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The Problem of the First Passage Time for Some Elliptic Pseudodifferential Operators Over the p-Adics

El problema del primer retorno para algunos operadores pseudo-diferenciables elípticos sobre los p-ádicos

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Abstract. In this article we study the problem of the first passage time associated to certain elliptic pseudodifferential operators in dimensions 4 and 2 over the p-adics. This type of problems appeared in connection with certain models of complex systems.

Key words and phrases. Random walks, Diffusion, Dynamics of disordered systems, Relaxation of complex systems, p-Adic numbers, Non-archimean analysis.

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Resumen. En este artículo se estudia el problema del primer retorno asociado a ciertos operadores pseudo-diferenciales elípticos en dimensiones 4 y 2 sobre los números p-ádicos. Este tipo de problemas está conectado con ciertos modelos de sistemas complejos.

Palabras y frases clave. Caminatas aleatorias, difusión, sistemas dinámicos desordenados, relajación en sistemas complejos, números p-ádicos, análisis no Arquimediano.

1. Introducción

During the last twenty-five years there has been a strong interest on random walks on ultrametric spaces mainly due to its connections with models of complex systems, such as glasses and proteins. Random walks on ultrametric spaces are very convenient for describing phenomena whose space of states display a

hierarchical structure, see e.g. [2, 1, 3, 8, 10, 11, 15, 12, 14, 17, 19, 20, 21], and the references therein. Avetisov et al. have constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes, see [2], [3]. These models can be applied, among other things, to the study of the relaxation of biological complex systems [4]. From a mathematical point of view, in these models the time-evolution of a complex system is described by a p-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

The problem of the first passage time was study in [5] for the dimension 1 and in arbitrary dimension in [7]. In [5] and [7] pseudodifferential operators with radial symbols were considered. In this article we consider operators over $\mathbb{Q}_p^4$ whose symbols are not radial functions. By using a similar techniques to those of [5] and [7], we study the problem of the first passage time for a random walk $X(t,\omega)$ (see Definition 3.4) on the ultrametric space $\mathbb{Q}_p^4$, whose distribution density $Z(x,t)$, $x \in \mathbb{Q}_p^4$, $t \in \mathbb{R}_+$, satisfies the ultrametric diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = -\frac{1}{\Gamma_p^2(-\alpha)} \int_{\mathbb{Q}_p^4} \frac{u(x-y,t)-u(x,t)}{|f(y)|_p^{\alpha+2}}\, d^4y,$$

where f is an elliptic quadratic form of dimension 4, see (2).

Our aim is to prove that the random walk $X(t,\omega)$ is recurrent if $\alpha \geq 2$ and transient when $\alpha < 2$, see Theorem 3.10. By using the same techniques, we obtain similar results for the problem of the first passage time over $\mathbb{Q}_p^2$, see Theorem 4.1.

The article is organized as follows. In Section 2, we review the basic notions of p-adic analysis and some results about elliptic pseudodifferential operators which were studied in [21] and [6]. In Section 3, we study the first passage time for a pseudodifferential operator attached to an elliptic quadratic form of dimension 4. In Section 4, we present similar results to those in Section 3, for the problem of the first passage time over $\mathbb{Q}_p^2$. Finally, in Section 5, by using the same technique used in [5] we find the asymptotic behavior for the survival probability.

2. Preliminaries

In this section we fix notation and collect some basic results on p-adic analysis that we will use through the article. For a detailed exposition on p-adic analysis the reader may consult [1], [18], [20].

2.1. The Field of p-Adic Numbers

Along this article p will denote a prime number different from 2. The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\bullet|_p$, which is defined as

$$|x|_p = \begin{cases} 0, & \text{if} \quad x = 0; \\ p^{-\gamma}, & \text{if} \quad x = p^{\gamma}\dfrac{a}{b}; \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order of x*. We extend the $p-$adic norm to $\mathbb{Q}_p^n$ by taking

$$\|x\|_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for} \qquad x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1\leq i\leq n}\big\{ord(x_i)\big\}$, then $\|x\|_p = p^{-ord(x)}$. Any $p-$adic number $x \neq 0$ has a unique expansion $x = p^{ord(x)}\sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if} \quad x = 0 \quad \text{or} \quad ord(x) \geq 0; \\ p^{ord(x)}\sum_{j=0}^{-ord(x)-1} x_j p^j, & \text{if} \quad ord(x) < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B_\gamma^n(a) = \big\{x \in \mathbb{Q}_p^n : \|x - a\|_p \leq p^\gamma\big\}$ *the ball of radius p^γ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$*, and by $S_\gamma(a) = \big\{x \in \mathbb{Q}_p^n : \|x - a\|_p = p^\gamma\big\}$ *the sphere of radius p^γ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$*, and take $B_\gamma^n(0) := B_\gamma^n$, $S_\gamma^n(0) := S_\gamma^n$. Note that $B_\gamma^n(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$, where $B_\gamma(a_i) := \big\{x \in \mathbb{Q}_p : |x - a_i|_p \leq p^\gamma\big\}$ is the one-dimensional ball of radius p^γ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^n(0)$ is equals the product of n copies of $B_0(0) := \mathbb{Z}_p$, *the ring of $p-$adic integers*. We denote by $\Omega\big(\|x\|_p\big)$ the characteristic function of $B_0^n(0)$. For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of A.

2.2. The Bruhat-Schwartz Space

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is *called locally constant* if for any $x \in \mathbb{Q}_p^n$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \quad \text{for} \quad x' \in B_{l(x)}^n. \tag{1}$$

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $S\big(\mathbb{Q}_p^n\big) := S$. For $\varphi \in S\big(\mathbb{Q}_p^n\big)$, the

largest of such number $l = l(\varphi)$ satisfying (1) is called *the exponent of local constancy of* φ.

Let $S'\big(\mathbb{Q}_p^n\big) := S'$ denote the set of all functionals (distributions) on $S\big(\mathbb{Q}_p^n\big)$. All functionals on $S\big(\mathbb{Q}_p^n\big)$ are continuous.

Set $\Psi(y) = \exp\big(2\pi i\{y\}_p\big)$ for $y \in \mathbb{Q}_p$. The map $\Psi(\bullet)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuos map from $\mathbb{Q}_p$ into the unit circle satisfying $\Psi(y_0+y_1) = \Psi(y_0)\Psi(y_1)$, y_0, $y_1 \in \mathbb{Q}_p$.

