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The Brauer Group of K3 Covers

El grupo de Brauer de K3 cubrimientos

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Abstract. In this paper we study the injectivity of the induced morphism on the Brauer groups $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ given by the K3 cover $\pi : X \to Y$ of the Enriques surface Y.

Key words and phrases. Brauer group, K3 surface, Hochschild–Serre spectral sequence.

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Resumen. En este artículo estudiamos la inyectividad del morfismo inducido sobre los grupos de Brauer $\pi^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ dado por el K3 cubrimiento $\pi : X \to Y$ de la superficie de Enriques Y.

Palabras y frases clave. Grupo de Brauer, superficie K3, sucesión espectral de Hochschild–Serre.

1. Introduction

Let Y be an Enriques surface and $\pi : X \to Y$ its K3 cover with the fixed point free involution τ compatible with π. Since the Brauer group $\mathrm{Br}(Y)$ is $\mathbb{Z}/2\mathbb{Z}$, it is natural to ask about the triviality of the morphism $\pi^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$. This question was first mentioned by Harari and Skorobogatov in [3] and later answered by Beauville in [2] where he proved that the morphism is trivial if and only if the period map $\wp(Y, \varphi)$ belongs to one of the hypersurfaces H_λ for some $\lambda \in \Lambda^-$ with $\lambda^2 \equiv 2 \bmod 4$ and where H_λ is the hypersurface of Ω (this is the domain given by the equations $\omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0, \omega \cdot \lambda \neq 0$ for all $\lambda \in \Lambda^-$ with $\lambda^2 = -2$) defined by the equation $\lambda \cdot \omega = 0$. We give some group cohomology conditions for the morphism π^* to be injective. Besides, we also establish the type of the Néron Severi group of the K3 cover X of Picard number 11 such that the morphism $\pi^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is injective.

2. Basic Facts about Enriques Surfaces

We briefly recall some fundamental facts about Enriques and K3 surfaces.

Definition 1. A K3 surface is a compact complex surface X with trivial canonical bundle, i.e. $\omega_X \cong \mathcal{O}_X$, and $H^1(X, \mathcal{O}_X) = 0$.

Definition 2. An Enriques surface is a compact complex surface X with $\omega_X^2 \cong \mathcal{O}_X$, $\omega_X \neq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

The second cohomoloy of a K3 surface $H^2(X, \mathbb{Z})$ endowed with the cup-product is an even unimodular lattice of rank 22 and signature $(3, 19)$, i.e.,

$$H^2(X, \mathbb{Z}) \cong E_8^{\oplus 2} \oplus U^{\oplus 3}$$

where E_8, U are the root and hyperbolic lattices respectively.

Let Y be a smooth Enriques surface, $\pi : X \to Y$ its K3 cover and $\tau : X \to X$ the corresponding fixed point free involution such that $X/\tau \cong Y$. Thus we obtain the following lemma

Lemma 3. $0 \to \langle \omega_Y \rangle \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)^\tau \to 0$ *is an exact sequence.*

Proof. Let $\mathcal{L}$ be a sheaf with $\pi^*(\mathcal{L}) = \mathcal{O}_X$. Then $\mathcal{L} \otimes (\mathcal{O}_Y \oplus \omega_Y) = \pi_*\big(\pi^*(\mathcal{L})\big) = \pi_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus \omega_Y$. Therefore $\mathcal{L}$ is either $\mathcal{O}_Y$ or ω_Y. On the other hand, if $\lambda_\tau : \mathcal{M} \to \tau^*(\mathcal{M})$ is an isomorphism for some line bundle $\mathcal{M} \in \mathrm{Pic}(X)$, then, since $\mathcal{M}$ is simple (because it is a line bundle), $\tau^*\lambda_\tau \circ \lambda_\tau = c.\,\mathrm{id}$ for some $c \in \mathbb{C}$. Thus, we can replace λ_τ by $\frac{1}{\sqrt{c}}\lambda_\tau$ to obtain a linearization on $\mathcal{M}$ (see Definition 7 below). Hence, there exists a line bundle $\mathcal{L}$ on Y such that $\pi^*\mathcal{L} = \mathcal{M}$. ☑

Lemma 4.

i) If X is a K3 surface, then $H_1(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{tors} = 0$ (see [1, Prop. 3.3]).

ii) If Y is an Enriques surface, then $H_1(Y, \mathbb{Z}) = H^2(Y\mathbb{Z})_{tors} = \mathbb{Z}/2\mathbb{Z}$.

Lemma 5. *If Y is an Enriques surface, then* $\mathrm{Br}'(Y) = H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Proof. By Serre duality and Lemma 4(i), it follows that $0 = b_1(Y) = b_3(Y)$ and $H^3(Y, \mathbb{Z})_{tors} = H^2(Y, \mathbb{Z})_{tors} = \mathbb{Z}/2\mathbb{Z}$ (see [1, page 15]). Since $p_g(Y) = 0$, the exponential sequence induces the following exact sequence

$$0 \to H^2(Y, \mathcal{O}_Y^*) \to H^3(Y, \mathbb{Z}) \to H^3(Y, \mathcal{O}_X).$$

Then, from the vanishing of $H^3(Y, \mathcal{O}_X)$, we conclude the isomorphism $\mathrm{Br}'(Y) = H^3(Y, \mathbb{Z})$ and from the vanishing $b_3(Y) = 0$, we deduce that $H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. ☑

3. The Kernel of $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$

We will study the kernel of the map $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ induced by the universal cover, $\pi : X \to Y$, of the Enriques surface Y. In a particular case we will be able to describe the non trivial element of $\mathrm{Br}'(Y)$ as a Brauer–Severi variety over Y. For the basic facts about group cohomology we refer to [11]. In order to describe $\ker(\pi^*)$, we use the Hochschild–Serre spectral sequence (see [5, Theorem 14.9])

$$E_2^{p,q} := H^p\big(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathcal{O}_X^*)\big) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y^*). \tag{1}$$

and the following theorem (cf. [11, Theorem 6.2.2]). First, we recall that for a cyclic group G of order m with a generator τ, the norm in $\mathbb{Z}G$ is the element $N = 1 + \tau + \cdots + \tau^{m-1}$.

Theorem 6. *If A is a G-module with G a cyclic group generated by τ, then*

$$H^n(G, A) = \begin{cases} A^G, & \text{if} \quad n = 0; \\ \big\{a \in A : Na = 0\big\}/(\tau - 1)A, & \text{if} \quad n \quad \text{is odd;} \\ A^G/NA, & \text{otherwise.} \end{cases}$$

The last theorem can be used to compute $E_2^{n,0}$ for all n. First, since the action of $\langle\tau\rangle = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^* = H^0(X, \mathcal{O}_X^*)$ is trivial, one has

$$E_2^{n,0} = H^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = 0 \tag{2}$$

for all even integers $n \neq 0$. On the other hand, if n is an odd integer and $a \in \mathbb{C}^*$ with $N(a) = 1$, it follows from the definition of the norm map that $1 = a\tau(a) = a^2$. Thus

$$E_2^{n,0} = H^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}. \tag{3}$$

Since $E_2^{2,0} = 0$, also $E_\infty^{2,0} = 0$ and the following exact sequence follows:

$$0 \to E_\infty^{1,1} \to H^2(Y, \mathcal{O}_Y^*) \to H^2(X, \mathcal{O}_X^*)^\tau. \tag{4}$$

Let us recall now a few facts about linearization for finite group actions. Let Z be a smooth projective variety with an action by a finite group G. Let $\sigma : G \times Z \to Z$ be the action on Z, $\mu : G \times G \to G$ be the multiplication map of G and $p_2 : G \times Z \to Z$, $p_{23} : G \times G \times Z \to G \times Z$ be the projections.

