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Quasi Partial Sums of Harmonic Univalent Functions

Sumas Cuasi-Parciales de Funciones Armónicas Univalentes

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Abstract. In this work, we obtain some conditions under which the quasi partial sums of the generalized Bernardi integral operator consisting of the harmonic univalent functions belongs to a similar class.

Key words and phrases. quasi-partial sums, integral operator, harmonic functions.

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Resumen. En este trabajo obtenemos algunas condiciones bajo las cuales las sumas cuasi-parciales del operador integral Bernardi generalizado que consiste de funciones armónicas univalentes pertenece a una clase similar.

Palabras y frases clave. Sumas cuasi-parciales, operador integral, funciones armónicas.

1. Introduction

Let A denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

In [7], Opoola defined the class $T_n^\alpha(\beta)$ to be a subclass of A consisting of analytic functions satisfying the condition

$$Re\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta, z \in \mathbb{U},\ z^\alpha \neq 0,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \leq \beta < 1, \alpha > 0,$$

where we take the principal value for z^α and D^n is the Salagean differential operator [10] defined as follows:

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z),$$
$$D^n f(z) = D\left(D^{n-1} f(z)\right) = z\left(D^{n-1} f(z)\right).$$

For $n = 1, \alpha = 1$, the functions in $T_1^1(\beta)$ are called functions of bounded turning [4]. From (1), we can write that

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha, \quad \alpha > 0 \tag{2}$$

Using binomial expansion of (2), we have

$$f(z)^\alpha = z^\alpha + \alpha a_2 z^{\alpha+1} + \left[\alpha a_3 + \frac{\alpha(\alpha-1)}{2!} a_2^3\right] z^{\alpha+2} + \left[\alpha a_4 + \frac{\alpha(\alpha-1)}{2!} 2a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} a_2^3\right] z^{\alpha+3} + \cdots.$$

Hence, we define the class of analytic functions of fractional power A_α as

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}, \alpha > 0. \tag{3}$$

Here, again we take the principal value for z^α throughout the article. Thus, we obtain the differential operator

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1} \tag{4}$$

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain D if both u and v are real harmonic in D. In any simply connected domain, we can write

$$f = h + \overline{g}, \tag{5}$$

where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'| > |g'|$ in D.

Denote by S_H the class of functions f of the form (5) that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U}$.

A function f of the form (5) in S_H is said to be in the class $HP(\beta)$, if and only if

$$Re\left(h'(z) + g'(z)\right) > \beta, \quad 0 \leq \beta < 1, z \in \mathbb{U}$$

The class $HP(\beta)$ was introduced and extensively studied by Karpuzoḡullari et al. [6]. Very recently, some other interesting properties such as generalized convolution and partial sums for this class $HP(\beta)$ have been studied by Porwal and Dixit in [8] and [9], respectively.

In the present paper, we generalize the classes $T_n^\alpha(\beta)$ and $HP(\beta)$, and for this purpose we may express the analytic functions h and g as

$$h(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}, g(z)^\alpha = \sum_{k=1}^{\infty} b_k(\alpha) z^{\alpha+k-1}, |b_1(\alpha)| < 1. \quad (6)$$

We denote by $TH_n^\alpha(\beta)$ the class of all functions of the form (12) that satisfy the condition

$$Re\left\{\frac{D^n h(z)^\alpha + D^n g(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta, 0 \le \beta < 1, \alpha > 0, n \in \mathbb{N}_0. \quad (7)$$

where D_n is the Salagean operator [10].

***Remark* 1.1.** **i** For $\alpha = 1, n = 1, g \equiv 0$ the class $TH_n^\alpha(\beta)$ reduces to $B(\beta)$, the class of bounded turning functions (see [4]).

ii For $g \equiv 0$, the class $TH_n^\alpha(\beta)$ reduces to $T_n^\alpha(\beta)$ studied by Opoola (see [7]).

iii For $\alpha = 1, n = 1$, the class $TH_n^\alpha(\beta)$ reduces to $HP(\beta)$ studied by Karpuzoḡullari et al. in [6], (see also [2]).