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \bullet x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in S\big(\mathbb{Q}_p^n\big)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-\xi \bullet x)\varphi(\xi)\, d^n x, \qquad \text{for} \qquad \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition $vol\big(B_0^n\big) = 1$. The Fourier transform is a linear isomorphism from $S\big(\mathbb{Q}_p^n\big)$ onto itself satisfying $\big(\mathcal{F}(\mathcal{F}\varphi)\big)(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

2.3. Fourier Transform

The Fourier transform $\mathcal{F}\big[f\big]$ of a distribution $f \in S'\big(\mathbb{Q}_p^n\big)$ is defined by

$$\big(\mathcal{F}\big[f\big], \varphi\big) = \big(f, \mathcal{F}\big[\varphi\big]\big) \qquad \text{for all} \qquad \varphi \in S\big(\mathbb{Q}_p^n\big).$$

The Fourier transform $f \to \mathcal{F}\big[f\big]$ is a linear isomorphism from $S'\big(\mathbb{Q}_p^n\big)$ onto $S'\big(\mathbb{Q}_p^n\big)$. Furthermore, $f = \mathcal{F}\big[\mathcal{F}\big[f\big]\big(-\xi\big)\big]$.

2.4. Elliptic Pseudo Differential Operators

Definition 2.1. Let $f(\xi) \in \mathbb{Q}_p^n\big[\xi_1, \ldots, \xi_n\big]$ be a non constant polynomial. We say that $f(\xi)$ is an elliptic polynomial of degree d, if it satisfies the following conditions (i) $f(\xi)$ is a homogeneous polynomial of degree d, and (ii) $f(\xi) = 0 \Leftrightarrow \xi = 0$.

We note that if $f(\xi)$ is elliptic, then $cf(\xi)$ is elliptic for any $c \in \mathbb{Q}_p^\times$. For this reason we will assume from now on that elliptic polynomials have coefficients in $\mathbb{Z}_p$.

Definition 2.2. Let $f(\xi) \in \mathbb{Z}_p^n\big[\xi_1, \ldots, \xi_n\big]$ be a non constant polynomial. A pseudo differential operator $f(D, \alpha)$, $\alpha > 0$, with symbol $\big|f(\xi)\big|_p^\alpha$, is an operator of the form

$$\big(f(D,\alpha)\varphi\big) := \mathcal{F}_{\xi\to x}^{-1}\big(|f|_p^\alpha \mathcal{F}_{x\to\xi}\varphi\big), \qquad \text{for} \qquad \varphi \in S\big(\mathbb{Q}_p^n\big).$$

If f is an elliptic polynomial, we said that $f(D,\alpha)$ is an *elliptic pseudodifferential operator*.

When $n=4$, Zúñiga-Galindo and Casas-Sánchez established the following formula

$$-\big(f(D,\alpha)\varphi\big)=-\frac{1}{\Gamma_p^2(-\alpha)}\int_{\mathbb{Q}_p^4}\frac{\varphi(x-y)-\varphi(x)}{\big|f(y)\big|_p^{\alpha+2}}\,d^4y,\qquad \text{for all}\qquad \varphi\in S\big(\mathbb{Q}_p^4\big),$$

where $\Gamma_p^2(\alpha):=\frac{1-p^{\alpha-2}}{1-p^{-\alpha}}$ is the p-adic Gamma function in dimension two, see [20].

The operator $f(D,\alpha)$ has a self-adjoint extension with dense domain in $L^2\big(\mathbb{Q}_p^4\big)$. A elliptic form $f(\xi)$ over $\mathbb{Q}_p^n$, with $p\neq 2$, is called quadratic form if $d=2$. It is known that all quadratic forms in five or more variables are not elliptic.

We set

$$f(x):=x_1^2-ax_2^2-px_3^2+pax_4^2,\qquad f^\circ(x):=apx_1^2-px_2^2-ax_3^2+x_4^2,\tag{2}$$

with $a\in\mathbb{Z}$ a quadratic non-residue module p (note that $|a|_p=1$). Then f, f° are elliptic polynomials of degree 2.

Lemma 2.3. *Let f, f° be as above. Then*

(i) $p^{-1}\|x\|_p^2\leq|f(x)|_p\leq\|x\|_p^2,\qquad$ for every $\qquad x\in\mathbb{Q}_p^4$;

(ii) $p^{-1}\|x\|_p^2\leq\big|f^\circ(x)\big|_p\leq\|x\|_p^2,\qquad$ for every $\qquad x\in\mathbb{Q}_p^4$.

Proof. If $x=0$ then the statement is obvious. If $x\neq 0$ then $x=p^{ord(x)}u$ where $\|u\|_p=1$ and $|f(x)|_p=p^{-2\,ord(x)}|f(u)|_p=\|x\|_p^2|f(u)|_p$ it is easy to check that $|f(u)|_p\in\big\{p^{-1},1\big\}$, which completes the proof for f. The proof for f° is similar. ☑

This lemma is a particular case of Lemma 1 in [21].

2.5. Some Results About the Solution of the Cauchy Problem for Elliptic Operators

We need some results of [21] for $n=4$. Consider the Cauchy problem

$$\begin{cases}\frac{\partial\varphi(x,t)}{\partial t} &=-\big(f(D,\alpha)\varphi\big)(x,t),\quad x\in\mathbb{Q}^4,\ t\in(0,T];\\ \varphi(x,0) &=\varphi_0(x).\end{cases}\tag{3}$$

Set

$$Z_t(x):=Z(x,t)=\int_{\mathbb{Q}_p^4}\Psi(\xi\bullet x)e^{-t|f^\circ(\xi)|_p^\alpha}\,d^4\xi.\tag{4}$$

Then $Z(x,t)$ is a fundamental solution of (3) and the solution for Cauchy problem (3) is given by

$$\varphi(x,t) = Z(x,t) * \varphi_0(x) = \int_{\mathbb{Q}_p^4} \Psi(\xi \bullet x)\varphi_0(\xi) e^{-t|f^\circ(\xi)|_p^\alpha}\, d^4\xi.$$

Some known results about the fundamental solution are collected in the following theorem.

Theorem 2.4. *The function $Z(x,t)$ has the following properties:*

(i) $Z(x,t) \geq 0$ for any $t > 0$;

(ii) $\int\limits_{\mathbb{Q}_p^n} Z(x,t)\, d^4x = 1$ for any $t > 0$;

(iii) $Z(x,t) \leq A\Big(\|x\|_p + t^{\frac{1}{2\alpha}}\Big)^{-4-2\alpha}$; here, A is a positive constant for any $t > 0$, and any $x \in \mathbb{Q}_p^4$;

(iv) $Z_t(x) * Z_{t'}(x) = Z_{t+t'}(x)$ for any t, $t' > 0$;

(v) $\lim\limits_{t\to 0^+} Z(x,t) = \delta(x)$ in $S'\big(\mathbb{Q}_p^4\big)$;

(vi) $Z_t(x) \in C\big(\mathbb{Q}_p^4, \mathbb{R}\big) \cap L^1\big(\mathbb{Q}_p^4\big) \cap L^2\big(\mathbb{Q}_p^4\big)$ for any $t > 0$.

