_2}\right] \\ &\underset{(i)}{=} \sup_{\alpha \in \mathbb{N}^n} \left[\left(\frac{\rho_2}{\rho_1}\right)^{|\alpha'|} \left(\frac{\rho_2}{\rho_1}\right)^{\alpha_j+1} \frac{(\alpha_j + 1)}{\rho_2}\right] \\ &\leq \sup_{k=1,2,\ldots} \left[\frac{k}{\rho_2} \left(\frac{\rho_2}{\rho_1}\right)^k\right], \end{aligned}$$

where, on the right-hand side of (i), $\alpha' \in \mathbb{N}^{n-1}$ *is* α *with the j-th coordinate removed, if* $n > 1$.

Let $\frac{\rho_2}{\rho_1} = a$, *and let us consider the function* $g(s) = sa^s$, *defined on* $[1, \infty)$.

If $a \leq \frac{1}{e}$, *the methods of Calculus show that* $g'(s) \leq 0$ *for* $s \in [1, \infty)$. *Therefore,* $g(s) \leq g(1) = a$, *or*

$$\sup_{k=1,2,\ldots} \left[\frac{k}{\rho_2} \left(\frac{\rho_2}{\rho_1}\right)^k\right] \leq \frac{1}{\rho_1} = \frac{1}{\rho_1} \frac{\rho_1 - \rho_2}{\rho_1 - \rho_2} = \frac{1 - \frac{\rho_2}{\rho_1}}{\rho_1 - \rho_2} \leq \frac{1}{\rho_1 - \rho_2}.$$

When $1 > a > \frac{1}{e}$, *the function* g *has a local maximum at* $-\frac{1}{\ln a} \in (1, \infty)$ *and* $g\left(-\frac{1}{\ln a}\right) = -\frac{1}{e \ln a}$.

Since g *increases on* $\left[1, -\frac{1}{\ln a}\right]$, *we have* $g\left(-\frac{1}{\ln a}\right) \geq g(1)$. *Moreover, L'Höpital's rule tells us that there is*

$$\lim_{s \to \infty} g(s) = 0.$$

Hence, g has a global maximum at $-\frac{1}{\ln a}$, when $1 > a > \frac{1}{e}$, and as a consequence,

$$\sup_{k=1,2,\ldots}\left[\frac{k}{\rho_2}\left(\frac{\rho_2}{\rho_1}\right)^k\right] \le -\frac{1}{e\rho_2 \ln\frac{\rho_2}{\rho_1}} = -\frac{1}{e}\frac{1-\frac{\rho_2}{\rho_1}}{\ln\frac{\rho_2}{\rho_1}}\frac{1}{\rho_1-\rho_2}.$$

Now, using L'Höpital's rule again, there is

$$\lim_{\frac{\rho_2}{\rho_1}\to 1^-} -\frac{1}{e}\frac{1-\frac{\rho_2}{\rho_1}}{\ln\frac{\rho_2}{\rho_1}} = \frac{1}{e}.$$

We conclude that there is a constant $C > 0$ such that

$$-\frac{1}{e}\frac{1-\frac{\rho_2}{\rho_1}}{\ln\frac{\rho_2}{\rho_1}} \le C$$

when $1 > \frac{\rho_2}{\rho_1} > \frac{1}{e}$.

So, the operator ∂_j is, indeed, singular on the scale $\{\mathcal{A}_\rho\}_{\rho>0}$.

In [42], Ovsjannikov solved an abstract initial value problem, for a class of operators that are singular on a Banach scale. Although the result, known as Ovsjannikov's theorem, is stated without proof, Ovsjannikov remarks that it is proved using successive approximations and a non-trivial estimate that is a power. Accordingly, in the next section we present a proof that mimics the proof of Theorem 14, with appropriate modifications.