Definition 7. A G-linearization of a coherent sheaf F is an isomorphism $\lambda : \sigma^* F \xrightarrow{\sim} p_2^* F$ of $\mathcal{O}_{G\times Z}$-modules that satisfies the cocycle condition $(\mu \times \mathrm{id}_Z)^*\lambda = p_{23}^*\lambda \circ (\sigma \times \mathrm{id}_G)^*\lambda$.

In the particular case that G is a finite group, the last definition can be reformulated as follows: A G-linearization of F is given by isomorphisms $\lambda_g : F \xrightarrow{\sim} g^*F$ for all $g \in G$ satisfying $\lambda_1 = \mathrm{id}_F$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$. If (F, λ) and (F', λ') are two G-linearised sheaves, then $\mathrm{Hom}(F, F')$ becomes a G-representation defined by the right action $g.f = (\lambda'_g)^{-1} \circ g^*f \circ \lambda_g$ for $f : F \to F'$.

Let Y be an Enriques surface and $\pi : X \to Y$ its universal cover map. We proceed to define the *relative norm homomorphis* $N_{X/Y}$. Let U_i be an open covering of Y such that $\widehat{U}_i := \pi^{-1}(U_i)$ consists of two copies of U_i. Take $f = (f_0, f_1) \in \mathcal{O}^*(\widehat{U}_i)$ and define the *sheaf relative norm map* by $f_0 f_1$. Thus, the relative norm homorphism induced in the Picard groups can be defined as follows: take a 1-cocycle $\big\{\widehat{\varphi}_i = (\varphi_0^i, \varphi_1^i)\big\}_i$ over X that represents a line bundle $\mathcal{L}$, and define our desired morphism by $N_{X/Y}\big(\big\{(\varphi_0^i, \varphi_1^i)\big\}_i\big) = \big\{\varphi_0^i \cdot \varphi_1^i\big\}_i$. This is also the cocycle defining the line bundle $\det(\pi_*(\mathcal{L}))$. Hence, we obtain $N_{X/Y}(-) = \det(\pi_*(-))$ and one can show the following lemma whose proof can be found in [2].

Lemma 8. *The kernel of* $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ *is*

$$\big(\ker N_{X/Y}\big)/\big((1-\tau)\,\mathrm{Pic}(X)\big).$$

Definition 9. Let X be a surface and $\mathcal{P}$ a $\mathbb{P}^1$-bundle on X. We say that $\mathcal{P}$ comes from a vector bundle if there exists a vector bundle E on X such that $\mathcal{P} \cong \mathbb{P}(E)$.

Lemma 10. *Let Y be an Enriques surface and $\pi : X \to Y$ its universal cover map. Let $\mathcal{L}$ be a line bundle satisfying $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, $N_{X/Y}(\mathcal{L}) = 0$, and such that $[\mathcal{L}]$ is nontrivial in $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big)$. Then $\mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ descends to a projective bundle that does not come from a vector bundle.*

Proof. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle with $N_{X/Y}(\mathcal{L}) = 0$ representing a nontrivial element in

$$\begin{aligned} E_2^{1,1} &= H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) \\ &= \frac{\{L \in \mathrm{Pic}(X) : \tau^*L \otimes L = \mathcal{O}_X\}}{\{\tau^*M \otimes M^\vee : M \in \mathrm{Pic}(X)\}}. \end{aligned}$$

We proceed to give a G-linearization on $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$:

$$\lambda_\tau : \mathbb{P}\big(\tau^*(\mathcal{O}_X \oplus \mathcal{L})\big) \longrightarrow \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}).$$

Since $N_{X/Y}(\mathcal{L}) = 0$ we can find a G-linearised isomorphism $i : \mathcal{L} \otimes \tau^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$ where we consider $\mathcal{O}_X$ endowed with the canonical G-linearization. We define λ_τ as the composition of morphisms

$$\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \to \mathbb{P}\big(\tau^*\mathcal{L} \oplus (\mathcal{L} \otimes \tau^*\mathcal{L})\big) \to \mathbb{P}(\tau^*\mathcal{L} \oplus \mathcal{O}_X) \to \mathbb{P}(\mathcal{O}_X \oplus \tau^*\mathcal{L})$$

$$[a:b]\mapsto[a\tau^*b:b\tau^*b]\mapsto\big[a\tau^*b:i(b\tau^*b)\big]\mapsto\big[i(b\tau^*b):a\tau^*b\big]$$

where a and b are sections of $\mathcal{O}_X$ and $\mathcal{L}$ respectively. Note that $\mathbb{P}(\mathcal{O}_X\oplus\tau^*\mathcal{L})=\mathbb{P}(\tau^*\mathcal{O}_X\oplus\tau^*\mathcal{L})$ because we consider the canonical linearization on $\mathcal{O}_X$, i.e. $\tau^*\mathcal{O}_X=\mathcal{O}_X$. Since i is a G-linearised isomorphism, it commutes with τ and from this we can check that $\lambda_\tau^2=\mathrm{id}$ as follows:

$$\begin{aligned}\lambda_\tau^2\big([a:b]\big)&=\lambda_\tau\big(\big[i(b\tau^*b):a\tau^*b\big]\big)\\&=\big[i\big((a\tau^*b)\tau^*(a\tau^*b)\big):i(b\tau^*b)\tau^*(a\tau^*b)\big]\\&=\big[a\tau^*a.i(b\tau^*b):i(b\tau^*b)\tau^*(a\tau^*b)\big]\\&=\big[a\tau^*a:\tau^*(a\tau^*b)\big]\\&=\big[a\tau^*a:b\tau^*a\big]\\&=[a:b].\end{aligned}$$

Hence, the projective bundle $\mathbb{P}(\mathcal{O}_X\oplus\mathcal{L})$ descends to a projective bundle $\mathcal{P}$ over Y. Now, we show that $\mathcal{P}$ does not come from a vector bundle on Y. Suppose $\mathcal{P}=\mathbb{P}(E)$ for some vector bundle E over Y and so $\mathbb{P}\big(\pi^*(E)\big)=\mathbb{P}(\mathcal{O}_X\oplus\mathcal{L})$. Thus, it follows that $\pi^*(E)=M\otimes(\mathcal{O}_X\oplus\mathcal{L})$, for some $M\in\mathrm{Pic}(X)$. By taking determinants on both sides of this isomorphism we get $\det\big(\pi^*(E)\big)=M^{\otimes 2}\otimes\mathcal{L}$.