For f of the form (1), the generalized Bernardi integral operator $F(z)^\alpha$ is given by

$$F(z)^\alpha = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} f(t)^\alpha dt, \alpha + c > 0 \quad (8)$$

$$= z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha + c}{\alpha + c + k - 1} a_k(\alpha) z^{\alpha+k-1}. \quad (9)$$

For f^α in S_H, where h^α and g^α are given by (6), the generalized Bernardi integral operator $F(z)^\alpha$ is given by

$$F(z)^\alpha = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} h(t)^\alpha dt + \overline{\frac{\alpha + c}{z^c} \int_0^z t^{c-1} g(t)^\alpha dt}$$
$$= z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha + c}{\alpha + c + k - 1} a_k(\alpha) z^{\alpha+k-1} + \sum_{k=1}^{\infty} \overline{\frac{\alpha + c}{\alpha + c + k - 1} b_k(\alpha) z^{\alpha+k-1}}.$$

Now we define the mth quasi-partial sums $F_m(z)^\alpha$ of the integral operator $F(z)^\alpha$ for functions f of the form (6) as follows

$$\begin{aligned} F_m(z)^\alpha = z^\alpha + \sum_{k=2}^{m} \frac{\alpha+c}{\alpha+c+k-1} a_k(\alpha) z^{\alpha+k-1} \\ + \sum_{k=2}^{m} \overline{\frac{\alpha+c}{\alpha+c+k-1} a_k(\alpha) z^{\alpha+k-1}} \quad (10) \\ = H_m(z)^\alpha + \overline{G_m(z)^\alpha}. \end{aligned}$$

In [1], Babalola defined a new concept of quasi-partial sums of the generalized Bernardi integral operator for analytic univalent functions and he extended an earlier result of Jahangiri and Farahmand [5]. Yet analogous results on harmonic univalent functions have not been so far explored. Recently, Porwal and Dixit [8] studied the partial sums of Libera integral operator for harmonic univalent functions and some interesting results were obtained. Motivated by the above mentioned work, an attempt has been made by using the concept of quasi-partial sums of the generalized Bernardi operator for harmonic functions to extend the results further.

2. Main Result

We first mention a sufficient condition for the function f^α of the form (12) belong to the class $TH_n^\alpha(\beta)$ given by the following result which can be established easily.

Theorem 2.1. *Let the function $f^\alpha = h^\alpha + \overline{g^\alpha}$, where h^α and g^α are defined as in (6) if*

$$\sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1} \left(\frac{\alpha+k-1}{\alpha}\right)^n |a_k(\alpha)| + \sum_{k=1}^{\infty} \frac{\alpha+c}{\alpha+c+k-1} \left(\frac{\alpha+k-1}{\alpha}\right)^n |b_k(\alpha)| \le 1-\beta \quad (11)$$

where $n \in \mathbb{N}_0, 0 \le \beta < 1, \alpha > 0$ and $c \ge -1$ then f^α is sense-preserving harmonic univalent in $\mathbb{U}$, and $f^\alpha \in TH_n^\alpha(\beta)$.

Proof. If $z_1^\alpha \neq z_2^\alpha$, then

$$\left|\frac{f(z_1)^\alpha - f(z_2)^\alpha}{h(z_1)^\alpha - h(z_2)^\alpha}\right| \ge 1 - \left|\frac{g(z_1)^\alpha - g(z_2)^\alpha}{h(z_1)^\alpha - h(z_2)^\alpha}\right|$$

$$= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(\alpha)(z_1^{\alpha+k-1} - z_2^{\alpha+k-1})}{(z_1^\alpha - z_2^\alpha) + \sum_{k=2}^{\infty} a_k(\alpha)(z_1^{\alpha+k-1} - z_2^{\alpha+k-1}}\right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty}(\alpha+k-1)b_k(\alpha)}{\alpha + \sum_{k=2}^{\infty}(\alpha+k-1)a_k(\alpha)}$$

$$\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\alpha+c}{\alpha+c+k-1} (\frac{\alpha+k-1}{\alpha})^n \times (1/(1-\beta))|b_k(\alpha)|}{1 + \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1} (\frac{\alpha+k-1}{\alpha})^n \times (1/(1-\beta))|a_k(\alpha)} \geq 0,$$

which proves the univalence. Note that f is sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)^\alpha| &\geq \alpha - \sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)||z|^{\alpha+k-1} \\
&> \alpha - \sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)| \\
&\geq \alpha^n - \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1} \frac{(\alpha+k-1)^n}{1-\beta} |a_k(\alpha)| \\
&\geq \sum_{k=1}^{\infty} \frac{\alpha+c}{\alpha+c+k-1} \frac{(\alpha+k-1)^n}{1-\beta} |b_k(\alpha)| \\
&\geq \sum_{k=1}^{\infty}(\alpha+k-1)|b_k(\alpha)| \\
&> \sum_{k=1}^{\infty}(\alpha+k-1)|b_k(\alpha)||z|^{\alpha+k-1} \\
&\geq |g(z)^\alpha|.
\end{aligned}$$