Proof. (i),(ii),(iii),(iv) follow from [21, Theorem 2, Proposition 2, Theorem 1, Proposition 2].

(v) By Lemma 2.3 $e^{-tf^\circ(\|\xi\|_p)} \in C\big(\mathbb{Q}_p^4, \mathbb{R}\big) \cap L^1$ for $t > 0$, then the inner product

$$\Big\langle e^{-tf^\circ(\|\xi\|_p)}, \phi \Big\rangle = \int_{\mathbb{Q}_p^4} e^{-tf^\circ(\|\xi\|_p)} \overline{\phi\big(\xi\big)}\, d^n\xi$$

defines a distribution on $\mathbb{Q}_p^4$; now, by the Dominated Converge Theorem,

$$\lim_{t\to 0^+} \Big\langle e^{-tf^\circ(\|\xi\|_p)}, \phi \Big\rangle = \langle 1, \phi \rangle$$

and thus

$$\lim_{t\to 0^+} \big\langle Z(x,t), \phi \big\rangle = \lim_{t\to 0^+} \Big\langle e^{-tf^\circ\|\xi\|_p}, \mathcal{F}^{-1}\phi \Big\rangle = \big\langle 1, \mathcal{F}^{-1}\phi \big\rangle = (\delta, \phi).$$

(vi) From Lemma 2.3, we have $Z_t(x) \in C\big(\mathbb{Q}_p^4, \mathbb{R}\big) \cap L^1\big(\mathbb{Q}_p^4\big)$, $t > 0$, and by (i) and (ii), $Z_t(x) \in L^2\big(\mathbb{Q}_p^n\big)$. ☑

2.6. Markov Processes over $\mathbb{Q}_p^4$

Along this section we consider $\left(\mathbb{Q}_p^4, \|\bullet\|_p\right)$ as complete non-Archimedean metric space and use the terminology and results of [9, Chapters Two, Three]. Let $\mathcal{B}$ denote the Borel $\sigma-$algebra of $\mathbb{Q}_p^4$. Thus $\left(\mathbb{Q}_p^4, \mathcal{B}, d^4x\right)$ is a measure space.

We set

$$p(t,x,y) \coloneqq Z(x-y,t) \quad \text{for} \quad t>0, \quad x,y \in \mathbb{Q}_p^4,$$

and

$$P(t,x,B) = \begin{cases} \int_B p(t,y,x)\, d^4y, & \text{for} \quad t>0, \quad x \in \mathbb{Q}_p^4, \quad B \in \mathcal{B}; \\ \mathbf{1}_B(x), & \text{for} \quad t=0. \end{cases}$$

Lemma 2.5. *With the above notation the following assertions hold:*

(i) $p(t,x,y)$ *is a normal transition density;*

(ii) $P(t,x,B)$ *is a normal transition function.*

Proof. The result follows from Theorem 2.4. See [9, Section 2.1] for further details. ☑

Lemma 2.6. *The transition function* $P(t,x,B)$ *satisfies the following two conditions:*

(i) for each $u \geq 0$ *and compact* B

$$\lim_{x\to\infty} \sup_{t\leq u} P(t,x,B) = 0, \qquad \text{[Condition } L(B)\text{]};$$

(i) for each $\epsilon > 0$ *and compact* B

$$\lim_{t\to 0^+} \sup_{x\in B} P\big(t,x,\mathbb{Q}_p^4 \smallsetminus B_\epsilon^4(x)\big) = 0, \qquad \text{[Condition } M(B)\text{]}.$$

Proof.

(i) By Theorem 2.4 (iii) and the fact that $\|\bullet\|_p$ is an ultranorm, we have

$$\begin{aligned} P(t,x,B) &\leq Ct \int_B \left(\|x-y\|_p + t^{\frac{1}{2\alpha}}\right)^{-4-2\alpha} d^4y \\ &= t\Big(\|x\|_p + t^{\frac{1}{2\alpha}}\Big)^{-4-2\alpha} vol(B), \quad \text{for} \quad x \in \mathbb{Q}_p^4 \smallsetminus B. \end{aligned}$$

Therefore $\lim_{x\to\infty} \sup_{t\leq u} P(t,x,B) = 0.$

(ii) By using Theorem 2.4 (iii), $\alpha > 0$, and the fact that $\|\bullet\|_p$ is an ultranorm, we have

$$\begin{aligned} P\big(t,x,\mathbb{Q}_p^4 \smallsetminus B_\epsilon^4(x)\big) &\leq Ct \int\limits_{\|x-y\|_p>\epsilon} \Big(\|x-y\|_p + t^{\frac{1}{2\alpha}}\Big)^{-4-2\alpha} d^4y \\ &= Ct \int\limits_{\|z\|_p>\epsilon} \Big(\|z\|_p + t^{\frac{1}{2\alpha}}\Big)^{-4-2\alpha} d^4z \\ &\leq Ct \int\limits_{\|z\|_p>\epsilon} \|z\|_p^{-4-2\alpha}\, d^4z \\ &= C'(\alpha,\epsilon)t. \end{aligned}$$

Therefore

$$\lim_{t\to 0^+} \sup_{x\in B} P\big(t,x,\mathbb{Q}_p^4 \smallsetminus B_\epsilon^4(x)\big) \leq \lim_{t\to 0^+} \sup_{x\in B} C'(\alpha,\epsilon)t = 0. \qquad ☑$$

Theorem 2.7. *$Z(x,t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.*

Proof. The result follows from [9, Theorem 3.6] by using that $\big(\mathbb{Q}_p^4, \|x\|_p\big)$ is semi-compact space, i.e. a locally compact Hausdorff space with a countable base, and $P(t,x,B)$ is a normal transition function satisfying conditions $L(B)$ and $M(B)$, c.f. Lemma 2.5, 2.6. ☑

3. The First Passage Time over $\mathbb{Q}_p^4$

Consider the following Cauchy problem

$$\begin{cases} \frac{\partial\varphi(x,t)}{\partial t} &= -\frac{1}{\Gamma_p^2(-\alpha)} \int_{\mathbb{Q}_p^4} \frac{\varphi(x-y,t)-\varphi(x,t)}{|f(y)|_p^{\alpha+2}}\, d^4y, \qquad x\in\mathbb{Q}_p^4,\ t\in(0,T] \\ \varphi(x,0) &= \Omega\big(\|x\|_p\big). \end{cases} \tag{5}$$

The solution of (5) is given by

$$\varphi(x,t) = \int\limits_{\mathbb{Q}_p^4} \Psi(\xi\bullet x)\Omega\big(\|\xi\|_p\big) e^{-t|f^o(\xi)|_p}\, d^4\xi. \tag{6}$$

