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Topological degree methods for a Strongly nonlinear $p(x)$-elliptic problem

Métodos de grado topológico para un problema $p(x)$-elíptico fuertemente no lineal

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Abstract. This article is devoted to study the existence of weak solutions for the strongly nonlinear $p(x)$-elliptic problem

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda|u|^{q(x)-2}u + f(x,u,\nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Our technical approach is based on the recent Berkovits topological degree.

Key words and phrases. Strongly nonlinear elliptic problem, Generalized Lebesgue and Sobolev spaces, $p(x)$-Laplacian, Topological Degree.

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Resumen. Este artículo está dedicado a estudiar la existencia de soluciones débiles para el problema $p(x)$-elíptico fuertemente no lineal

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda|u|^{q(x)-2}u + f(x,u,\nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Nuestro enfoque técnico se basa en el reciente grado topológico de Berkovits.

Palabras y frases clave. Problema elíptico fuertemente no lineal, espacios generalizados de Lebesgue y Sobolev, $p(x)$-Laplaciano, grado topológico.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. The specific attention accorded to such kind of problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomenon which arise in elastic mechanics or electrorheological fluids (sometimes referred to as "smart fluids")(see [14, 19]). Many results have been obtained for this kind of problems, for instance we here cite [3, 4, 5, 8, 9, 10].

We consider the following nonlinear $p(x)$-elliptic problem

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda|u|^{q(x)-2}u + f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $-\Delta_{p(x)}(u) = -div(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p(\cdot), q(\cdot) \in C(\bar{\Omega})$ and λ is a real parameter. We assume also that $p(\cdot)$ is log-Hölder continuous function (in a sense to be precised in section 3 below) and $2 < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < \infty$.

For $\lambda = 0$ and f independent of ∇u, Fan and Zhang (in [10]) present several sufficient conditions for the existence of solutions for the problem. Their discussion is based on the theory of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. The same problem is studied by P.S. Iliaş (in [12]) who discusses sufficient conditions which allow to use variational and topological methods to prove the existence of weak solutions.

For $f \equiv 0$ and $p(\cdot) = q(\cdot)$, X. Fan and et. al. (in [18]) study the eigenvalues of the problem. The present some sufficient conditions for $\inf \Lambda = 0$ and for $\inf \Lambda > 0$, respectively where Λ is the set of eigenvalues.

R. Alsaedi (in [1]) establishes sufficient conditions for the existence of non-trivial weak solutions for the following problem:

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u + \mu|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The proofs combine the Ekeland variational principle, the mountain pass theorem and energy arguments.

In this paper, we will generalize these works, by proving, under conditions on the functions p and q and a suitable growth condition of f, the existence of weak solutions for the problem (1). Our technical approach is based on the recent Berkovits topological degree.

The paper is divided into four sections. In the second section, we introduce some classes of mappings of generalized (S_+) type and the recent Berkovits degree. In the third section, some basic properties of variable Lebesgue and Sobolev spaces and several important properties of $p(x)$-Laplacian operator are presented. Finally, in the fourth section, we give the assumptions and our main results concerning the weak solutions of problem (1).

2. Classes of mappings and topological degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle \cdot\,,\,\cdot \rangle$ and let Ω be a nonempty subset of X. The symbol $\rightarrow$ ($\rightharpoonup$) stands for strong (weak) convergence.

Let Y be a real Banach space. We recall that a mapping $F : \Omega \subset X \to Y$ is *bounded,* if it takes any bounded set into a bounded set. F is said to be *demicontinuous,* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$. F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+), if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$. F is said to be *quasimonotone* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$. We say that F has the property $(QM)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $limsup\langle Fu_n, y - y_n\rangle \geq 0$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \,|\, F \text{ is bounded, demicontinuous and satifies condition } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \,|\, F \text{ is bounded, demicontinuous and satifies condition } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \,|\, F \text{ is demicontinuous and satifies condition } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \,|\, G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\bar{G})$ is called an *essential inner map* to F.

Lemma 2.1. [2, Lemmas 2.2 and 2.4] *Suppose that $T \in \mathcal{F}_1(\bar{G})$ is continuous and $S : D_S \subset X^* \to X$ is demicontinuous such that $T(\bar{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statements are true:*

(i) If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\bar{G})$, where I denotes the identity operator.

(ii) If S is of class (S_+), then $SoT \in \mathcal{F}_T(\bar{G})$.