7 Ovsjannikov's theorem

Let $\{X_\rho\}_{\rho>0}$ be a Banach scale. In what follows, if $0 < \rho_2 < \rho_1$, $L(X_{\rho_1}, X_{\rho_2})$ will denote the Banach space of linear and continuous operators from X_{ρ_1} into X_{ρ_2} with the operator norm. Given an interval I, we consider an operator $A(t)$, $t \in I$, singular on the scale in such a manner that $A : I \to L(X_{\rho_1}, X_{\rho_2})$ is continuous for each $0 < \rho_2 < \rho_1$ and there is $N > 0$ such that

$$(\rho_1 - \rho_2)\,\|A(t)\|_{L(X_{\rho_1}, X_{\rho_2})} \le N, \tag{35}$$

for every $t \in I$ and all $\rho_1, \rho_2 > 0$ with $\rho_2 < \rho_1$.

Furthermore, given $\rho_0 > 0$, we fix a function $f : I \to X_{\rho_0}$ that is continuous, and we also fix $t_0 \in I$ and $y_0 \in X_{\rho_0}$.

Then, we can state Ovsjannikov's theorem.

Theorem 57. *Under the conditions above, if we fix $0 < \rho < \rho_0$, there is an interval $J = J(\rho) \subseteq I$ containing t_0 and a continuously differentiable function $y : J \to X_\rho$ that solves the problem*

$$\begin{cases} y' = A(t)(y) + f(t) \\ y(t_0) = y_0 \end{cases} \tag{36}$$

for $t \in J$.

Moreover, the solution is unique near t_0.

Proof. Let us begin by proving local existence.

We fix ρ' such that $0 < \rho < \rho' < \rho_0$. Given $k \geq 1$, we also fix positive numbers $\varepsilon_1, \ldots, \varepsilon_k$ so that

$$\varepsilon_1 = \cdots = \varepsilon_k = \frac{\rho_0 - \rho'}{k}.$$

Therefore,

$$\begin{aligned} \rho' &< \rho' + \varepsilon_1 < \cdots < \rho' + \varepsilon_1 + \cdots + \varepsilon_{k-1} \\ &< \rho' + \varepsilon_1 + \cdots + \varepsilon_k = \rho' + \frac{k}{k}(\rho_0 - \rho') = \rho_0. \end{aligned}$$

Let

$$y_0(t) = y_0 + \int_{t_0}^{t} f(s)\, ds,$$

for $t \in I$.

Then, the function y_0 is continuously differentiable from I into X_{ρ_0}. We claim that $A(s)(y_0(s))$ is continuous from I into $X_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}}$. Indeed, if $\{s_j\}_{j\geq 1}$ is any sequence in I converging to $s \in I$,

$$\begin{aligned} & \left\| A(s_j)(y_0(s_j)) - A(s)(y_0(s)) \right\|_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}} \\ \leq\; & \underbrace{\left\| A(s_j) \right\|_{L\left(X_{\rho_0}, X_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}}\right)}}_{(i)} \underbrace{\left\| y_0(s_j) - y_0(s) \right\|_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}}}_{(ii)} \\ & \underbrace{+\left\| A(s_j) - A(s) \right\|_{L\left(X_{\rho_0}, X_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}}\right)}}_{(iii)} \left\| y_0(s) \right\|_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}}. \end{aligned}$$

By hypothesis, (ii) and (iii) converge to zero as $j \to \infty$, while (i) is bounded. Therefore, the function

$$y_1(t) = y_0 + \int_{t_0}^{t} A(s)(y_0(s))\, ds + \int_{t_0}^{t} f(s)\, ds$$

is continuously differentiable from I into $X_{\rho'+\varepsilon_1+\cdots+\varepsilon_{k-1}}$.

Proceeding in this fashion, we arrive at the function

$$y_k(t) = y_0 + \int_{t_0}^{t} A(s)(y_{k-1}(s))\, ds + \int_{t_0}^{t} f(s)\, ds, \tag{37}$$

which is continuously differentiable from I into $X_{\rho'}$. In particular, the sequence $\{y_k\}_{k\geq 1}$ consists of functions that are continuous from I into $X_{\rho'}$.