Lemma 3.1. *The function $\varphi(x,t)$ is infinitely differentiable in the time $t\geq 0$ and its derivative is given by*

$$\frac{\partial^m\varphi}{\partial t^m}(x,t) = (-1)^m \int\limits_{\mathbb{Q}_p^4} |f^\circ(\xi)|_p^m \Psi(\xi\bullet x)\Omega\big(\|\xi\|_p\big) e^{-t|f^\circ(\xi)|_p}\, d^4\xi, \quad \text{for} \quad m\in\mathbb{N}.$$

Proof. Note that for $t \geq 0$ and $m \in \mathbb{N}$, $|f^{\circ}(\xi)|_p^m \Psi(\xi \bullet x)\Omega\big(\|\xi\|_p\big)e^{-t|f^{o}(\xi)|_p} \in L^1\big(\mathbb{Q}_p^4\big)$. The announced formula is obtained by induction on m by applying the Lebesgue Dominated Convergence Theorem. ☑

Lemma 3.2. [6, Lemma 7] *Let* $f(x) = x_1^2 - ax_2^2 - px_3^2 + apx_4^2$, $\alpha \in \mathbb{C}$, *then*

$$\int\limits_{\|y\|_p \geq 1} \frac{1}{|f(y)|_p^{\alpha+2}} d^4y = \frac{p^{-2\alpha}\big(1-p^{-2}\big)(1+p^{\alpha})}{1-p^{-2\alpha}}, \qquad \mathrm{Re}(\alpha) > 0.$$

***Remark* 3.3.** We note

$$-\frac{1}{\Gamma_p^2(-\alpha)} \int\limits_{\|y\|_p > 1} \frac{d^4y}{|f(y)|_p^{\alpha+2}} \leq -\frac{1}{\Gamma_p^2(-\alpha)} \int\limits_{\|y\|_p \geq 1} \frac{d^4y}{|f(y)|_p^{\alpha+2}} \leq 1. \tag{7}$$

Set Υ to be the space of all paths of the random process $X(t,\omega)$. Then there exists a probability space $(\Upsilon, \mathcal{B}, P)$, where P is a probability measure on Υ. The construction of this probability space follows from classical arguments, see e.g. [16, pp. 338–339] or [13, proof of Theorem 5.9]. The argument uses the one-point compactification $\overline{\mathbb{Q}}_p^4$ of $\mathbb{Q}_p^4$ by a point, and that $\Upsilon = \prod\limits_{0 \leq t < \infty} \overline{\mathbb{Q}}_p^4(t)$, where the $\overline{\mathbb{Q}}_p^4(t)$ are copies of $\overline{\mathbb{Q}}_p^4$. The construction of P follows from the Stone-Weierstrass Theorem and Riesz-Markov Theorem like in the Archimedean case. The probability $P(d\omega)$ is roughly $\prod\limits_{i=1}^{+\infty} \big[Z_{t_i}(x_i) * \Omega\big(\|x_i\|_p\big)\big]\, dx_i$.

This section is dedicated to the study of the following random variable.

Definition 3.4. The random variable $\tau_{\mathbb{Z}_p^4} : \Upsilon \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$\inf\big\{t > 0; X(t,\omega) \in \mathbb{Z}_p^4 : \text{there exists } t' \text{ such that } 0 < t' < t \text{ and } X(t',\omega) \notin \mathbb{Z}_p^4\big\}$$

is called the first passage time of a path of the random process $X(t,\omega)$ entering the domain $\mathbb{Z}_p^4$.

Note that the initial condition in (5) implies that

$$P\big(\{\omega \in \Upsilon : X(0,\omega) \in \mathbb{Z}_p^4\}\big) = 1.$$

Definition 3.5. We say that $X(t,\omega)$ is recurrent with respect to $\mathbb{Z}_p^4$ if

$$P\big(\{\omega \in \Upsilon : \tau_{\mathbb{Z}_p^4}(\omega) < +\infty\}\big) = 1. \tag{8}$$

Otherwise we say that $X(t,\omega)$ is transient with respect to $\mathbb{Z}_p^4$.

The meaning of (8) is that every path of $X(t,\omega)$ is sure to return to $\mathbb{Z}_p^4$. If (8) does not hold, then there exist paths of $X(t,\omega)$ that abandon $\mathbb{Z}_p^4$ and never go back.

Lemma 3.6. *The probability density function for a path of $X(t,\omega)$ to enter into $\mathbb{Z}_p^4$ at the instant of time t, with the condition that $X(0,\omega)\in\mathbb{Z}_p^4$ is given by*

$$g(t) = -\frac{1}{\Gamma_p^2(-\alpha)}\int_{\mathbb{Q}_p^4\smallsetminus\mathbb{Z}_p^4}\frac{\varphi(x,t)}{|f(x)|_p^{\alpha+2}}\,d^4x. \tag{9}$$

Proof. The survival probability, by definition

$$S(t) \coloneqq S_{\mathbb{Z}_p^4}(t) = \int_{\mathbb{Z}_p^4}\varphi(x,t)\,d^4x,$$

is the probability that a path of $X(t,\omega)$ remains in $\mathbb{Z}_p^4$ at the time t. Because there are no external forces acting on the random walk, we have

$$S'(t) = \begin{array}{l}\text{Probability that a path of } X(t,\omega)\\ \text{goes back to } \mathbb{Z}_p^4 \text{ at the time } t\end{array} - \begin{array}{l}\text{Probability that a path of } X(t,\omega)\\ \text{exits } \mathbb{Z}_p^4 \text{ at the time } t\end{array}$$

$$= g(t) - C\bullet S(t) \quad \text{with} \quad 0 < C \leq 1. \tag{10}$$

By using Lemma 3.1 and (5) we have

$$\begin{aligned} S'(t) = \int\limits_{\mathbb{Z}_p^4}\frac{\partial\varphi(x,t)}{\partial t}\,d^4x &= \frac{-1}{\Gamma_p^2(-\alpha)}\int\limits_{\mathbb{Z}_p^4}\int\limits_{\mathbb{Q}_p^4}\frac{\varphi(x-y,t)-\varphi(x,t)}{|f(y)|_p^{\alpha+2}}\,d^4y\,d^4x \\ &= \frac{-1}{\Gamma_p^2(-\alpha)}\int\limits_{\mathbb{Z}_p^4}\int\limits_{\mathbb{Z}_p^4}\frac{\varphi(x-y,t)-\varphi(x,t)}{|f(y)|_p^{\alpha+2}}\,d^4y\,d^4x\ + \\ &\qquad \frac{-1}{\Gamma_p^2(-\alpha)}\int\limits_{\mathbb{Z}_p^4}\int\limits_{\mathbb{Q}_p^4\smallsetminus\mathbb{Z}_p^4}\frac{\varphi(x-y,t)-\varphi(x,t)}{|f(y)|_p^{\alpha+2}}\,d^4y\,d^4x. \end{aligned}$$