In particular, this implies that M is not invariant. Indeed, if M is an invariant line bundle, $\mathcal{L}=\det\big(\pi^*(E)\big)\otimes(M^\vee)^{\otimes 2}$ is an invariant bundle. Hence $\mathcal{L}\cong\mathcal{O}_X$ because $\tau^*\mathcal{L}\otimes\mathcal{L}=\mathcal{O}_X$, a contradiction. Since $M^{\otimes 2}\otimes\mathcal{L}$ is invariant and $\tau^*\mathcal{L}\otimes\mathcal{L}=\mathcal{O}_X$, one has

$$M^{\otimes 2}\otimes\mathcal{L}=\tau^*(M^{\otimes 2}\otimes\mathcal{L})=\tau^*M^{\otimes 2}\otimes\mathcal{L}^\vee$$

and so, $\tau^*M^{\otimes 2}=M^{\otimes 2}\otimes\mathcal{L}^{\otimes 2}$. Hence, from the torsion freeness of $\mathrm{Pic}(X)$ we obtain $\tau^*M=M\otimes\mathcal{L}$, i.e., $\mathcal{L}=\tau^*M\otimes M^\vee$, but this contradicts the assumption that $\mathcal{L}$ defines a non trivial element in $E_2^{1,1}$. ☑

Lemma 11. *Let $\pi:X\to Y$ be the universal cover of an Enriques surface Y with $\rho(X)=10$. Then $\pi^*:\mathrm{Br}'(Y)\to\mathrm{Br}'(X)$ is a nontrivial homomorphism.*

Proof. We show that $\rho(X)=10$ implies $\mathrm{Pic}(X)^\tau=\mathrm{Pic}(X)$, i.e., all the line bundles on X are invariant. Since $\rho(X)=10$, $\mathrm{Pic}(X)^\tau\subseteq\mathrm{Pic}(X)$ is a sublattice of finite index. Thus, if $\mathcal{L}$ is a line bundle, there exists a positive integer r with $\mathcal{L}^{\otimes r}\in\mathrm{Pic}(X)^\tau$, i.e.,

$$\tau^*\mathcal{L}^{\otimes r}=\mathcal{L}^{\otimes r}.$$

Hence

$$(\tau^*\mathcal{L}\otimes\mathcal{L}^\vee)^{\otimes r}=\mathcal{O}_X.$$

Since $\mathrm{Pic}(X)$ is torsion free, we obtain

$$\tau^*\mathcal{L}\otimes\mathcal{L}^\vee=\mathcal{O}_X,$$

i.e., $\mathcal{L}$ is an invariant line bundle. Thus, the group $H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big)$ vanishes and the lemma holds. ☑

Example 12. In this example we show the existence of a K3 surface X with $\rho(X) = 10$ that covers an Enriques surface. First, we find a K3 surface with Picard number 10. Let us define $\Lambda := E_8 \oplus E_8 \oplus U \oplus U \oplus U$ and an involution ρ of L by

$$\rho : \Lambda \to \Lambda, (e_1, e_2, h_1, h_2, h_3) \mapsto (e_2, e_1, -h_1, h_3, h_2).$$

Note that this involution is the universal action (cf. [1, Ch. VIII, Lemma 19.1]), i.e. whenever $\pi : X \to Y$ is the universal covering of an Enriques surface Y with $\tau : X \to X$ the covering involution, then there exists an isometry $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ such that $\phi \circ \tau^* = \rho \circ \phi$. The ρ-invariant sublattice of Λ is

$$\Lambda^+ = \{x \in \Lambda : \rho(x) = x\} = \{(e, e, 0, h, h) : e \in E_8, h \in U\},$$

which is isometric to $E_8(2) \oplus U(2)$, where the isometry is given as follows

$$\rho^+ : \Lambda^+ \to E_8(2) \oplus U(2), \qquad (e, e, 0, h, h) \mapsto (e, h).$$

Hence, $E_8(2) \oplus U(2) \hookrightarrow E_8^{\oplus 2} \oplus U^{\oplus 3}$ is a primitive embedding. Since this lattice has Picard number 10 and signature (1,9), by [6, Cor. 2.9] we can find an algebraic K3 surface X with $\mathrm{NS}(X) = E_8(2) \oplus U(2)$. Now, we show that X has a fixed point free involution. The isometry ρ^+ also yields an isomorphism

$$\big(\Lambda^+\big)^\vee / \Lambda^+ \cong (\mathbb{Z}/2\mathbb{Z})^{10}.$$

It means that Λ^+ is a 2-elementary lattice with $l(A_{\Lambda^+}) = 10$. This gives us an involution

$$\tau^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

which is the identity on Λ^+ and acts like multiplication by (-1) on $T_X = (\Lambda^+)^\perp = \big(\mathrm{NS}(X)\big)^\perp$ where the orthogonal complement is taken in $H^2(X, \mathbb{Z})$. Since τ^* is the identity on Λ^+ (=NS(X) through the isometry ρ^+), it is effective and so it maps a Kähler class to a Kähler class. By the global Torelli Theorem for K3 surfaces, there exists a unique involution $\tau : X \to X$ which induces τ^* on $H^2(X, \mathbb{Z})$. Then it follows from [8, Thm. 4.2.2], that the set of fixed points X^τ is empty. It means that the involution τ is fixed point free, hence X/τ is an Enriques surface.

Now, we introduce the following spectral sequence

$$E_{2,\mathbb{Z}}^{p,q} := H^p\big(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathbb{Z})\big) \Rightarrow H^{p+q}(Y, \mathbb{Z}) \tag{5}$$

associated to the covering map $\pi : X \to Y$ of an Enriques surface Y and we compute some terms of this. Since X is a K3 surface, the vanishing $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$ implies

$$E_{2,\mathbb{Z}}^{n,1} = E_{2,\mathbb{Z}}^{n,3} = 0 \tag{6}$$

for all integers n. Now, we compute the terms $E_{2,\mathbb{Z}}^{n,0}$ for all integers n. First, we note that the action of $\mathbb{Z}/2\mathbb{Z}$ is trivial on $\mathbb{Z}$. Since the term $E_{2,\mathbb{Z}}^{0,0}$ corresponds to the invariant elements of $\mathbb{Z}$ under the action of $\mathbb{Z}/2\mathbb{Z}$ we obtain that $E_{2,\mathbb{Z}}^{0,0} = \mathbb{Z}$. Now, let us compute the terms $E_{2,\mathbb{Z}}^{n,0}$ for odd n. Since the action is trivial, we deduce that

$$0 = N(m) = \tau^*(m) + m = 2m.$$

Then it follows that $m = 0$ and hence by Theorem 6 that $E_{2,\mathbb{Z}}^{n,0} = 0$. On the other hand, if n is an even number we can see that $E_{2,\mathbb{Z}}^{n,0} = \mathbb{Z}/2\mathbb{Z}$. Summarizing,

$$E_{2,\mathbb{Z}}^{n,0} = \begin{cases} \mathbb{Z}, & \text{if} \quad n = 0; \\ 0, & \text{if} \quad n \quad \text{is odd}; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if} \quad n \quad \text{is even}, n \neq 0. \end{cases} \tag{7}$$

From (6) and (7) we deduce

$$E_{\infty,\mathbb{Z}}^{0,3} = E_{\infty,\mathbb{Z}}^{2,1} = E_{\infty,\mathbb{Z}}^{3,0} = 0$$

and this implies

$$E_{\infty,\mathbb{Z}}^{1,2} = \mathbb{Z}/2\mathbb{Z}. \tag{8}$$

The homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ induces a homomorphism $C : E_2^{1,1} \to E_{2,\mathbb{Z}}^{1,2}$ which can be easily described using Theorem 6 as

$$C : \frac{\{L \in \mathrm{Pic}(X) : \tau^* L \otimes L \cong \mathcal{O}_X\}}{\{\tau^* M \otimes M^\vee : M \in \mathrm{Pic}(X)\}} \to \frac{\{\ell \in H^2(X,\mathbb{Z}) : \tau^*\ell + \ell = 0\}}{\{\tau^* m - m : m \in H^2(X,\mathbb{Z})\}}, \tag{9}$$

sending $[L]$ to $[c_1(L)]$.