We assume now that I is a finite interval of the form $[t_0, b]$ for some $b > t_0$.

$$
\begin{aligned}
\|y_k(t) - y_{k-1}(t)\|_{\rho'} &= \left\| \int_{t_0}^{t} A(s_1)\left(y_{k-1}(s_1) - y_{k-2}(s_1)\right) ds_1 \right\|_{\rho'} \\
&\leq \frac{N}{\varepsilon_1} \int_{t_0}^{t} \|y_{k-1}(s_1) - y_{k-2}(s_1)\|_{\rho'+\varepsilon_1} ds_1 \\
&= \frac{N}{\varepsilon_1} \int_{t_0}^{t} ds_1 \left\| \int_{t_0}^{s_1} A(s_2)\left(y_{k-2}(s_2) - y_{k-3}(s_2)\right) ds_2 \right\|_{\rho'+\varepsilon_1} \\
&\leq \frac{N^2}{\varepsilon_1 \varepsilon_2} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} \|y_{k-2}(s_2) - y_{k-3}(s_2)\|_{\rho'+\varepsilon_1+\varepsilon_2} ds_2
\end{aligned}
$$

for $t \in I$ and $k \geq 2$ in $\mathbb{N}$, where N is the constant in (35).

Continuing in this manner, we will have

$$
\begin{aligned}
\cdots &\leq \frac{N^{k-1}}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k-1}} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \cdots \\
&\quad \cdots \int_{t_0}^{s_{k-2}} \|y_1(s_{k-1}) - y_0(s_{k-1})\|_{\rho'+\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{k-1}} ds_{k-1} \\
&= \frac{N^{k-1}}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k-1}} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \cdots \\
&\quad \cdots \int_{t_0}^{s_{k-2}} ds_{k-1} \left\| \int_{t_0}^{s_{k-1}} A(s_k)\left(y_0(s_k)\right) ds_k \right\|_{\rho'+\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{k-1}} \\
&\leq \frac{N^{k}}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k}} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \cdots \\
&\quad \cdots \int_{t_0}^{s_{k-2}} (s_{k-1} - t_0)\, ds_{k-1} \sup_{s_k \in I} \|y_0(s_k)\|_{\rho_0}.
\end{aligned}
$$

Let

$$
\begin{aligned}
\sup_{s_k \in I} \|y_0(s_k)\|_{\rho_0} &= \sup_{s_k \in I} \left\| y_0 + \int_{t_0}^{s_k} f(s)\, ds \right\|_{\rho_0} \\
&\leq \|y_0\|_{\rho_0} + \int_I \|f(s)\|_{\rho_0} ds = C.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|y_k(t) - y_{k-1}(t)\|_{\rho'} \\
&\leq C \frac{N^k}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \cdots \int_{t_0}^{s_{k-2}} (s_{k-1} - t_0)\, ds_{k-1} \\
&= C \frac{N^k}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k} \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \cdots \int_{t_0}^{s_{k-3}} \frac{(s_{k-2} - t_0)^2}{2} ds_{k-2}.
\end{aligned}
$$

After these successive integrations,

$$\|y_k(t) - y_{k-1}(t)\|_{\rho'} \le C \frac{N^k}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k} \frac{(t-t_0)^k}{k!}$$

or

$$\|y_k(t) - y_{k-1}(t)\|_{\rho'} \le C \frac{N^k k^k}{(\rho_0 - \rho')^k} \frac{(t-t_0)^k}{k!} \tag{38}$$

for $t \in I$. Likewise, when $I = [a, t_0]$ for some $a < t_0$,

$$\|y_k(t) - y_{k-1}(t)\|_{\rho'} \le C \frac{N^k k^k}{(\rho_0 - \rho')^k} \frac{(t_0 - t)^k}{k!}$$

for $t \in I$.

Finally,

$$\|y_k(t) - y_{k-1}(t)\|_{\rho'} \le C \frac{N^k k^k}{(\rho_0 - \rho')^k} \frac{|t-t_0|^k}{k!}$$

for $t \in I$.