The integral over $\mathbb{Z}_p^4\times\mathbb{Z}_p^4$ is zero since the parameter of constancy of φ is greater that 1. Thus

$$\begin{aligned} S'(t) = \frac{-1}{\Gamma_p^2(-\alpha)}\int\limits_{\mathbb{Z}_p^4}\int\limits_{\mathbb{Q}_p^4\smallsetminus\mathbb{Z}_p^4}\frac{\varphi(x-y,t)}{|f(y)|_p^{\alpha+2}}\,d^4y\,d^4x\ + \\ \frac{1}{\Gamma_p^2(-\alpha)}\int\limits_{\mathbb{Z}_p^4}\int\limits_{\mathbb{Q}_p^4\smallsetminus\mathbb{Z}_p^4}\frac{\varphi(x,t)}{|f(y)|_p^{\alpha+2}}\,d^4y\,d^4x. \end{aligned} \tag{11}$$

Now by substituting (6) in the first integral in (11) and by using Lemma 2.3, Fubini's Theorem and fact that $\int_{\mathbb{Z}_p^4} \Psi(\xi \bullet y)\, d^4y = \Omega\big(\|\xi\|_p\big)$, we have

$$S'(t) = \frac{-1}{\Gamma_p^2(-\alpha)} \int\limits_{\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4} \frac{\varphi(y,t)}{|f(y)|_p^{\alpha+2}}\, d^4y - \frac{-1}{\Gamma_p^2(-\alpha)} \left(\int\limits_{\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4} \frac{1}{|f(y)|_p^{\alpha+2}}\, d^4y \right) S(t).$$

Take $C = \frac{-1}{\Gamma_p^2(-\alpha)} \int_{\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4} \frac{1}{|f(y)|_p^{\alpha+2}}\, d^4y \leq 1$, c.f. (7). Finally, by using (10), one gets

$$g(t) = \frac{-1}{\Gamma_p^2(-\alpha)} \int\limits_{\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4} \frac{\varphi(x,t)}{|f(x)|_p^{\alpha+2}}\, d^4x. \qquad ☑$$

Proposition 3.7. *The probability density function $f(t)$ of the random variable $\tau(\omega)$ satisfies the non-homogeneous Volterra equation of second kind*

$$g(t) = \int_0^\infty g(t-\tau) f(\tau)\, d\tau + f(t). \tag{12}$$

Proof. The result follows from Lemma 3.6 by using the argument given in the proof of Theorem 1 in [5]. ☑

Lemma 3.8. *Let $f^\circ(x) = apx_1^2 - px_2^2 - ax_3^2 + x_4^2$ be with $a \in \mathbb{Z}$ a quadratic non-residue module and $\gamma \in \mathbb{Z}$, $\alpha > 0$, $\mathrm{Re}(s) > 0$. Then the following formulas hold:*

(i)
$$\int_{S_0^4} \frac{1}{s + p^{-2\gamma\alpha} |f^\circ(y)|_p^\alpha}\, d^4y = \frac{1-p^{-2}}{s+p^{-2\gamma\alpha}} + \frac{(1-p^{-2})p^{-2}}{s+p^{-2\gamma\alpha-\alpha}}.$$

(ii) If $|\xi|_p \geq p$, then there exist constants C_1 and C_2 such that

$$\int_{S_0^4} \frac{\Psi(y \bullet \xi)}{s + p^{-2\gamma\alpha} |f^\circ(y)|_p^\alpha}\, d^4y = \begin{cases} \frac{C_1}{s+p^{-2\gamma\alpha}} - \frac{C_2}{s+p^{-2\gamma\alpha-\alpha}}, & \text{if} \quad \|\xi\|_p = p; \\ 0, & \text{if} \quad \|\xi\|_p > p. \end{cases}$$

Proof. Let $S_0^4(0) = \sqcup_i U^{(i)}$ be where $U^{(i)} = U_1^{(i)} \times U_2^{(i)} \times U_3^{(i)} \times U_4^{(i)}$ and

$$U_j^{(i)} := \begin{cases} p^{i_j}\mathbb{Z}_p, & \text{if} \quad i_j = 1; \\ \mathbb{Z}_p^*, & \text{if} \quad i_j = 0; \end{cases}$$

for $i = (i_1, i_2, i_3, i_4) \in \{0,1\}^4 \smallsetminus \big\{(1,1,1,1)\big\}$. Thus

$$\int_{S_0^4} \frac{1}{s+p^{-2\gamma\alpha}|f^\circ(y)|_p^\alpha}\, d^4y = \sum_i \int_{U^{(i)}} \frac{1}{s+p^{-2\gamma\alpha}|f^\circ(y)|_p^\alpha}\, d^4y := \sum_i Z_i(\alpha). \quad (13)$$

Taking into account that $|f^\circ(y)|_p = \max\left\{p^{-1}|x_1|_p^2, p^{-1}|x_2|_p^2, |x_3|_p^2, |x_4|_p^2\right\}$ is constant on each $U_j^{(i)}$, the calculation is reduced to the calculation of $vol(U^{(i)})$. The following table summarizes the calculations:

Index i	$\boldsymbol{Z_i(\alpha)}$
$(1,1,1,0)$, $(1,1,0,1)$	$\frac{(1-p^{-1})p^{-3}}{s+p^{-2\gamma\alpha}}$
$(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$	$\frac{(1-p^{-1})^2p^{-2}}{s+p^{-2\gamma\alpha}}$
$(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$	$\frac{(1-p^{-1})^3p^{-1}}{s+p^{-2\gamma\alpha}}$
$(0,0,0,0)$	$\frac{(1-p^{-1})^4}{s+p^{-2\gamma\alpha}}$
$(1,0,1,1)$, $(0,1,1,1)$	$\frac{(1-p^{-1})p^{-3}}{s+p^{-2\gamma\alpha-\alpha}}$
$(0,0,1,1)$	$\frac{(1-p^{-1})^2p^{-2}}{s+p^{-2\gamma\alpha-\alpha}}$

(i) It follows from (13) by using the above table.