Theorem 13 (Schwarzenberger, [10])**.** *Let X be a projective surface. A topological complex vector bundle admits a holomorphic structure if and only if its first Chern class belongs to the Neron–Severi group of the surface.*

Lemma 14. *Let Y be an Enriques surface. Then every topological vector bundle on Y has a holomorphic structure.*

***Proof*.** Let E be a $\mathcal{C}_X$–bundle on Y. Since Y is an Enriques surface then $\mathrm{NS}(Y) \cong H^2(Y,\mathbb{Z})$. Hence $c_1(E) \in \mathrm{NS}(Y)$ and by Theorem 13, E has a holomorphic structure. ☑

Lemma 15. *The homomorphism C is injective.*

Proof. Let $[L]$ be the class of a line bundle L such that $\tau^*L \otimes L = \mathcal{O}_X$. Suppose that $C(L) = 0$. Thus, there exists a topological line bundle M such that $L = M^\vee \otimes \tau^*M$ and so

$$-c_1(M) + c_1(\tau^*M) = c_1(M^\vee \otimes \tau^*M) = c_1(L) \in \mathrm{NS}(X). \tag{10}$$

On the other hand, since the topological rank 2 vector bundle $\tau^*M \oplus M$ has a linearization (i.e. the trivial linearization), there exists a topological vector bundle E on Y such that $\pi^*E = \tau^*M \oplus M$. By Lemma 14, E has a holomorphic structure and induces one on $\tau^*M \oplus M$. Thus, by Theorem 13,

$$c_1(\tau^*M \oplus M) \in \mathrm{NS}(X). \tag{11}$$

Therefore, by (10) and (11), $2c_1(\tau^*M) = (c_1(\tau^*M) - c_1(M)) + c_1(\tau^*M \otimes M) \in \mathrm{NS}(X)$. Since X is a K3 surface, $c_1 : \mathrm{Pic}(X) \hookrightarrow H^2(X,\mathbb{Z})$ is injective and so

$$\frac{H^2(X,\mathbb{Z})}{\mathrm{NS}(X)} \hookrightarrow H^2(X, \mathcal{O}_X).$$

Thus $c_1(\tau^*M) \in \mathrm{NS}(X)$ because $2c_1(\tau^*M) \in \mathrm{NS}(X)$ and $H^2(X, \mathcal{O}_X)$ is torsion free, and so we conclude $[L] = 0$ in $E_2^{1,1}$. ☑

In Example 12 we have introduced the involution ρ on the K3 lattice $\Lambda := (E_8)^{\oplus 2} \oplus U^{\oplus 3}$ and also defined the invariant lattice Λ^+. We define similarly the ρ-anti-invariant sublattice of Λ by

$$\Lambda^- := \{\ell \in \Lambda : \rho(\ell) = -\ell\}.$$

Given $\ell = (x, y, z_1, z_2, z_3) \in \Lambda$, we get $\rho(\ell) = -\ell$ if and only if

$$\ell = (x, -x, z_1, z_2, -z_2).$$

Let $m = (m_1, m_1, n_1, n_2, n_3) \in \Lambda$, then

$$\rho(m) - m = \big(m_2 - m_1, -(m_2 - m_1), -2n_1, n_3 - n_2, -(n_3 - n_2)\big).$$

this yields that

$$\ell = (x, -x, z, y, -y) \in \Lambda^-$$

can be written as $\rho(m) - m$ for some $m \in \Lambda$ if and only if $z = -2n$ for some $n \in U$.

Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. Consider the spectral sequence $E_{2,\mathbb{Z}}^{1,2}$ associated to this (see (5)). Let $\ell \in H^2(X,\mathbb{Z})$ such that $\tau^*\ell = -\ell$. Thus, $2\ell = \ell - \tau^*\ell$, i.e. $[2\ell] = 0$ in $E_{2,\mathbb{Z}}^{1,2} =$

$H^1\big(\mathbb{Z}/2\mathbb{Z}, H^2(X,\mathbb{Z})\big)$. Therefore, any element in $E^{1,2}_{2,\mathbb{Z}} = H^1\big(\mathbb{Z}/2\mathbb{Z}, H^2(X,\mathbb{Z})\big)$ is 2-torsion.

By definition, $E^{1,2}_{3,\mathbb{Z}} = \ker\left(d_2^{1,2} : E^{1,2}_{2,\mathbb{Z}} \to E^{3,1}_{2,\mathbb{Z}}\right)$. Thus

$$E^{1,2}_{3,\mathbb{Z}} = E^{1,2}_{2,\mathbb{Z}}$$

because $E^{3,1}_{2,\mathbb{Z}} = H^3\big(\mathbb{Z}/2\mathbb{Z}, H^1(X,\mathbb{Z})\big) = 0$. Since

$$\mathbb{Z}/2\mathbb{Z} = E^{1,2}_{\infty,\mathbb{Z}} = \ker\left(d_3^{1,2} : E^{1,2}_{3,\mathbb{Z}} \to E^{4,0}_{3,\mathbb{Z}}\right),$$

we have only the following two options:

(i) $E^{1,2}_{2,\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $d_3^{1,2} \neq 0$,

(ii) $E^{1,2}_{2,\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$ and $d_3^{1,2} = 0$.

Now, we show that (ii) can not occur.

Lemma 16. *Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. Then the $d_3^{1,2}$ of the spectral sequence $E^{p,q}_{2,\mathbb{Z}}$ associated to the morphism $\pi : X \to Y$ is not* 0*.*

***Proof*.** First, we compute the term $E^{0,4}_{\infty,\mathbb{Z}}$. Since

$$E^{1,3}_{\infty,\mathbb{Z}} = E^{3,1}_{\infty,\mathbb{Z}} = 0,$$

$E^{4,0}_{2,\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$ and $E^{2,2}_{2,\mathbb{Z}}$ is a torsion group, one finds

$$E^{0,4}_{\infty,\mathbb{Z}} = \mathbb{Z}.$$

Suppose that $d_3^{1,2} = 0$. Since X is a K3 surface,

$$E^{0,3}_{2,\mathbb{Z}} = H^0\big(\mathbb{Z}/2\mathbb{Z}, H^3(X,\mathbb{Z})\big) = 0 \tag{12}$$

$$E^{2,1}_{2,\mathbb{Z}} = H^2\big(\mathbb{Z}/2\mathbb{Z}, H^1(X,\mathbb{Z})\big) = 0. \tag{13}$$