Now, we write

$$y_k(t) = y_0(t) + \sum_{l=1}^{k} [y_l(t) - y_{l-1}(t)]$$

and

$$\begin{aligned} y_{k+m}(t) - y_k(t) &= \sum_{l=1}^{k+m} [y_l(t) - y_{l-1}(t)] - \sum_{l=1}^{k} [y_l(t) - y_{l-1}(t)] \\ &= \sum_{l=k+1}^{k+m} [y_l(t) - y_{l-1}(t)], \end{aligned}$$

for $k, m \ge 1$.

The estimate (38) implies that

$$\|y_{k+m}(t) - y_k(t)\|_{\rho'} \le \sum_{l=k+1}^{k+m} \|y_l(t) - y_{l-1}(t)\|_{\rho'} \le C \sum_{l=k+1}^{k+m} \frac{N^l l^l}{(\rho_0 - \rho')^l} \frac{|t-t_0|^l}{l!}.$$

If we apply the ratio test to the series with general term $\frac{N^l l^l}{(\rho_0-\rho')^l}\frac{|t-t_0|^l}{l!}$ we obtain,

$$\begin{aligned} & \frac{N^{l+1}(l+1)^{l+1}|t-t_0|^{l+1}}{(l+1)!(\rho_0-\rho')^{l+1}} \frac{l!(\rho_0-\rho')^l}{N^l l^l |t-t_0|^l} \\ = \quad & \frac{N}{\rho_0-\rho'}\left(1+\frac{1}{l}\right)^l |t-t_0| \underset{l\to\infty}{\to} \frac{eN}{\rho_0-\rho'}|t-t_0|. \end{aligned}$$

Therefore, the sequence $\{y_k(t)\}_{k\geq 1}$ converges in $X_{\rho'}$, for each $t\in I$ satisfying

$$|t-t_0| < \frac{\rho_0-\rho'}{eN}, \tag{39}$$

to a limit $y(t)$.

Moreover, the convergence is uniform when

$$\frac{eN}{\rho_0-\rho'}|t-t_0| \leq c,$$

for any $0<c<1$.

Let J_c be the interval containing t_0, determined by the conditions $t\in I$ and $|t-t_0| < c\frac{\rho_0-\rho'}{eN}$. Then, the function $y: J_c \to X_{\rho'}$ is continuous. Furthermore,

$$\sup_{s\in J}\|A(s)(y_{k-1}(s)-y(s))\|_\rho \leq \frac{N}{\rho'-\rho}\sup_{s\in J}\|y_{k-1}(s)-y(s)\|_{\rho'} \underset{k\to\infty}{\to} 0.$$

Therefore, we can take the limit in X_ρ, on both sides of (37), obtaining

$$y(t) = y_0 + \int_{t_0}^t A(s)(y(s))\,ds + \int_{t_0}^t f(s)\,ds$$

for $t\in J$ for any $0<c<1$. That is to say, the function $y(t)$ solves the problem (36) for each $t\in I$ satisfying (39).

To prove local uniqueness, let $y: I\to X_\rho$ be a solution, in the same interval $I=[t_0,b]$ for some $b>t_0$, of the initial value problem

$$\begin{cases} y' = A(t)(y) \\ y(t_0) = 0\,. \end{cases} \tag{40}$$

If we fix $0<\rho''<\rho$, we can write, in $X_{\rho''}$,

$$y(t) = \int_{t_0}^t A(s_1)\,y(s_1)\,ds_1.$$

For $k\geq 1$, we partition the interval $[\rho'',\rho]$ in k equal subintervals using the intermediate points

$$\rho_j = \rho'' + \frac{j}{k}(\rho-\rho'').$$

Then,

$$\begin{aligned} \|y(t)\|_{\rho''} &\leq \frac{kN}{\rho-\rho''}\int_{t_0}^{t}\|y(s_1)\|_{\rho_1}\,ds_1 \\ &= \frac{kN}{\rho-\rho'}\int_{t_0}^{t}ds_1\left\|\int_{t_0}^{s_1}A(s_2)\,y(s_2)\right\|_{\rho_1}ds_2 \\ &\leq \frac{k^2N^2}{(\rho-\rho')^2}\int_{t_0}^{t}ds_1\int_{t_0}^{s_1}\|y(s_2)\|_{\rho_2}\,ds_2. \end{aligned}$$