(ii) It follows by using the same decomposition as above, and the following fact

$$\int_{B_{-1}^1} \Psi(y\xi)\, dy = \begin{cases} p^{-1}, & \text{if} \quad |\xi|_p \le p; \\ 0, & \text{if} \quad |\xi|_p > p. \end{cases}$$

$$\int_{S_0^1} \Psi(y\xi)\, dy = \begin{cases} \left(1-p^{-1}\right), & \text{if} \quad |\xi|_p \le 1; \\ -p^{-1}, & \text{if} \quad |\xi|_p = p; \\ 0, & \text{if} \quad |\xi|_p > p. \end{cases}$$

We have

$$\int_{S_0^4} \frac{\Psi(y\bullet\xi)}{s+p^{-2\gamma\alpha}|f^\circ(y)|_p^\alpha}\, d^4y = \sum_i \int_{U^{(i)}} \frac{\Psi(y\bullet\xi)}{s+p^{-2\gamma\alpha}|f^\circ(y)|_p^\alpha}\, d^4y = \begin{cases} \frac{C_1}{s+p^{-2\gamma\alpha}} - \frac{C_2}{s+p^{-2\gamma\alpha-\alpha}}, & \text{if} \quad \|\xi\|_p = p; \\ 0, & \text{if} \quad \|\xi\|_p > p. \end{cases} \quad ☑$$

Proposition 3.9. *The Laplace transform $G(s)$ of $g(t)$ is given by $G(s) = G_1(s) + G_2(s)$, where*

$$G_1(s) = \\ -\frac{\big(1-p^{-2}\big)\big(1+p^{\alpha}\big)}{\Gamma_p^2(-\alpha)} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=\nu}^{\infty} p^{-4\gamma} \left(\frac{1-p^{-2}}{s+p^{-2\gamma\alpha}} + \frac{\big(1-p^{-2}\big)p^{-2}}{s+p^{-2\gamma\alpha-\alpha}} \right),$$

and

$$G_2(s) = \frac{-1}{\Gamma_p^2(-\alpha)} \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} p^{-4(\nu-1)} \left(\frac{C_1}{s+p^{-2(\nu-1)\alpha}} - \frac{C_2}{s+p^{-2(\nu-1)\alpha-\alpha}} \right).$$

Proof. We first note that, if $\mathrm{Re}(s) > 0$, then

$$\frac{e^{-st} e^{-t|f^{\circ}(\xi)|_p^{\alpha}} \Omega\big(\|\xi\|_p\big)}{|f(x)|_p^{\alpha+2}} \in L^1\big((0,\infty) \times \mathbb{Q}_p^4 \times \mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4, dt\, d^4\xi\, d^4x\big). \tag{14}$$

We compute the Laplace transform $G(s)$ of $g(t)$ substituting (6) into (9) and interchanging the iterated integrals in a suitable form, which is allowed by (14) via Fubini's Theorem. In this way one gets

$$G(s) = \int_{\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4} \int_{\mathbb{Z}_p^4} \frac{\Psi(\xi \bullet x)}{\big(s + |f^{\circ}(\xi)|_p^{\alpha}\big)|f(x)|_p^{\alpha+2}}\, d^4\xi\, d^4x, \quad \text{for} \quad \mathrm{Re}(s) > 0.$$

We now assert that $G(s)$ is convergent for $\mathrm{Re}(s) > 0$. Indeed, since

$$\big|s + |f^{\circ}(\xi)|_p^{\alpha}\big| \geq \mathrm{Re}(s) + |f^{\circ}(\xi)|_p^{\alpha} > |f^{\circ}(\xi)|_p^{\alpha} \quad \text{for} \quad \mathrm{Re}(s) > 0. \tag{15}$$

By Lemma 2.3, we have

$$\frac{1}{\big|s + |f^{\circ}(\xi)|_p^{\alpha}\big||f(x)|_p^{\alpha+2}} \in L^1\big(\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4 \times \mathbb{Z}_p^4, d^4x\, d^4\xi\big), \quad \text{for} \quad \mathrm{Re}(s) > 0,$$

then

$$\begin{aligned} G(s) &= \frac{-1}{\Gamma_p^2(-\alpha)} \int_{\mathbb{Q}_p^4 \smallsetminus \mathbb{Z}_p^4} \frac{1}{|f(x)|_p^{\alpha+2}} \int_{\mathbb{Z}_p^4} \frac{\Psi(\xi \bullet x)}{s + |f^{\circ}(\xi)|_p^{\alpha}}\, d^4\xi\, d^4x \\ &= \frac{-1}{\Gamma_p^2(-\alpha)} \sum_{\nu=1}^{\infty} \int_{S_{\nu}^4} \frac{1}{|f(x)|_p^{\alpha+2}} \sum_{\gamma=0}^{\infty} \int_{S_{-\gamma}^4} \frac{\Psi(\xi \bullet x)}{s + |f^{\circ}(\xi)|_p^{\alpha}}\, d^4\xi\, d^4x. \end{aligned}$$

By making the following change of variables

$$\begin{cases} \xi = p^{\gamma} y', & d^4\xi = p^{-4\gamma} d^4y'; \\ x = p^{-\nu} y, & dx = p^{4\nu} d^4y. \end{cases}$$

one gets

$$G(s) = \frac{-1}{\Gamma_p^2(-\alpha)} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{S_0^4} \frac{1}{|f(y)|_p^{\alpha+2}} \sum_{\gamma=0}^{\infty} p^{-4\gamma} \int_{S_0^4} \frac{\Psi\big(p^{\gamma-\nu}y'\bullet y\big)}{s+p^{-2\gamma\alpha}|f^\circ(y')|_p^\alpha}\, d^4y'\, d^4y \quad (16)$$

To calculate the integral (16) we divide the interior integral as follows:

$$\int_{S_0^4} \frac{\Psi\big(p^{\gamma-\nu}y'\bullet y\big)}{s+p^{-2\gamma\alpha}|f^\circ(y')|_p^\alpha}\, d^4y' = \int_{\substack{S_0^4 \\ |p^{\gamma-\nu}y'\bullet y|_p\le 1}} \frac{\Psi(p^{\gamma-\nu}y'\bullet y)}{s+p^{-2\gamma\alpha}|f^\circ(y')|_p^\alpha}\, d^4y' + \int_{\substack{S_0^4 \\ |p^{\gamma-\nu}y'\bullet y|_p> 1}} \frac{\Psi(p^{\gamma-\nu}y'\bullet y)}{s+p^{-2\gamma\alpha}|f^\circ(y')|_p^\alpha}\, d^4y'$$

and define for $|p^{\gamma-\nu}y'\bullet y|_p \le 1$,

$$G_1(s) := \frac{-1}{\Gamma_p^2(-\alpha)} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{S_0^4} \frac{1}{|f(y)|_p^{\alpha+2}} \sum_{\gamma=v}^{\infty} p^{-4\gamma} \int_{S_0^4} \frac{1}{s+p^{-2\gamma\alpha}|f^\circ(y')|_p^\alpha}\, d^4y'\, d^4y,$$

and for $|p^{\gamma-\nu}y'\bullet y|_p > 1$,

$$G_2(s) := \frac{-1}{\Gamma_p^2(-\alpha)} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{S_0^4} \frac{1}{|f(y)|_p^{\alpha+2}} \sum_{\gamma=0}^{v-1} p^{-4\gamma} \int_{S_0^4} \frac{\Psi\big(p^{\gamma-\nu}y'\bullet y\big)}{s+p^{-2\gamma\alpha}|f^\circ(y')|_p^\alpha}\, d^4y'\, d^4y.$$