By definition of the terms of the spectral sequence,

$$E^{4,0}_{3,\mathbb{Z}} = \frac{E^{4,0}_{2,\mathbb{Z}}}{\operatorname{im}\left(d_2^{2,1} : E^{2,1}_{2,\mathbb{Z}} \to E^{4,0}_{2,\mathbb{Z}}\right)}$$

and by (13), $E^{4,0}_{3,\mathbb{Z}} = E^{4,0}_{2,\mathbb{Z}}$. Since $d_3^{1,2} = 0$,

$$E^{4,0}_{4,\mathbb{Z}} = \frac{E^{4,0}_{3,\mathbb{Z}}}{\operatorname{im}\left(d_3^{1,2} : E^{1,2}_{3,\mathbb{Z}} \to E^{4,0}_{3,\mathbb{Z}}\right)} = E^{4,0}_{3,\mathbb{Z}},$$

and finally by (12)

$$E^{4,0}_{\infty,\mathbb{Z}} = E^{4,0}_{5,\mathbb{Z}} = \frac{E^{4,0}_{4,\mathbb{Z}}}{\operatorname{im}\left(d_4^{0,3} : E^{0,3}_{4,\mathbb{Z}} \to E^{4,0}_{4,\mathbb{Z}}\right)} = E^{4,0}_{4,\mathbb{Z}}.$$

Hence we conclude $E^{4,0}_{\infty,\mathbb{Z}} = E^{4,0}_{2,\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$, a contradiction. ☑

4. More about the Morphism $\mathbf{Br}'(Y) \to \mathbf{Br}'(X)$

We recall the following two results due to Beauville:

Proposition 17. ([2, Cor. 5.6 and its proof]) *Let $\lambda = (\alpha, \alpha', \beta) \in H^2(X, \mathbb{Z})$ such that $\alpha, \alpha' \in E_8 \oplus U$ and $\beta \in U$ and ε the class of $e+f$ in $U_2 := U/2U$ where $\{e, f\}$ is the basis of the hyperbolic lattice U. Then the following conditions are equivalent:*

i) $\pi_*\lambda = 0$ *and* $\lambda \notin (1-\tau^*)\big(H^2(X,\mathbb{Z})\big)$;

ii) $\tau^*\lambda = -\lambda$ *and* $\lambda^2 \equiv 2 \bmod 4$.

iii) *the class* $\overline{\beta} = \varepsilon$ *and* $\alpha' = -\alpha$.

Corollary 18. $\pi : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ *is trivial if and only if there exists a line bundle L on X with $\tau^* L = L^\vee$ and $c_1(L)^2 \equiv 2 \bmod 4$.*

Now, we quickly recall a kind of divisors in the period domain Ω of $E_8(2) \oplus U(2)$-polarized marked K3 surfaces. If we fix the unique primitive embedding of $E_8(2) \oplus U(2)$ in the K3 lattice Λ, then Ω is by definition

$$\Omega := \Big\{[\omega] \in \mathbb{P}\Big(\big(E_8(2) \oplus U(2)\big)^{\perp}_{\mathbb{C}}\Big) : \omega^2 = 0,\ \omega\overline{\omega} > 0\Big\}.$$

Let $S \subset \Lambda$ be a primitive sublattice of rank 11 containing the lattice $E_8(2) \oplus U(2)$. Then the subset

$$\Omega(S) := \Big\{[\omega] \in \mathbb{P}\big(S^{\perp}_{\mathbb{C}}\big) : \omega^2 = 0,\ \omega\overline{\omega} > 0\Big\}$$

is called the Heegner divisor of type S in Ω.

Proposition 19. ([9, Proposition. 3.1]) *If X corresponds to a very general point of $\Omega(S)$, i.e. in the complement of a union of countably many proper closed analytic subset of $\Omega(S)$, then we have $\mathrm{NS}(X) = S$.*

***Remark* 20.** Ohashi proved in [9, Theorem. 3.4], that for a lattice $S = U(2) \oplus E_8(2) \oplus \langle -2N \rangle$ with $N \equiv 0 \bmod 4$, there exists a K3 surface X with an Enriques quotient and such that $\mathrm{NS}(X) = S$.

Example 21. Now, we will show the existence of a K3 surface X covering an Enriques surface Y with $\rho(X) = 11$ and $E_2^{1,1} = 0$ which from (4) implies that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. Let $\alpha \in \Lambda$, defined by (see [7])

$$\alpha = \left(\sum_{i \text{ odd}} a_i e_i, -\sum_{i \text{ odd}} a_i e_i, 0, f_1 - f_2, -f_1 + f_2 \right),$$

where the $a_i's$ are integers. This is a primitive element, $\alpha = \beta - \rho(\beta)$ where

$$\beta = (a_1 e_1 + a_3 e_3, -a_5 e_5 - a_7 e_7, 0, f_1, f_2)$$

and

$$\alpha^2 = -4 \sum_{i \text{ odd}} a_i^2 = -4m.$$

Thus, $E_8(2) \oplus U(2) \oplus \alpha\mathbb{Z} \hookrightarrow E_8^{\oplus 2} \oplus U^{\oplus 3}$ is a primitive embedding (note that $E_8(2) \oplus U(2)$ diagonally embeds in $E_8^{\oplus 2} \oplus U^{\oplus 3}$). Note that by the Lagrange's four-square Theorem ([4, Proposition 17.7.1]), m can take any positive integer value. By Proposition 19 and Remark 20, there exists a K3 surface X with an Enriques quotient Y and such that

$$\mathrm{NS}(X) = E_8(2) \oplus U(2) \oplus \alpha\mathbb{Z}$$

and by [1, Lemma 19.1] there exists an isometry $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ such that $\phi \circ \tau^* = \rho \circ \phi$. Now, we take a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \phi^{-1}(\alpha)$. Then,

$$\begin{aligned} \alpha &= -\rho(\alpha) \\ &= -\rho\big(\phi\big(\phi^{-1}(\alpha)\big)\big) \\ &= -\phi\big(\tau^*\big(\phi^{-1}(\alpha)\big)\big) \\ &= -\phi\big(\tau^*\big(c_1(\mathcal{L})\big)\big) \\ &= -\phi\big(c_1(\tau^*\mathcal{L})\big) \\ &= \phi\big(c_1\big(\tau^*\mathcal{L}^\vee\big)\big). \end{aligned}$$

Then, from the injectivity of ϕ, it follows that

$$c_1(\tau^*\mathcal{L} \otimes \mathcal{L}) = 0,$$

and since X is a K3 surface we deduce

$$\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X,$$

i.e. $[\mathcal{L}] \in E_2^{1,1}$. Now, since $\alpha = \beta - \rho(\beta)$ and $E_2^{1,1} \subseteq E_{2,\mathbb{Z}}^{1,2}$ (Lemma 15), then $[\mathcal{L}] = 0$ in $E_2^{1,1}$.