Continuing in this fashion as we did before, we end up with the estimate

$$\|y(t)\|_{\rho''} \leq \sup_{t\in I}\|y(t)\|_{\rho}\frac{(Nk(t-t_0))^k}{k!\,(\rho-\rho'')^k}$$

or, in general,

$$\|y(t)\|_{\rho''} \leq \sup_{t\in I}\|y(t)\|_{\rho}\frac{(Nk\,|t-t_0|)^k}{k!\,(\rho-\rho'')^k}$$

for $t\in I$ and $k=1,2,\ldots$.

The ratio test tells us that the series

$$\sum_{k\geq 1}\frac{(Nk\,|t-t_0|)^k}{k!\,(\rho-\rho'')^k}$$

converges when

$$\frac{eN\,|t-t_0|}{(\rho-\rho'')} < 1. \tag{41}$$

Therefore, there is

$$\lim_{k\to\infty}\frac{(Nk\,|t-t_0|)^k}{k!\,(\rho-\rho'')^k} = 0$$

and, as a consequence, the function $y(t)$ is zero for $t\in I$ satisfying condition (41).

This completes the proof of the theorem. □

Remark 58. *Conditions (39) and (41) will become the same if we pick ρ' and ρ'' in such a manner that $\rho'-\rho=\rho-\rho''$. Therefore, there is uniqueness in the same interval containing t_0, where we proved existence.*

Remark 59. *Let us observe that, in Theorem 57, we only assume that f is continuous because we quickly resort to working on a compact interval. If we choose to work with a possibly unbounded interval, it is natural to assume that the continuous function f is also integrable on I. Therefore, the difference is merely of a technical nature.*

In Theorem 57 we work with an operator $A(t)$ that is singular on the scale. This is a hypothesis specific to non-trivial Banach scales. However, if we choose the scale $\{X_\rho\}_{\rho>0}$

for which X_ρ is equal to X for a given Banach space X, and we assume that $A(t)$ is bounded on the scale, Theorem 14 follows from Theorem 57.

Let us observe that if the operator $A(t)$ satisfies (33) uniformly in $t \in I$, that is, if there is a constant $N > 0$ so that

$$\sup_{t\in I} \|A(t)\|_{L\left(X_{\rho_1}, X_{\rho_2}\right)} \leq N$$

for every $0 < \rho_2 < \rho_1$, then, the problem (36) has a solution defined on I. Indeed, it is enough to observe that the j-th step in the proof of Theorem 57) contributes a factor N, instead of a factor $\frac{N}{\varepsilon_j}$, yielding the estimate

$$\|y_k(t) - y_{k-1}(t)\|_{\rho'} \leq CN^k \frac{|t-t_0|^k}{k!}$$

for $t \in I$ and $k = 1, 2, \ldots$

Therefore,

$$\sum_{l=k+1}^{k+m} \|y_l(t) - y_{l-1}(t)\|_{\rho'} \leq C \sum_{l=k+1}^{k+m} N^l \frac{|t-t_0|^l}{l!},$$

which implies that the sequence $\{y_k(t)\}_{k\geq 0}$ converges in X_ρ, for each $t \in I$. Moreover, the convergence is uniform with respect to t, in any compact subinterval of I, $|t-t_0| \leq c$. Hence, the limit function is a solution of the initial value problem (36) in I.

Remark 60. *Theorem 57 was extended in [15] to the initial value problem*

$$\begin{cases} y' = A(t)(y) + f(t,y) \\ y(t_0) = y_0 \end{cases}$$

with the added assumption that the function f is Lipschitz in y, uniformly with respect to t.