Hence $G(s) = G_1(s) + G_2(s)$. By using Lemma 3.8, we have

$$G_1(s) = -\frac{(1-p^{-2})(1+p^\alpha)}{\Gamma_p^2(-\alpha)} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=\nu}^{\infty} p^{-4\gamma} \left(\frac{1-p^{-2}}{s+p^{-2\gamma\alpha}} + \frac{(1-p^{-2})p^{-2}}{s+p^{-2\gamma\alpha-\alpha}} \right)$$

and

$$G_2(s) = \frac{-1}{\Gamma_p^2(-\alpha)} \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} p^{-4(\nu-1)} \left(\frac{C_1}{s + p^{-2(\nu-1)\alpha}} - \frac{C_2}{s + p^{-2(\nu-1)\alpha-\alpha}} \right). \quad ☑ \tag{17}$$

Theorem 3.10.

(i) If $\alpha \geq 2$, then $X(t, \omega; \boldsymbol{W})$ is recurrent with respect to $\mathbb{Z}_p^n$.

(ii) If $2 > \alpha$, then $X(t, \omega; \boldsymbol{W})$ is transient with respect to $\mathbb{Z}_p^n$.

Proof. By Proposition 3.7, the Laplace transform of $F(s)$ of $f(t)$ equals $\frac{G(s)}{1+G(s)}$, where $G(s)$ is the Laplace transform of $g(t)$, and thus

$$F(0) = \int_0^{\infty} f(t)\, dt = 1 - \frac{1}{1 + G(0)}.$$

Hence, in order to prove that $X(t, \omega; \boldsymbol{W})$ is recurrent, it is sufficient to show that $G(0) = \lim_{s\to 0} G(s) = \infty$, and to prove that it is transient that $G(0) = \lim_{s\to 0} G(s) < \infty$.

(i) Take $s \in \mathbb{R}$, $s > 0$ and set $s = p^{-2\nu\alpha} = p^{-2\gamma\alpha}$, note that $s \to 0^+ \Leftrightarrow v \to \infty$ $(v = \gamma)$. Now taking only first term of $G_1(s)$ we have

$$G(s) > -\frac{\left(1 - p^{-2}\right)\left(1 + p^{\alpha}\right)}{\Gamma_p^2(-\alpha)} p^{-2\alpha} \times \sum_{\gamma=1}^{\infty} p^{-4\gamma} \left(\frac{1 - p^{-2}}{s + p^{-2\gamma\alpha}} + \frac{\left(1 - p^{-2}\right)p^{-2}}{s + p^{-2\gamma\alpha-\alpha}} \right) + G_2(s).$$

We get $G_2\left(p^{-2\nu\alpha}\right) < \infty$, but the first sum diverges if $\alpha \geq 2$. Then

$$\lim_{s\to 0^+} G(s) = \infty.$$

(ii) Now

$$|G(s)| \leq -\frac{\left(1 - p^{-2}\right)\left(1 + p^{\alpha}\right)}{\Gamma_p^2(-\alpha)} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=\nu}^{\infty} p^{-4\gamma} \left(\frac{1 - p^{-2}}{p^{-2\gamma\alpha}} + \frac{\left(1 - p^{-2}\right)p^{-2}}{p^{-2\gamma\alpha-\alpha}} \right) + G_2(0).$$

One sees easily that $G_2(0)$ converges and that the double series converges if $\alpha > 2$. Therefore $\lim_{s\to 0^+} G(s) < \infty$. ☑

4. The First Passage Time over $\mathbb{Q}_p^2$

We set $f(x) = x_1^2 - \eta x_2^2$, $\eta = \epsilon$, p, $p\epsilon$, where ϵ is unit which is not square in $\mathbb{Q}_p$. Then f is a quadratic form over $\mathbb{Q}_p^2$.

We now consider pseudodifferential operators whose symbols involve elliptic quadratic forms of dimension 2 over $\mathbb{Q}_p^2$.

$$\big(f(D,\alpha)\varphi\big)(x) := \begin{cases} \frac{1}{\Gamma_p^2(-2\alpha)} \int_{\mathbb{Q}_p^2} \frac{\varphi(y)-\varphi(x)}{|f(x-y)|_p^{\alpha+1}}\, dy, & \eta = \epsilon \\ \frac{1}{\Gamma_p^1(-\alpha)} \int_{\mathbb{Q}_p^2} \frac{\varphi(y)-\varphi(x)}{|f(x-y)|_p^{\alpha+1}}\, dy, & \eta = p,\ \epsilon p, \end{cases}$$

for $\varphi \in S\big(\mathbb{Q}_p^2\big)$, and $\alpha > 0$, see Proposition 1 of [6].

By using the technique of Section 3 we prove the following results.

Theorem 4.1.

(i) If $\alpha \geq 1$ and $\eta = \epsilon$, ϵp, p then $X(t,\omega;\boldsymbol{W})$ is recurrent with respect to $\mathbb{Z}_p^2$.

(ii) If $1 > \alpha$, and $\eta = \epsilon$, ϵp, p then $X(t,\omega;\boldsymbol{W})$ is transient with respect to $\mathbb{Z}_p^2$.

5. Survival Probability

The survival probability is given by

$$S_{\mathbb{Z}_p^4}(t) := \int_{\mathbb{Z}_p^4} \varphi(x,t)\, d^4x$$

where $\varphi(x,t)$ is given by (6). By Fubini's Theorem and Lemma 2.3, we have

$$\big(1-p^{-4}\big)\sum_{i=0}^{\infty} e^{-tp^{-2i-1}} p^{-4i} \leq S_{\mathbb{Z}_p^4}(t) \leq \big(1-p^{-4}\big)\sum_{i=0}^{\infty} e^{-tp^{-2i}} p^{-4i}. \tag{18}$$

We now give a simple generalizing of [5, Lemma A1], to dimension n.

Lemma 5.1. *If $\alpha > 0$ then*

$$\frac{p^{-n}}{t^{\frac{n}{\alpha}}\alpha \ln p}\gamma\left(\frac{n}{\alpha}, tp^{\alpha}\right) \leq \sum_{i=0}^{\infty} \frac{e^{-tp^{-i\alpha}}}{p^{ni}} \leq \frac{p^n}{t^{\frac{n}{\alpha}}\alpha \ln p}\gamma\left(\frac{n}{\alpha}, t\right),$$

where $\gamma(a,b) = \int_0^b e^{-z} z^{a-1}\, dz$ is the incomplete Gamma function.