Now, we show that for any line bundle $\mathcal{M}$ such that $\tau^*\mathcal{M} \otimes \mathcal{M} = \mathcal{O}_X$, there exists an integer n such that $\mathcal{M} = \mathcal{L}^{\otimes n}$. By construction of the above primitive embedding, we have that the action of τ^* on $E_8(2) \oplus U(2)$ is the identity. Thus, if $\mathcal{M}$ is a line bundle, it can be written as $\mathcal{M} = \mathcal{L}^{\otimes n} \otimes \mathcal{F}$ for some invariant line bundle $\mathcal{F}$. Hence

$$\mathcal{O}_X = \tau^*\mathcal{M} \otimes \mathcal{M} = \tau^*\mathcal{L}^{\otimes n} \otimes \tau^*\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{F} = \mathcal{F}^{\otimes 2}.$$

Hence $\mathcal{F} = \mathcal{O}_X$ because $\mathrm{Pic}(X)$ is torsion free and thus $\mathcal{M} = \mathcal{L}^{\otimes n}$. Thus, we have shown that $E_2^{1,1} = 0$.

Example 22. Let E_1, E_2 be elliptic curves over k (a field of characteristic 0) which are not isogeneous over $\overline{k}$ and such that their points of order 2 are defined over k. For $i = 1, 2$, let D_i be a principal homogeneous space of E_i whose class in $H^1\big(Gal(\overline{k}/k), E_i\big)$ has order 2. The antipodal involution $P \mapsto -P$ defines an involution on D_1 and on D_2, and defines a Kummer surface X by considering the minimal desingularization of the quotient of $D_1 \times D_2$ by the simultaneous antipodal involution. Since X is a Kummer surface, it covers an Enriques surface Y. Harari and Skorobogatov were able to prove that for this example the morphism $\pi^* : \mathrm{Br}'\left(\overline{Y}\right) \to \mathrm{Br}'\left(\overline{X}\right)$ is injective (see [3, Corollary 2.8]) where $\overline{X}$ and $\overline{Y}$ are the surfaces over $\overline{k}$ obtained from X and Y respectively by extending the ground field from k to $\overline{k}$. We also know from Corollary 4.4 in [6] that $\rho\big(\overline{X}\big) \geq 17$ because X is a Kummer surface.

Let $\pi : X \to Y$ be the universal covering map of an Enriques surface Y and let τ be the fixed point free involution of X associated to π. We proceed to study how τ acts on $H^2\big(X, \mathcal{O}_X^*\big)$ and on $H^3\big(X, \mathcal{O}_X^*\big)$.

Lemma 23. *Let X be a K3 surface with a fixed point free involution τ. The involution τ acts on $H^2\big(X, \mathcal{O}_X^*\big)$ as $\tau^*\alpha = \alpha^{-1}$.*

Proof. The involution τ acts on $H^2(X, \mathcal{O}_X)$ as $-\mathrm{id}$. Indeed, since $H^2\big(X, \mathcal{O}_X\big)$ is one dimensional then the action τ on this is $\pm\mathrm{id}$. If θ is a 2−form and $\tau^*\theta = \theta$, the form descends to a 2−form on $Y := X/\tau$. This is a contradiction because for any Enriques surface, $h^{0,2}(Y) = 0$. From the exponential sequence we get

$$\begin{array}{ccc} H^2\big(X, \mathcal{O}_X\big) & \twoheadrightarrow & H^2\big(X, \mathcal{O}_X^*\big) \\ {\scriptstyle -\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \tau^*} \\ H^2\big(X, \mathcal{O}_X\big) & \twoheadrightarrow & H^2\big(X, \mathcal{O}_X^*\big) \end{array}$$

Hence for every $\alpha \in H^2\big(X, \mathcal{O}_X^*\big)$, $\tau^*\alpha = \alpha^{-1}$. ☑

Lemma 24. *Let X be a K3 surface. Any element in the Brauer group* $\mathrm{Br}'(X)$ *is* $2-$*divisible.*

Proof. From the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

we get

$$0 \to \mathrm{Br}'(X)_2 \to \mathrm{Br}'(X) \to \mathrm{Br}'(X) \to 0$$

because $H^3(X, \mathbb{Z}/2\mathbb{Z}) = 0$. ☑

***Remark* 25.** Let $\rho := \rho(X)$ denote the Picard number of a surface X. Let X be a K3 surface with an involution τ that has no fixed points. For any invariant line bundle L under τ, there is a line bundle M on the Enriques surface $Y := X/\tau$ such that $\pi^*M = L$. This is no longer true for Brauer classes. Indeed, by Lemma 23, the invariant elements of $\mathrm{Br}'(X)$ under τ consist of all the 2-torsion elements of $\mathrm{Br}'(X)$. Since X is a K3 surface, $\mathrm{Br}'(X)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Hence, since $\rho \leq 20$, there exists an element $\alpha \in \mathrm{Br}'(X)$ such $\tau^*\alpha = \alpha$ which is not in the image $\mathrm{im}\big(\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)\big)$. In conclusion, you may have picked α that happens to be in the image, but since $22 - \rho \geq 2$, there is always another one.

Now, let us compute some elements of the spectral sequence $E_2^{p,q}$ introduced in (1), associated to the universal covering map $\pi : X \to Y$ of an Enriques surface Y. First, we know from the exponential sequence that

$$H^3(Y, \mathcal{O}_Y^*) \cong H^4(Y, \mathbb{Z}) = \mathbb{Z}. \tag{14}$$

***Remark* 26.** By Theorem 6,

$$E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) = \frac{\big\{L \in \mathrm{Pic}(X) : \tau^*L \otimes L^\vee = \mathcal{O}_X\big\}}{\big\{\tau^*M \otimes M : M \in \mathrm{Pic}(X)\big\}}$$

and

$$E_2^{1,2} = H^1\big(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathcal{O}_X^*)\big) = \frac{\big\{\alpha \in H^2\big(X, \mathcal{O}_X^*\big) : \tau^*(\alpha).\alpha = 1\big\}}{\big\{\tau^*(\beta).\beta^{-1} : \beta \in H^2\big(X, \mathcal{O}_X^*\big)\big\}}.$$

By Lemmas 23 and 24, $E_2^{1,2} = 0$. Now, if $L \in \mathrm{Pic}(X)$ with $\tau^*L \otimes L^\vee = \mathcal{O}_X$. Then $[L^{\otimes 2}] = [\tau^*(L) \otimes L]$, i.e. $[L]$ is a 2-torsion element in $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big)$.

Thus $E_2^{1,2}=0$, $E_2^{3,0}=\mathbb{Z}/2\mathbb{Z}$ (cf. (3)) and $E_2^{2,1}$ is a torsion group (by (26)). In conclusion, we get from the (14) which says that $E^3=\mathbb{Z}$, that

$$E_\infty^{0,3}=\mathbb{Z}, \tag{15}$$

$$E_\infty^{1,2}=E_\infty^{2,1}=E_\infty^{3,0}=0. \tag{16}$$

The action τ on $H^3\big(X,\mathcal{O}_X^*\big)=H^4(X,\mathbb{Z})=\mathbb{Z}$ is $\pm\,\mathrm{id}$. If $\tau^*=-\,\mathrm{id}$, then $E_2^{0,3}=H^0\big(\mathbb{Z}/2\mathbb{Z},H^3\big(X,\mathcal{O}_X^*\big)\big)=H^3\big(X,\mathcal{O}_X^*\big)^\tau=0$, but this contradicts the fact $E_\infty^{0,3}=\mathbb{Z}$. Thus, we have shown the following lemma. (Note that this lemma trivially follows only from the fact that $H^3\big(X,\mathcal{O}_X^*\big)=H^4(X,\mathbb{Z})=\mathbb{Z}$ and the action on the last cohomology group is id, but the computations above are needed).