Several references (see, for instance, [23] and [34]), point out that the scales of Banach spaces defined by Ovsjannikov, are not related in any way to the scales appearing in connection with the theory of interpolation spaces. To be sure, in both cases, we have families of Banach spaces, on which certain operators act. However, in a Banach scale as defined by Ovsjannikov, given $0 < \rho_2 < \rho_1$, there is a non-expansive embedding from X_{ρ_1} into X_{ρ_2}. Moreover, the operators acting on Ovsjannikov's scales, are assumed to belong to $L(X_{\rho_1}, X_{\rho_2})$ for any $0 < \rho_2 < \rho_1$. This is not generally the case with interpolation scales. For example the family $\{L^p(\mathbb{R}^n)\}_{p\geq 1}$ is an interpolation scale frequently used in Harmonic Analysis, in connection with important classes of operators, such as Calderón-Zygmund operators. However, there is no embedding between these spaces. As for the operators acting on this scale, we put forth, as an example, the simple case of the Fourier transform.

Theorem 61. *(Hausdorff-Young) Given $1 \leq p \leq 2$ fixed, if $f \in L^p(\mathbb{R}^n)$, the Fourier transform $\widehat{f}$ is well defined and belongs to $L^q(\mathbb{R}^n)$, where q is the conjugate exponent of p, that is, $q = \frac{p-1}{p}$.*

Moreover, there is $C_p > 0$ so that

$$\left\|\widehat{f}\right\|_{L^q} \leq C_p \|f\|_{L^p}.$$

This result is optimal in the following sense:

Given $p > 2$, there is $f \in L^p(\mathbb{R}^n)$, so that there is no value of q for which we can say that $\widehat{f}$ is a function in $L^q(\mathbb{R}^n)$.

For more on the definition of the Fourier transform and on the Hausdorff-Young theorem see, for instance, ([1], pp. 122-123).

There are numerous applications, extensions, and variations, of the results discussed in this section. For instance, we cite [51], [52], [43], [47], [44], [19], and the many references therein.

As promised, we end with a brief overview of the notion of generic property, as it relates to local existence and local uniqueness of solutions for an initial value problem in one variable. We will state a few definitions and results, referring to the original sources for the details. Even in this limited form, we believe that the notion of generic property gives a perspective, from a theoretical viewpoint, to the results discussed in our exposition.

8 Convergence of successive approximations, local existence, and local uniqueness, as generic properties

We begin with a little bit of background material that, for our purposes, it will suffice to place in the context of a metric space M.

The following definitions are due to René-Louis Baire, who included them in his doctoral dissertation, published in 1899 in *Annali di Matematica Pura ed Applicata*.

Definition 62.
1. *A subset N of M is called nowhere-dense in M, if its closure $\overline{N}$ in M has empty interior.*
2. *A subset N of M is of first category in M if it can be written as a countable union of nowhere-dense sets.*
3. *A subset N of M is of second category in M, if it is not of first category in M.*

In Walter Rudin's words ([49], p. 42), "This terminology (due to Baire) is admittedly rather bland and unsuggestive. Meager and nonmeager have been used instead in some texts. But "category arguments" are so entrenched in the mathematical literature and are so well known that it seems pointless to insist on a change."

Example 63.
1. *The set $\mathbb{N}$ of non-negative integers is nowhere dense in $\mathbb{R}$.*
2. *Since each of the sets $\{q\}$, for q rational, is nowhere dense in $\mathbb{R}$, we conclude that the rational numbers form a subset of $\mathbb{R}$ of first category in $\mathbb{R}$ that, in addition, is dense in $\mathbb{R}$. The set of irrational numbers is of second category and dense, in $\mathbb{R}$.*

As we are working with metric spaces, we state the following result in its "historical" form:

Theorem 64. *(Baire's category theorem) (for the proof see, for instance, [7], p. 77, Theorem 6.1) A complete metric space is of second category in itself.*

Remark 65.
1. *By its construction, the Cantor set is of first category in $\mathbb{R}$. However, since it is a closed subset of $\mathbb{R}$, it is a complete metric subspace of $\mathbb{R}$ and, therefore, it is of second category in itself.*
2. *Likewise, $\mathbb{N}$ is of second category in itself.*
3. *The set of rational numbers, with the relative euclidean metric, is of first category in itself.*

In what follows, we will always consider sets that are subsets of a clearly defined complete metric space. Therefore, most of the time, we will just say "... is of first category", "... is nowhere dense", "... is of second category", etc.