By (7), we have

$$\left\{ \frac{D^n h(z)^\alpha + D^n g(z)^\alpha}{\alpha^n z^\alpha} \right\} > \beta.$$

Using the fact that $Re(w) > \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$\left|1 - \beta + \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right| - \left|1 - \beta - \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right| \geq 0. \tag{12}$$

That is,

$$|(1-\beta)\alpha^n z^n + D^n f(z)^\alpha| - |(1-\beta)\alpha^n z^n - D^n f(z)^\alpha| \geq 0,$$

$$\begin{aligned}
&|(1-\beta)\alpha^n z^n + D^n h(z)^\alpha + D^n g(z)^\alpha| \\
&\qquad - |(1-\beta)\alpha^n z^n - D^n h(z)^\alpha - D^n g(z)^\alpha|
\end{aligned}$$

$$\Big|\alpha^n z^n - \beta\alpha^n z^n + \alpha^n z^n + \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}$$
$$+ \sum_{k=1}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n b_k(\alpha) z^{\alpha+k-1}\Big|$$
$$- \Big|\alpha^n z^n + \beta\alpha^n z^n - \alpha^n z^n$$
$$- \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}$$
$$- \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n b_k(\alpha) z^{\alpha+k-1}\Big|,$$

$$= \Big|\alpha^n(2-\beta)z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}$$
$$+ \sum_{k=1}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n b_k(\alpha) z^{\alpha+k-1}\Big|$$
$$- \Big|\beta\alpha^n z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}$$
$$- \sum_{k=1}^{\infty} \frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n b_k(\alpha) z^{\alpha+k-1}\Big|$$
$$\geq 2\alpha^n(1-\beta)|z|^\alpha$$
$$+ \sum_{k=2}^{\infty} 2\frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n |a_k(\alpha)||z|^{\alpha+k-1}$$
$$- \sum_{k=1}^{\infty} 2\frac{\alpha+c}{\alpha+c+k-1}(\alpha+k-1)^n |b_k(\alpha)||z|^{\alpha+k-1}$$
$$\times 2\alpha^n(1-\beta)\Big[1 + \sum_{k=2}^{\infty} \frac{1}{1-\beta}\frac{\alpha+c}{\alpha+c+k-1}(\frac{\alpha+k-1}{\alpha})^n |a_k(\alpha)|$$
$$+ \sum_{k=1}^{\infty} \frac{1}{1-\beta}\frac{\alpha+c}{\alpha+c+k-1}(\frac{\alpha+k-1}{\alpha})^n |b_k(\alpha)|\Big].$$

This last expression is nonnegative by (11), and so the proof is complete. ☑

The result of Theorem 2.1 is sharp for the harmonic univalent function of the form

$$f^{\alpha}(z) = z^{\alpha} + \sum_{k=2}^{\infty} \frac{(\alpha+c+k-1)(1-\beta)}{\alpha+c} \left(\frac{\alpha}{\alpha+k-1}\right)^{n} x_k z^{\alpha+k-1}$$
$$+ \sum_{k=1}^{\infty} \frac{(\alpha+c+k-1)(1-\beta)}{\alpha+c} \left(\frac{\alpha}{\alpha+k-1}\right)^{n} \overline{y_k z^{\alpha+k-1}}, \quad (13)$$

where $n \in \mathbb{N}_0, 0 \leq \beta < 1, \alpha > 0,\ c \geq -1$ and

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$$

shows that the coefficient bound given by (11) is sharp.

If we put $n = 2, c = 0$ and $\alpha = 1$ in Theorem 2.1, then we obtain the following result of Karpuzoğullari et al. [6].

Corollary 2.2. *Let the function $f = h + g$ be such that h and g are given by (5). Furthermore, let*

$$\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq 1 - \beta,$$

where $0 \leq \beta < 1$. Then f is harmonic univalent, sense-preserving in $\mathbb{U}$ and $f \in HP(\beta)$.