Lemma 27. *Let X be a K3 surface with a fixed point free involution τ. Then the action of τ on $H^3\big(X,\mathcal{O}_X^*\big)$ is trivial.*

***Remark* 28.** Let L be a line bundle such that $\tau^*L\otimes L=\mathcal{O}_X$. Thus, $L^{\otimes 2}=L\otimes(\tau^*L)^\vee$, i.e. $[L]\otimes[L]=[L^{\otimes 2}]=0$ in $E_2^{1,1}=H^1(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X))$. Since

$$E_2^{0,2}=H^0\big(\mathbb{Z}/2\mathbb{Z},H^2\big(X,\mathcal{O}_X^*\big)\big)=H^2\big(X,\mathcal{O}_X^*\big)^\tau,$$

by Lemma 23, $E_2^{0,2}=\mathrm{Br}'(X)_2$. Indeed, if $\alpha\in\mathrm{Br}'(X)$ with $\tau^*\alpha=\alpha$, then by Lemma 23, $\alpha=\tau^*\alpha=\alpha^{-1}$, i.e. α is a 2−torsion element of $\mathrm{Br}'(X)$. On the other hand, if $\alpha\in\mathrm{Br}'(X)_2$, then by Lemma 23, $\alpha=\alpha^{-1}=\tau^*\alpha$. Finally, by Remark 25, $E_2^{0,2}=\mathrm{Br}'(X)_2=(\mathbb{Z}/2\mathbb{Z})^{22-\rho}$.

Since any element in $E_2^{1,1}$ is a 2-torsion element, we have only the following four cases:

i) $E_2^{1,1}=0$ or

ii) $E_2^{1,1}=\mathbb{Z}/2\mathbb{Z}$, $\quad d_2^{1,1}=\mathrm{id}$, $\quad$ i.e. $\quad E_\infty^{1,1}=0$ or

iii) $E_2^{1,1}=\mathbb{Z}/2\mathbb{Z}$, $\quad d_2^{1,1}=0$, $\quad$ i.e. $\quad E_\infty^{1,1}=\mathbb{Z}/2\mathbb{Z}$ or

iv) $E_2^{1,1}=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $\quad d_2^{1,1}\neq 0$, $\quad$ i.e. $\quad 0\to\mathbb{Z}/2\mathbb{Z}\to E_2^{1,1}\xrightarrow{d_2^{1,1}}E_2^{3,0}\to 0$.

Lemma 29. *Let Y be an Enriques surface, $\pi:X\to Y$ the universal covering map of Y and τ the fixed point free involution given by π. If $E_2^{1,1}=H^1\big(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X)\big)=0$. Then $E_2^{2,1}=H^2\big(\mathbb{Z}/2\mathbb{Z},\mathrm{Pic}(X)\big)=(\mathbb{Z}/2\mathbb{Z})^{20-\rho}$.*

***Proof*.** Since $E_2^{1,1}=0$,

$$E_3^{3,0}=\frac{E_2^{3,0}}{\mathrm{im}\left(d_2^{1,1}:E_2^{1,1}\to E_2^{3,0}\right)}=E_2^{3,0}=\mathbb{Z}/2\mathbb{Z}$$

and by (16)

$$0 = E_\infty^{3,0} = E_4^{3,0} = \frac{E_3^{3,0}}{\operatorname{im}\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right)}.$$

Thus $d_3^{0,2}$ is surjective. Since $E_2^{1,1} = 0$,

$$\mathbb{Z}/2\mathbb{Z} = E_\infty^{0,2} = E_4^{0,2} = \ker\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right), \tag{17}$$

and since $E_3^{3,0} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ and all elements in $E_2^{0,2}$ are 2-torsion,

$$E_3^{0,2} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{18}$$

By (16)

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)},$$

and thus the morphism $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ is surjective. Hence, by (17) and the fact that any element in $E_2^{0,2}$ is a $2-$torsion element (cf. Remark 28),

$$E_2^{0,2} = E_3^{0,2} \times E_2^{2,1}.$$

From $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$ (cf. Remark 28) and (18), $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$. ☑

Lemma 30. *Let Y be an Enriques surface, $\pi : X \to Y$ the universal covering map of Y and τ the fixed point free involution given by π. If $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{1,1} = 0$. Then $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$.*

Proof. Since $E_2^{1,1} \neq 0$ and $E_\infty^{1,1} = 0$, $\operatorname{im}\left(d_2^{1,1}\right) = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ (cf. (3)). Thus

$$E_3^{3,0} = \frac{E_2^{3,0}}{\operatorname{im}\left(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}\right)} = 0. \tag{19}$$

By Remark 26, any element in $E_2^{2,1}$ is $2-$torsion. Then there is an integer m such that $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^m$. By (16),

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\operatorname{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)}$$

and thus $\operatorname{im}\left(d_2^{0,2}\right) = (\mathbb{Z}/2\mathbb{Z})^m$. Hence

$$\ker\left(d_2^{0,2}\right) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}$$

because $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Since $E_\infty^{0,2} = \mathbb{Z}/2\mathbb{Z}$,

$$\mathbb{Z}/2\mathbb{Z} = E_\infty^{0,2} = E_4^{0,2} = \ker\left(d_3^{0,2} : \ker\left(d_2^{0,2}\right) \to E_3^{3,0}\right)$$

and from (19)

$$\mathbb{Z}/2\mathbb{Z} = \ker\left(d_2^{0,2}\right) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}$$

and so $m = 21 - \rho$. ☑

Lemma 31. *Let X be a K3 surface that covers an Enriques surface Y and such that its spectral sequence satisfies $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. Then $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$.*

Proof. By assumptions $d_2^{1,1}$ is trivial and so

$$E_3^{3,0} = \frac{E_2^{3,0}}{\mathrm{im}\left(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}\right)} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$$

and by definition

$$E_4^{3,0} = \frac{E_3^{3,0}}{\mathrm{im}\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right)}. \tag{20}$$

On the other hand,

$$0 = E_\infty^{0,2} = \ker\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right)$$

because $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. Hence $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0} = \mathbb{Z}/2\mathbb{Z}$ is injective and this and (20) imply the following equivalence:

$$E_3^{0,2} = \mathbb{Z}/2\mathbb{Z} \quad \text{if and only if} \quad E_\infty^{3,0} = E_4^{3,0} = 0. \tag{21}$$

By (16), $E_\infty^{3,0} = 0$. Thus, the equivalence (21) implies $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$. Since by Remark 26, any element in $E_2^{2,1}$ is a 2-torsion element, there exists an integer m such that $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^m$. By (16),

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\mathrm{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)},$$

and thus

$$\mathrm{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right) = (\mathbb{Z}/2\mathbb{Z})^m,$$

i.e. the map $d_2^{0,2}$ is surjective. Since $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$ (cf. Remark 28), $E_3^{0,2} = \ker\left(d_2^{0,2}\right)$, it yields from the surjectivity of $d_2^{0,2}$ that

$$E_3^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}.$$

Thus, $m = 21 - \rho$ because $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$. Hence

$$E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}.$$ ☑

Lemma 32. *Let Y be an Enriques surface and $\pi : X \to Y$ the universal covering map of Y such that $E_2^{1,1} = H^1\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^2$. Then $E_2^{2,1} = H^2\big(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Moreover $\rho(X) \geq 12$.*