Remark 66.
1. *The following equivalence justifies the name "nowhere dense": A subset N of M is nowhere dense if, and only if, for each $U \subseteq M$ open, $U \bigcap N$ is not dense in U.*
2. *It should be clear, by definition, that the empty set is of first category.*
3. *If M is a complete metric space and $N \subset M$ is of first category, the complement, $M \backslash N$, must be of second category. Otherwise, $M = N \bigcup (M \backslash N)$ could be written as the countable union of nowhere-dense sets, which is not possible.*

Definition 67. *The complement of a set of first category or meager, is called co-meager or residual set.*

Remark 68. *Let us observe that a set that is not meager is not necessarily co-meager. There are sets that are neither meager nor co-meager. For instance, as subsets of $\mathbb{R}$, the interval $[0, 1]$ is not meager, and its complement is not meager either.*

Pretty much every book on Functional Analysis has one or more sections dedicated to category, as it pertains, in its modern version, to topological spaces (see, for instance, ([49], Chapter 2).

Somewhat informally, we can say that category allows us to talk about "small" and "large" sets in a topological sense. To be sure, we can talk about the "size" of a set in other ways. One of them is within measure theory. For a very lucid discussion, we refer to Professor Terence Tao's blog [50].

The proofs of many fundamental theorems in the theory of topological linear spaces, in particular in the theory of Banach spaces, use category arguments in a crucial manner ([49], Chapter 2; [50]).

Intuitively, a generic property is a property that is "typical", in the sense that it holds much more often than not. Formally, the idea of a generic property can mean different things, depending on the context [58]. Therefore, it is not quite possible to give a "one-size-fits-all" definition. Soon, we will restrict ourselves to the context of ordinary differential equations. For now, we present two simple, but illustrative, examples:

Example 69. *Let $C\,[0,1]$ be the linear space of continuous functions $f:[0,1]\to\mathbb{R}$. It is a Banach space with the sup norm $\|\cdot\|_\infty$. Therefore, it is a complete metric space with the induced distance.*

Consider the subset R of $C\,[0,1]$ defined as

$$R=\{f\in C\,[0,1]: f \text{ has a finite right derivative at some } x\in[0,1]\}\,.$$

It is proved in ([7], Section 6.3) that R is of first category. Since $C\,[0,1]$ is of second category in itself by Baire's category theorem, the complement of R in $C\,[0,1]$ must be of second category, according to 3) in Remark 66. That is, the set of continuous functions that have a finite right derivative at some $x\in[0,1]$ is topologically "small", while its complement is topologically "large".

We say that not having a finite right derivative at any $x\in[0,1]$ is a generic property of the functions in $C\,[0,1]$.

By the way, the subset of $C\,[0,1]$ consisting of functions that are not differentiable at any $x\in[0,1]$ contains $C\,[0,1]\setminus R$ and, therefore, it is of second category, which implies that it is not empty.

In the 1830s, Bernard Bolzano devised explicit examples of continuous functions nowhere differentiable. However, the first published example, due to Karl Weierstrass, appeared in 1872 in page 97 of his Abhandlungen aus der Functionenlehre. *For a detailed examination of Weierstrass's construction, using Fourier series, see, for instance, [55] and the references therein. Godfrey H. Hardy conducted in [26] a thorough investigation of the properties of the Weierstrass's function.*

When these "pathological" functions began to make their appearance, they were met with fierce resistance by prominent mathematicians ([31], p. 42). For instance, Charles Hermite talked about "this dreadful plague of continuous nowhere differentiable functions" and referred to them as a "lamentable scourge". Jules Henri Poincaré had this to say: "In the last half century we have seen a rabble of functions arise whose only job, it seems, is to look as little as possible like decent and useful functions. No more continuity, or perhaps continuity but no derivatives." He goes on "Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they will never have any other use." Henri Lebesgue acknowledged having some difficulty in publishing an article which contained a nowhere differentiable function.