Definition 2.2. Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\bar{G})$ be continuous and let $F, S \in \mathcal{F}_T(\bar{G})$. The affine homotopy

$H : [0,1] \times \bar{G} \to X$ defined by

$$H(t,u) := (1-t)Fu + tSu \text{ for } (t,u) \in [0,1] \times \bar{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

***Remark* 2.3.** [2] The above affine homotopy satisfies condition $(S_+)_T$.

We introduce the topological degree for the class $\mathcal{F}_B(X)$ due to Berkovits [2].

Theorem 2.4. *There exists a unique degree function*

$$d : \{(F,G,h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_{T,B}(\bar{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties:

(1) (Existence) if $d(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.

(2) (Additivity) Let $F \in \mathcal{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have

$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$

(3) (Homotopy invariance) If $H : [0,1] \times \bar{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,.), G, h(t))$ is constant for all $t \in [0,1]$.

(4) (Normalization) For any $h \in G$, we have $d(I,G,h) = 1$.

3. Variable Lebesgue and Sobolev spaces and Properties of $p(x)-$Laplacian operator

In the sequel, we consider a natural number N and a bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$.

We introduce the setting of our problem with some auxiliary results of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [11, 7, 13] for more details.

Denote

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) | \inf_{x \in \bar{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ := max\{h(x), x \in \bar{\Omega}\}, h^- := min\{h(x), x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}$$

endowed with *Luxemburg norm*

$$|u|_{p(x)} = \inf\{\lambda > 0 / \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\}.$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega).$$

$(L^{p(x)}(\Omega), |\ .\ |_{p(x)})$ is a Banach space [7, Theorem 2.5], separable and reflexive [7, Corollary 2.7]. Its conjugate space is $L^{p'(x)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, Hölder inequality holds [7, Theorem 2.1]

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2}$$

Notice that if (u_n) and $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold true (see [11])

$$|u|_{p(x)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1 (= 1; > 1),$$

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{3}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{4}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{5}$$

From (3) and (4), we can deduce the inequalities

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{6}$$

$$\rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} + |u|_{p(x)}^{p^+}. \tag{7}$$

If $p_1, p_2 \in C_+(\bar{\Omega}), p_1(x) \leq p_2(x)$ for any $x \in \bar{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) / |\nabla u| \in L^{p(x)}(\Omega)\}.$$

It is a Banach space under the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(.)}(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\ \cdot\ ||$. If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e., there is a constant $\alpha > 0$ such that for every $x, y \in \Omega, x \neq y$ with $|x-y| \leq \frac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x-y|}, \tag{8}$$

then we have the Poincaré inequality (see [15, 16]), i.e., the exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{9}$$

In particular, the space $W_0^{1,p(.)}(\Omega)$ has a norm $|\ \cdot\ |$ given by

$$|u|_{1,p(x)} = |\nabla u|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

which is equivalent to $||\cdot||$. In addition, we have the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$(see [7]). The space $(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)})$ is a Banach space, separable and reflexive (see [11, 7]). The dual space of $W_0^{1,p(x)}(\Omega)$, denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|v|_{-1,p'(x)} = \inf\{|v_0|_{p'(x)} + \sum_{i=1}^{N} |v_i|_{p'(x)}\},$$

where the infinimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, \ldots, v_N) \in (L^{p'(x)}(\Omega))^N$.

Next, we discuss the $p(x)-$Laplacian operator

$$-\Delta_{p(x)} u := -div(|\nabla u|^{p(x)-2}\nabla u).$$

Consider the following functional:

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, \quad u \in W_0^{1,p(x)}(\Omega).$$

We know that (see [6]) $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and the $p(x)-$Laplacian operator is the derivative operator of J in the weak sense.

We denote $L = J' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, then

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Theorem 3.1. [6, Theorem 3.1]

(i) $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator;*

(ii) L *is a mapping of class* (S_+)*;*

(iii) L *is a homeomorphism.*

4. Assumptions and Main Results

In this section, we study the strongly nonlinear problem (1) based on the degree theory in Section 2, where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\bar{\Omega})$ satisfies the log-Hölder continuity condition (8), $q \in C_+(\bar{\Omega})$, $2 < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a real-valued function such that:

(f_1) f satisfies the Carathéodory condition, that is, $f(.,\eta,\zeta)$ is measurable on Ω for all $(\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x,.,.)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$.

(f_2) f has the growth condition

$$|f(x,\eta,\zeta)| \leq c(k(x) + |\eta|^{r(x)-1} + |\zeta|^{r(x)-1})$$

for a.e. $x \in \Omega$ and all $(\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^N$, where c is a positive constant, $k \in L^{p'(x)}(\Omega)$, $k(x) \geq 0$ and $r \in C_+(\bar{\Omega})$ with $2 < r^- \leq r(x) \leq r^+ < p^-$.