The aim of the next result is to obtain conditions under which the quasi-partial sums of the generalized Bernardi integral operator of functions in the class $TH_n^{\alpha}(\beta)$ consisting of harmonic univalent functions belong to the similar class. For this purpose, the following lemmas will be needed.

Lemma 2.3. [1] *For $z \in \mathbb{U}$, $-1 < \gamma \leq A = 4.5678018\ldots$*

$$Re\left(\sum_{k=1}^{l} \frac{z^k}{k+\gamma}\right) \geq -\frac{1}{1+\gamma}$$

Lemma 2.4. [4] *Let $P(z)$ be analytic in $\mathbb{U}$, $P(0) = 1$ and $Re(P(z)) > \frac{1}{2}$ in $\mathbb{U}$. For functions Q analytic in $\mathbb{U}$ the convolution function $P * Q$ takes values in the convex hull of the image on $\mathbb{U}$ under Q.*

The operator " $*$ " stands for the Hadamard product or convolution of two power series $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ denoted by $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$.

Theorem 2.5. *If f of the form (6) with $b_1 = 0$ and $f \in TH_n^\alpha(\beta)$, then $F_m^\alpha \in TB_n^\alpha(1 - \frac{2(1-\beta)(\alpha+c)}{\alpha+c+1})$, $\quad \alpha + c \leq 4.5678018....$*

Furthermore if $\beta \geq \frac{1}{2}\frac{\alpha+c-1}{\alpha+c}$, then $F_m(z)^\alpha$ belongs to some subclasses of the class $TH_n^\alpha(\beta)$.

Proof. Let f^α be of the form (6) belonging to $TH_n^\alpha(\beta)$ for $0 \leq \beta < 1$.

The mth quasi-partial sums $F_m(z)^\alpha$ of the integral operator $F(z)^\alpha$ for functions f of the form (6) are as follows

$$F_m(z)^\alpha = z^\alpha + \sum_{k=2}^{m} \frac{\alpha+c}{\alpha+c+k-1} a_k(\alpha) z^{\alpha+k-1} + \sum_{k=2}^{m} \overline{\frac{\alpha+c}{\alpha+c+k-1} a_k(\alpha) z^{\alpha+k-1}}$$
$$= H_m(z)^\alpha + \overline{G_m(z)^\alpha}.$$

We have to show that

$$Re\left\{\frac{D^n H_m(z)^\alpha + D^n G_m(z)^\alpha}{\alpha^n z^\alpha}\right\} > 1 - \frac{2(1-\beta)(\alpha+c)}{\alpha+c+1}.$$

Since

$$Re\left\{\frac{D^n h(z)^\alpha + D^n g(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta,$$

we have

$$Re\left\{1 + \frac{1}{2(1-\beta)}\left(\sum_{k=2}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k(\alpha) z^{k-1} + \sum_{k=2}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n b_k(\alpha) z^{k-1}\right)\right\} > \frac{1}{2}. \tag{14}$$

Applying the convolution properties of power series to $\frac{D^n H_m(z)^\alpha + D^n G_m(z)^\alpha}{\alpha^n z^\alpha}$, we may write

$$\frac{D^n H_m(z)^\alpha + D^n G_m(z)^\alpha}{\alpha^n z^\alpha} = 1 + \sum_{k=2}^{m}\left(\frac{\alpha+k-1}{\alpha}\right)^n \frac{\alpha+c}{\alpha+c+k-1} a_k(\alpha) z^{k-1} + \sum_{k=2}^{m}\left(\frac{\alpha+k-1}{\alpha}\right)^n \frac{\alpha+c}{\alpha+c+k-1} b_k(\alpha) z^{k-1}$$

$$= \left(1 + \frac{1}{2(1-\beta)}\left(\sum_{k=2}^{m}\left(\frac{\alpha+k-1}{\alpha}\right)^n (a_k(\alpha) + b_k(\alpha))\, z^{k-1}\right)\right) * \left(1 + 2(1-\beta)\sum_{k=2}^{m} \frac{\alpha+c}{\alpha+c+k-1} z^{k-1}\right)$$

$$= P(z) * Q(z). \tag{15}$$

From (8) we have $\alpha + c > 0$. Now we suppose $\alpha + c \leq 4.5678018$, and using Lemma 2.3 with $l = m - 1$, we obtain

$$Re\left(\sum_{k=1}^{m-1} \frac{z^k}{\alpha + c}\right) > -\frac{1}{1 + \alpha + c}. \tag{16}$$