After all this backlash, it is ironical to arrive to a point where, among continuous functions, those that have a derivative somewhere are the rarity, in category terms.

Example 70. *Let* $L(C[0,1])$ *be the linear space consisting of linear and continuous operators* $T: C[0,1] \to C[0,1]$. *It is a Banach space with the operator norm. Let*

$$P=\{T : C[0,1] \to C[0,1] : T \text{ is the pointwise limit of a sequence in } L(C[0,1])\}.$$

It should be clear that $\mathcal{P}$ *consists of linear operators.*

Furthermore, as a consequence of ([49], p. 45, Theorem 2.7), $P \subseteq L(C[0,1])$. *Roughly speaking, we can say that the continuity of the pointwise limit is, trivially, a generic property, in this context.*

With these examples in mind, we attempt the following definition of generic property, which is suitable for our purpose:

Definition 71. *Let* M *be a complete metric space and let* $\mathcal{P}$ *be a property that makes sense to test on the elements of* M. *The property* $\mathcal{P}$ *is generic if the subset of* M *consisting of those elements for which* $\mathcal{P}$ *holds, is of second category in* M.

After this preparatory detour, we begin by looking at the convergence of successive approximations, as a generic property, in the sense of Definition 71.

In Remark 9, we mentioned that continuity of the right-hand side of the differential equation does not guarantee the convergence of the successive approximations. However, convergence turns out to be a generic property, as follows [57]:

Theorem 72. *Let* R *be the closed rectangle* $|t - t_0| \leq a$, $|y - y_0| \leq b$. *Let* M *be the set of continuous functions* $f : R \to \mathbb{R}$, *for which* $2a \sup_{(t,y)\in R} |f(t,y)| \leq b$. *The set* M *is a complete metric space with the sup distance* $\|\cdot\|_\infty$.

Then, there is a dense subset M^* *of* M *that is of second category in* M. *Moreover, for each* $f \in M^*$, *the successive approximations*

$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\, ds$$

converge uniformly to a unique function $y = y(t,f)$, *for* $|t - t_0| \leq a$.

Remark 73. *As Giovanni Vidossich observes in [57], Theorem 72 does not state that the initial value problem*

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \tag{42}$$

has a unique solution for every $f \in M^*$. *However, a much earlier result, published by Władysław Orlicz in 1932 in the* Bulletin de l'Academie Polonaise des Sciences, *shows that uniqueness of solutions is generic in a certain space of continuous and bounded functions, such as the space considered in the statement of Theorem 72.*

Both, Orlicz and Vidossich, state and prove their results for a system of equations. That is, they consider functions $f : R \to \mathbb{R}^n$, R *being the "rectangle"* $|t - t_0|$, $\|y - y_0\| \leq b$, *where* $\|\cdot\|$ *is the euclidean norm in* $\mathbb{R}^n$.

Peano's theorem tells us that the existence of local solutions is, trivially, a generic property.

A starting point in the proof of Theorem 72 is the fact that Lipschitz functions are dense in M (see [56], Theorem 2). However, there are many rather subtle and technical pieces, for which we refer to [57].

Let us recall that Example 19 showed that Peano's theorem does not extend generally to the case of Banach-space valued equations. Nevertheless, and quite surprisingly, not only the existence of local solutions, but also the convergence of successive approximations and the uniqueness of local solutions, are generic properties in a suitable space of continuous functions (see [35], [14], and the references therein).

Finally, for the problem

$$\begin{cases} y' = A(t)(y) + f(t, y) \\ y(t_0) = y_0, \end{cases}$$

with values in a Banach scale, mentioned in Remark 60, the existence of a unique solution is a generic property for f in a certain space of continuous functions [17].

Acknowledgement

I thank the anonymous referee for the comments on a first version of this article.

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Recibido en agosto de 2020. Aceptado para publicación en mayo de 2021.

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