Definition 4.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega (\lambda|u|^{q(x)-2}u + f(x,u,\nabla u))v dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

***Remark* 4.2.** Note that $\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \langle Lu, v\rangle$ as defined in section 2, $\lambda|u|^{q(x)-2}u \in L^{p'(x)}(\Omega)$ and $f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$ and the given hypotheses about the exponents p, q and r and condition (f_2) because: $k \in L^{p'(x)}(\Omega)$, $\alpha(x) = (q(x)-1)p'(x) \in C_+(\bar{\Omega})$ with $\alpha(x) < p(x)$ and $\beta(x) = (r(x)-1)p'(x) \in C_+(\bar{\Omega})$ with $\beta(x) < p(x)$. Then, we can conclude by the continuous embedding, $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ and $L^{p(x)} \hookrightarrow L^{\alpha(x)}$.

Since $v \in L^{p(x)}(\Omega)$, we have $(\lambda|u|^{q(x)-2}u + f(x,u,\nabla u))v \in L^1(\Omega)$. Then, the integral $\int_\Omega (\lambda|u|^{q(x)-2}u + f(x,u,\nabla u))v dx$ exist.

Lemma 4.3. *Under assumptions (f_1) and (f_2), the operator $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ setting by*

$$\langle Su, v\rangle = -\int_\Omega (\lambda|u|^{q(x)-2}u + f(x,u,\nabla u))v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega)$$

is compact.

***Proof*.** **Step 1**

Let $\phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be the operator defined by

$$\phi u(x) := -\lambda|u(x)|^{q(x)-2}u(x) \text{ for } u \in W_0^{1,p(x)}(\Omega) \text{ and } x \in \Omega.$$

It is obvious that ϕ is continuous. We prove that ϕ is bounded.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have by inequalities (6) and (7) that

$$\begin{aligned}
|\phi u|_{p'(x)} &\leq \rho_{p'(x)}(\phi u) + 1 \\
&= \int_\Omega |\lambda |u|^{q(x)-1}|^{p'(x)}\, dx + 1 \\
&\leq (|\lambda|^{p'^-} + |\lambda|^{p'^+})\rho_{\alpha(x)}(u) + 1 \\
&\leq (|\lambda|^{p'^-} + |\lambda|^{p'^+})(|u|_{\alpha(x)}^{\alpha^-} + |u|_{\alpha(x)}^{\alpha^+}) + 1.
\end{aligned}$$

By the continuous embedding $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ and the Poincaré inequality (9) we have

$$|\phi u|_{p'(x)} \leq const(|u|_{1,p(x)}^{\alpha^-} + |u|_{1,p(x)}^{\alpha^+}) + 1.$$

This implies that ϕ is bounded on $W_0^{1,p(x)}(\Omega)$.

Step 2

Let $\psi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\psi u(x) := -f(x, u, \nabla u) \text{ for } u \in W_0^{1,p(x)}(\Omega) \text{ and } x \in \Omega.$$

We prove that ψ is bounded and continuous.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have, by the growth condition (f_2) and the inequalities (6) and (7) that

$$\begin{aligned}
|\psi u|_{p'(x)} &\leq \rho_{p'(x)}(\phi u) + 1 \\
&= \int_\Omega |f(x, u(x), \nabla u(x)|^{p'(x)}\, dx + 1 \\
&\leq const(\rho_{p'(x)}(k) + \rho_{\beta(x)}(u) + \rho_{\beta(x)}(\nabla u)) + 1 \\
&\leq const(|k|_{p'(x)}^{p'^-} + |k|_{p'(x)}^{p'^+} + |u|_{\beta(x)}^{\beta^-} + |u|_{\beta(x)}^{\beta^+} + |\nabla u|_{\beta(x)}^{\beta^-} + |\nabla u|_{\beta(x)}^{\beta^+}) + 1.
\end{aligned}$$

By the continuous embedding $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and the Poincaré inequality (9) we have

$$|\psi u|_{p'(x)} \leq const(|k|_{p'(x)}^{p'^-} + |k|_{p'(x)}^{p'^+} + |u|_{1,p(x)}^{\beta^-} + |u|_{1,p(x)}^{\beta^+}) + 1.$$

This implies that ψ is bounded on $W_0^{1,p(x)}(\Omega)$.