***Proof*.** Since $E_2^{1,1} = (\mathbb{Z}/2\mathbb{Z})^2$ and $E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$, the map $d_2^{1,1} \neq 0$. Hence $E_\infty^{1,1} = E_3^{1,1} = \ker\left(d_2^{1,1}\right)$ is nontrivial, and it must be $\mathbb{Z}/2\mathbb{Z}$. By definition,

$$E_3^{3,0} = \frac{E_2^{3,0}}{\mathrm{im}\left(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}\right)} = 0 \tag{22}$$

and by (16)

$$E_\infty^{2,1} = E_3^{2,1} = \frac{E_2^{2,1}}{\mathrm{im}\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)} = 0. \tag{23}$$

Since $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$, then

$$0 = E_\infty^{0,2} = E_4^{0,2} = \ker\left(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}\right).$$

Thus, by (22), $E_3^{0,2} = 0$. By definition,

$$E_3^{0,2} = \ker\left(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}\right)$$

and then $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ is injective. Hence, by (23),

$$E_2^{2,1} = E_2^{0,2}.$$

Since

$$E_2^{0,2} = \mathrm{Br}'(X)_2 = (\mathbb{Z}/2\mathbb{Z})^{22-\rho},$$

one finds

$$E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}.$$

Since $E_2^{2,1}$ is a quotient of $\mathrm{Pic}(X)^\tau$ and thus of $\mathrm{Pic}(Y) = \mathbb{Z}^{10} \times \mathbb{Z}/2\mathbb{Z}$, one finds $22 - \rho \leq 10$ (note that $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Pic}(Y)$ goes to zero in $E_2^{2,1}$). Thus $\rho \geq 12$ ☑

In conclusion, by lemmas 29, 30, 31, 32 and the statement before Lemma 29, we only have the following four cases:

i) $E_2^{1,1} = 0, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$ or

ii) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_\infty^{1,1} = 0, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$ or

iii) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$ or

iv) $E_2^{1,1} = (\mathbb{Z}/2\mathbb{Z})^2, \quad E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}, \quad E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$.

Note that in the cases ii) and iii) we have that $\rho \geq 11$.

Theorem 33. *Let X be a K3 surface with a fixed point free involution τ and Picard number ρ such that $H^2\big(\mathbb{Z}/2\mathbb{Z}\operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Then the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial, where $Y := X/\langle\tau\rangle$.*

Proof. Since $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$, we are in case iv). Hence $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. By (4), the morphism $\pi : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial. ☑

Theorem 34. *Let X be a K3 surface with a fixed point free involution τ and Picard number ρ such that $H^2\big(\mathbb{Z}/2\mathbb{Z}\operatorname{Pic}(X)\big) = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$. Then the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is nontrivial, where $Y := X/\langle\tau\rangle$.*

Proof. Since $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$, we are in case i). Hence $E_\infty^{1,1} = 0$. By (4), the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. ☑

Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. We know that if X is as in the first case above, then $\rho(X) \geq 10$, and if X is one of the cases ii) or iii), then $\rho(X) \geq 11$ and if X is as in the case iv), then $\rho(X) \geq 12$. Thus, if $\rho(X) = 10$, the K3 surface X satisfies the conditions of the first case and we obtain $E_2^{1,1} = 0$. Hence, by (4), the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. This is another proof of the same result obtained before out Lemma 11.

Proposition 35. *Let X be a K3 cover of an Enriques surface Y such that $\rho(X) = 11$ and $\mathrm{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N\rangle$, where $N \geq 2$. Then $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective if and only if N is an even number.*

Proof. Note that $\mathrm{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N\rangle = \pi^*\,\mathrm{NS}(Y) \oplus \langle -2N\rangle$ (because, as in Example 12, $\Lambda^+ \cong U(2) \oplus E_8(2)$ and this is diagonally embedded in the K3 lattice), i.e. τ^* acts trivially on $U(2) \oplus E_8(2)$. Now, we show that τ acts as $-\,\mathrm{id}$ on $\langle -2N\rangle$. Let $L \in \mathrm{NS}(X)$ denote the generator of $\langle -2N\rangle$, i.e. $c_1^2(L) = -2N$. Thus,

$$\tau^* L = I \otimes L^{\otimes k} \tag{24}$$

for some integer k and invariant line bundle I and since τ is an involution:

$$\begin{aligned} L = \tau^*\tau^* L &= \tau^* I \otimes \tau^* L^{\otimes k} \\ &= I \otimes \tau^* L^{\otimes k} \\ &= I \otimes \big(I \otimes L^{\otimes k}\big)^{\otimes k} \\ &= I^{\otimes(k+1)} \otimes L^{\otimes k^2}. \end{aligned}$$

Hence

$$L^{\otimes(k^2-1)} \otimes I^{\otimes(k+1)} = \mathcal{O}_X \tag{25}$$

and we find that $L^{\otimes(k^2-1)}$ is an invariant line bundle. Thus,

$$\mathcal{O}_X = L^{\otimes(-k^2+1)} \otimes \tau^* L^{\otimes(k^2-1)} = \left(L^\vee \otimes \tau^* L\right)^{\otimes(k^2-1)}$$

and if $k \neq 1, -1$, then $\mathcal{O}_X = L^\vee \otimes \tau^* L$ (because $\mathrm{Pic}(X)$ is a free torsion group), i.e. L is an invariant line bundle which contradicts our assumption about L. If $k = 1$, then from (25) we get $I = \mathcal{O}_X$ and then by (24), L is an invariant bundle which contradicts our assumption on L. Thus $k = -1$ and from (25), $I = \mathcal{O}_X$ and from (24) we obtain $\tau^* L \otimes L = \mathcal{O}_X$, i.e. τ acts as $-\,\mathrm{id}$ in $\langle -2N \rangle$.

Now, we show that if M is a line bundle such that $\tau^* M \otimes M = \mathcal{O}_X$, then $M = L^{\otimes m}$ for some integer m. Indeed, if $M = L^{\otimes m} \otimes F$ where F is an invariant line bundle, then

$$\mathcal{O}_X = \tau^* M \otimes M = \tau^* L^{\otimes m} \otimes \tau^* F \otimes L^{\otimes m} \otimes F = F^{\otimes 2}.$$

Hence $F = \mathcal{O}_X$ because $\mathrm{Pic}(X)$ is torsion free and thus $M = L^{\otimes m}$.

Suppose that N is an even number and that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial. By Corollary 18, there exists a line bundle $M = L^{\otimes m}$ for some integer m such that $c_1(M)^2 \equiv 2 \bmod 4$. Thus $-2m^2 N \equiv 2 \bmod 4$, which implies that $m^2 N$ is an odd number and thus N is an odd number, a contradiction. On the other hand, let us suppose that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. By Corollary 18, $c_1^2(L) \not\equiv 2 \bmod 4$. Hence, $(1-N) \not\equiv 0 \bmod 2$ and thus N is an even number. ☑

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