By applying a simple algebra to inequality (16) and $Q(z)$ in (15), one may obtain

$$Re\,(Q(z)) = Re\left(1 + 2(1-\beta)\sum_{k=2}^{m} \frac{\alpha + c}{\alpha + c + k - 1} z^{k-1}\right) > 1 - \frac{2(1-\beta)(\alpha + c)}{\alpha + c + 1}.$$

On the other hand, the power series $P(z)$ in (15) in conjunction with condition (14) yields

$$Re\,(P(z)) > \frac{1}{2}.$$

Thus, by Lemma 2.4, the geometric quantity $\frac{D^n H_m(z)^\alpha + D^n G_m(z)^\alpha}{\alpha^n z^\alpha}$ takes values in the convex hull of $Q(z)$ so that,

$$\frac{D^n H_m(z)^\alpha + D^n G_m(z)^\alpha}{\alpha^n z^\alpha} = Re\,(Q(z)) > 1 - \frac{2(1-\beta)(\alpha + c)}{\alpha + c + 1}.$$

Observe that for $\beta \geq \frac{1}{2}\frac{\alpha + c - 1}{\alpha + c}$ the value $1 - \frac{2(1-\beta)(\alpha + c)}{\alpha + c + 1}$ is nonnegative. So, only in this case $F_m(z)^\alpha$ belongs to some subclasses of $TH_n^\alpha(\beta)$. This completes the proof. ☑

***Remark* 2.6.** *For $n = 0, \alpha = 1, c = 0$ in Theorem 2.5 and $f \in TH_0^1(\beta)$, $F_m \in TH(\frac{4\alpha - 1}{3})$. This result is obtained by Porwal and Dixit [8].*

***Remark* 2.7.** *For $\alpha = 1, c = 1$ and f of the form (1), the partial sums*

$$F_m(z) = z + \sum_{k=2}^{m} \frac{2}{k+1} a_k z^k \tag{17}$$

of the integral operator

$$F(z) = \frac{2}{z}\int_0^z f(t)dt \tag{18}$$

for each $f \in T(\beta)$ belongs to the class $T(\beta)$. In particularly, the partial sums (17) of the Libra integral (18) of a function of bounded turning is also of bounded turning which obtained by Jahangiri and Farahmand [5].

We note that some other results related to inclusion problems can also be found (see for example [3]).

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References

[1] K. O. Babalola, *Quasi-partial sums of the generalized Bernardi integral of certain analytic functions*, J. Nigerian Assoc. Math. Phy. **11** (2007), 67–70.

[2] K. K. Dixit and S. Porwal, *A subclass of harmonic univalent functions with positive coeficients*, Tamkang J. Math. **41** (2010), no. 3, 261–269.

[3] E. A. Eljamal and M. Darus, *Inclusion properties for certain subclasses of p-valent functions associated with new generalized derivative operator*, Vladikavkaz Mathematical Journal **15** (2013), no. 2, 27–34.

[4] W. Goodman, *Univalent functions, Vol. I, II*, Marnier Publishing, Florida, 1983.

[5] J. M. Jahangiri and K. Farahmand, *Partial sums of functions of bounded turning*, J. Inequal. Pure Appl. Math., **4** (2003), no. 4, 1–3, Art. 79.

[6] S. Y. Karpuzogullari, M. Ozturk, and M. Yamankaradeniz, *A subclass of harmonic univalent functions with negative coeficients*, Appl. Math. Comput. **142** (2003), 469–476.

[7] T. O. Opoola, *On a new subclasses of univalent functions*, Mathematica(Cluj)(36) **59** (1994), no. 2, 195–200.

[8] S. Porwal and K. K. Dixit, *Partial sums of harmonic univalent functions*, Studia Univ. Babes Bolayi **28** (2013), no. 1, 15–21.

[9] ———, *Some properties of generalized convolution of harmonic univalent functions*, Demonstratio Math. **43** (2013), no. 1, 63–74.

[10] G. S. Salagean, *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian-Finnish Seminar. Springer, Berlin, Heidelberg, 1983.

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