To show that ψ is continuous, let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Then $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{p(x)}(\Omega)$ and g in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \text{ and } \nabla u_k(x) \to \nabla u(x),$$

$$|u_k(x)| \leq h(x) \text{ and } |\nabla u_k(x)| \leq |g(x)|$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since f satisfies the Carathéodory condition, we obtain that

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

It follows from (f_2) that

$$|f(x, u_k(x), \nabla u_k(x))| \leq c(k(x) + |h(x)|^{r(x)-1} + |g(x)|^{r(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since
$$k + |h|^{r(x)-1} + |g(x)|^{r(x)-1} \in L^{p'(x)}(\Omega)$$

and taking into account the equality

$$\rho_{p'(x)}(\psi u_k - \psi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

the dominated convergence theorem and the equivalence (5) imply that

$$\psi u_k \to \psi u \text{ in } L^{p'(x)}(\Omega).$$

Thus the entire sequence (ψu_n) converges to ψu in $L^{p'(x)}(\Omega)$ and then ψ is continuous.

Step 3

Since the embedding $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is also compact. Therefore, the compositions $I^* o\phi$ and $I^* o\psi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ are compact. We conclude that $S = I^* o\phi + I^* o\psi$ is compact. This completes the proof. ☑

Theorem 4.4. *Under assumptions* (f_1) *and* (f_2)*, problem (1) has a weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

Proof. Let $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be as in Lemma 4.3 and $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, as in section 3.2, given by

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if and only if

$$Lu = -Su. \tag{10}$$

Thanks to the properties of the operator L seen in Theorem 3.1 and in view of Minty-Browder Theorem (see [17], Theorem 26A), the inverse operator

$T := L^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 4.3 that the operator S is bounded, continuous and quasimonotone.

Consequently, equation (10) is equivalent to

$$u = Tv \text{ and } v + SoTv = 0. \tag{11}$$

Following the terminology of [17], the equation $v + SoTv = 0$ is an *abstract Hammerstein equation* in the reflexive Banach space $W^{-1,p'(x)}(\Omega)$.

To solve equations (11), we will apply the degree theory introduced in section 2. To do this, we first claim that the set

$$B := \{v \in W^{-1,p'(x)}(\Omega) | v + tSoTv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $|Tv|_{1,p(x)} = |\nabla u|_{p(x)}$.

If $|\nabla u|_{p(x)} \le 1$, then $|Tv|_{1,p(x)}$ is bounded.

If $|\nabla u|_{p(x)} > 1$, then we get by, the implication (3), the growth condition (f_2), the Hölder inequality (2), the inequality (7) and the Young inequality, the estimate

$$\begin{aligned}
|Tv|_{1,p(x)}^{p^-} &= |\nabla u|_{p(x)}^{p-} \\
&\le \rho_{p(x)}(\nabla u) \\
&= \langle Lu, u\rangle \\
&= \langle v, Tv\rangle \\
&= -t\langle SoTv, Tv\rangle \\
&= t\int_\Omega (\lambda|u|^{q(x)-2}u + f(x, u, \nabla u))u\,dx \\
&\le const(|\lambda|\rho_{q(x)}(u) + \int_\Omega |k(x)u(x)|dx + \rho_{r(x)}(u) + \int_\Omega |\nabla u|^{r(x)-1}|u|dx) \\
&\le const(|u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+} + |k|_{p'(x)}|u|_{p(x)} + |u|_{r(x)}^{r^-} + |u|_{r(x)}^{r^+} \\
&\qquad + \frac{1}{r'^-}\rho_{r(x)}(\nabla u) + \frac{1}{r^-}\rho_{r(x)}(u)) \\
&\le const(|u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+} + |u|_{p(x)} + |u|_{r(x)}^{r^-} + |u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^+}).
\end{aligned}$$

From the Poincaré inequality (9) and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$ and $L^{p(x)} \hookrightarrow L^{r(x)}$, we can deduct the estimate

$$|Tv|_{1,p(x)}^{p^-} \le const(|Tv|_{1,p(x)}^{q^+} + |Tv|_{1,p(x)} + |Tv|_{1,p(x)}^{r^+}).$$

It follows that $\{Tv | v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (11) that the set B is bounded in $W^{-1,p'(x)}(\Omega)$. Consequently, there exists $R > 0$ such that

$$|v|_{-1,p'(x)} < R \text{ for all } v \in B.$$

This says that

$$v + tSoTv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0,1].$$

From Lemma 2.1 it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = LoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + SoT$ is also bounded. We conclude that

$$I + SoT \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega)$ given by

$$H(t,v) := v + tSoTv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem(2.4), we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1$$

and hence, there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

We conclude that $u = Tv$ is a weak solution of (1). This completes the proof. ☑

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