Proof. We know that $e^{-tp^{-x\alpha}}$ is an increasing function and p^{-nx} is a decreasing function in the variable x. Thus, we have on the interval $i \leq x \leq i+1$ that

$$\frac{e^{-tp^{-(x-1)\alpha}}}{p^{nx}} \leq \frac{e^{-tp^{-i\alpha}}}{p^{ni}} \leq \frac{e^{-tp^{-x\alpha}}}{p^{n(x-1)}}.$$

Integrating in the variable x from i to $i+1$, we get

$$p^{-n}\int_i^{i+1}\frac{e^{-tp^{-(x-1)\alpha}}}{p^{n(x-1)}}\,dx \le \frac{e^{-tp^{-i\alpha}}}{p^{ni}} \le p^n\int_i^{i+1}\frac{e^{-tp^{-x\alpha}}}{p^{nx}}\,dx.$$

Now by summing with respect to i from 0 to ∞ we have

$$p^{-n}\int_0^{\infty}\frac{e^{-tp^{-(x-1)\alpha}}}{p^{n(x-1)}}\,dx \le \sum_{i=0}^{\infty}\frac{e^{-tp^{-i\alpha}}}{p^{ni}} \le p^n\int_0^{\infty}\frac{e^{-tp^{-x\alpha}}}{p^{nx}}\,dx.$$

Finally, by changing variables as $z = tp^{-(x-1)\alpha}$ in the left side and $z = tp^{-x\alpha}$ in the right side, we have

$$\frac{p^{-n}}{t^{\frac{n}{\alpha}}\alpha lnp}\int_0^{tp^{\alpha}} e^{-z}z^{\frac{n}{\alpha}-1}\,dz \le \sum_{i=0}^{\infty}\frac{e^{-tp^{-i\alpha}}}{p^{ni}} \le \frac{p^n}{t^{\frac{n}{\alpha}}\alpha lnp}\int_0^{t} e^{-z}z^{\frac{n}{\alpha}-1}\,dz. \qquad ☑$$

Theorem 5.2. *The function $S_{\mathbb{Z}_p^4}(t)$ satisfies*

$$\frac{\big(1-p^{-4}\big)p^{-4}}{t^2\ln p}\big(1+o(1)\big) \le S_{\mathbb{Z}_p^4}(t) \le \frac{\big(1-p^{-4}\big)p^{6}}{t^2\ln p}\big(1+o(1)\big), \quad \text{for} \quad t >> 1.$$

Proof. It follows by applying twice Lemma 5.1 to (18) with $n=4$, $\alpha=2$ and using that $\gamma(a,b) = \Gamma(a)\big(1+o(1)\big)$, if $b >> 1$. ☑

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References

[1] S. Albeverio, A. Yu. Khrennikov, and V. M. Shelkovich, *Theory of p-Adic Distributions: Linear and Nonlinear Models*, Cambridge University Press, 2010.

[2] V. A. Avetisov and A. Kh. Bikulov, *On the Ultrametricity of the Fluctuation Dynamic Mobility of Protein Molecules*, Tr. Mat. Inst. Steklova **265** (2009), 75–81, (Russian).

[3] V. A. Avetisov, A. Kh. Bikulov, and S. V. Kozyrev, *Description of Logarithmic Relaxation by a Model of a Hierarchical Random Walk*, Dokl. Akad. Nauk 368 **2** (1999), 164–167, (Russian).

[4] V. A. Avetisov, A. Kh. Bikulov, and V. A. Osipov, *p-Adic Description of Characteristic Relaxation in Complex Systems*, J. Phys. A 36 **15** (2003), 4239–4246.

[5] V. A. Avetisov, A. Kh. Bikulov, and A. P. Zubarev, *First Passage Time Distribution and the Number of Returns for Ultrametric Random Walks*, J. Phys. **42** (2009), 18.

[6] O. F. Casas-Sánchez and W. A. Zúñiga-Galindo, *Riesz Kernels and Pseudodifferential Operators Attached to Quadratic Forms Over p-Adic Fields*, p-Adic Numbers, Ultrametric Anal. Appl. **5** (2013), no. 3, 177–193.

[7] L. F. Chacón-Cortes and W. A. Zúñiga-Galindo, *Nonlocal Operators, Parabolic-Type Equations, and Ultrametric Random Walks*, J. Math. Phys. **54** (2013).

[8] B. Dragovich, A. Yu. Khrennikov, S. V. Kozyrev, and I. V. Volovich, *On p-Adic Mathematical Physics*, p-Adic Numbers Ultrametric Anal Appl **1** (2009), 1–17.

[9] E. B. Dynkin, *Markov Processes*, vol. I, Springer-Verlag, 1965.

[10] J. Galeano-Peñaloza and W. A. Zúñiga-Galindo, *Pseudo-Differential Operators with Semi-Quasielliptic Symbols over p-Adic Fields*, J. Math. Anal. Appl. **1** (2012), no. 386, 32–49.

[11] W. Karwowski, *Diffusion Processes With Ultrametric Jumps*, Rep. Math. Phys. **60** (2007), 221–235.

[12] A. Yu. Khrennikov and S. V. Kozyrev, *p-Adic Pseudodifferential Operators and Analytic Continuation of Replica Matrices*, Theoret. and Math. Phys. **144** (2005), no. 2, 1166–1170.

[13] József Lörinczi, Fumio Hiroshima, and Volker Betz, *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space: With Applications to Rigorous Quantum Field Theory*, vol. 34, Walter de Gruyter, 2011.

[14] Marc Mézard, Giorgio Parisi, and Miguel Angel Virasoro, *Spin Glass Theory and Beyond*, World Scientific (1987).

[15] Kochubei Anatoly N., *Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields*, Marcel Dekker, Inc., New York, 2001.

[16] Edward Nelson, *Feynman Integrals and the Schrödinger Equation*, Journal of Mathematical Physics **5** (1964), no. 3, 332–343.

[17] R. Rammal, G. Toulouse, and M. A. Virasoro, *Ultrametricity for Physicists*, Rev. Modern Phys. **58** (1986), 765–788.

[18] M. H. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, 1975.

[19] V. S. Varadarajan, *Path Integrals for a Class of p-Adic Schrödinger Equations*, Lett. Math. Phys. **39** (1997), 97–106.

[20] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, World Scientific, 1994.

[21] W. A. Zúñiga-Galindo, *Parabolic Equations and Markov Processes over p-Adic Fields*, Potential Anal. **28** (2008), no. 2, 185–200.

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