Lecturas Matemáticas
Volumen **32** *(2011), páginas 5–22*
ISSN 0120–1980

Good λ Inequalities for Multilinear Integral Operators[1]

Lanzhe Liu
Changsha University of Science and Technology, Changsha, P. R. of China

Abstract. In this paper, some multilinear operators associated to certain sublinear integral operators are introduced. These operators include the important operators in harmonic analysis, such as Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator. The good λ inequalities for the multilinear operators are obtained. Using this result, the boundedness of the multilinear operators on Lebesgue spaces are obtained.

Key words and phrases. Multilinear operator; Littlewood-Paley operator; Marcinkiewicz operator; Bochner-Riesz operator; BMO function; Lipschitz function; Good λ inequality.

2010 AMS Mathematics Subject Classification. 42B20, 42B25.

Resumen. En este artículo se introducen algunos operadores asociados con ciertos operadores integrales sublineales. Estos operadores incluyen operadores importantes en análisis atmónico, tales como los operadores de Littlewood-Paley, los operadores de Marcinkiewicz y el operador de Bochner-Riesz. Se obtienen las buenas desigualdades λ para estos operadores multilineales. Usando este resultado, la acotación fe estos operadores multilineas se obtiene para los espacios de Lebesgue.

1. Introduction and Notations

As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [1]–[7], [15]. In [3], [6], Chen and Cohen proved the good λ inequalities for the multilinear singular integral operator, and the boundedness of the multilinear operators on Lebesgue spaces

[1]Supported by the NNSF (Grant: 10671117)

are obtained. It is known that the singular integral operators are the linear operators. The Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator are the important sublinear operators in harmonic analysis. The purpose of this paper is to introduce some multilinear operators associated to certain sublinear integral operators, the operators include the Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator. And we prove the good λ inequalities for the multilinear operators. Under this result, we get the boundedness of the multilinear operators on Lebesgue spaces.

First, let us introduce some notations (see [2], [8], [15], [16]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [8][16])

$$f^{\#}(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. For $0 < \beta < 1$, the Lipschitz space $\dot{\wedge}_\beta$ is the space of functions f such that

$$||f||_{\dot{\wedge}_\beta} = \sup_{\substack{x,\, h \in R^n \\ h \neq 0}} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [15]). Set, for $1 \le p < \infty$ and $0 \le \mu < n$,

$$M_{\mu,p}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-p\mu/n}} \int_Q |f(y)|^p dy \right)^{1/p},$$

we denote $M_r(f) = M_{\mu,r}(f)$ if $\mu = 0$, which is the Hardy-Littlewood maximal function when $r = 1$.

2. Theorems

In this paper, we will study a class of multilinear operators associated to some integral operators, whose definition follows.

Let m_j be the positive integers$(j = 1, \cdots, l)$, $m_1 + \cdots + m_l = m$ and A_j be the functions on R^n $(j = 1, \cdots, l)$. Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Let $F_t(x-y)$ be defined on $R^n \times [0,+\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x-y)f(y)dy,$$

$$F_t^{A_1,\cdots,A_l}(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} F_t(x-y)f(y)dy$$

and

$$F_{t,\varepsilon}^{A_1,\cdots,A_l}(f)(x) = \int_{|x-y|>\varepsilon} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} F_t(x-y)f(y)dy$$

for every bounded and compactly supported function f. Let $H, ||\cdot||$ be the Banach space $H = \{h : ||h|| < \infty\}$ of functions h, and assume that for any fixed $x \in R^n$, $F_t(f)(x)$, $F_{t,\varepsilon}^{A_1,\cdots,A_l}(f)(x)$ and $F_t^{A_1,\cdots,A_l}(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators associated to F_t are defined by

$$T^{A_1,\cdots,A_l}(f)(x) = ||F_t^{A_1,\cdots,A_l}(f)(x)||,$$

$$T_\varepsilon^{A_1,\cdots,A_l}(f)(x) = ||F_{t,\varepsilon}^{A_1,\cdots,A_l}(f)(x)||$$

and $T_\star^{A_1,\cdots,A_l}(f)(x) = \sup_{\varepsilon>0} T_\varepsilon^{A_1,\cdots,A_l}(f)(x)$, where F_t satisfies: for fixed $\delta > 0$,

$$||F_t(x-y)|| \le C|x-y|^{-n}$$

and

$$||F_t(y-x) - F_t(z-x)|| \le C|y-z|^\delta |x-z|^{-n-\delta}$$

if $2|y-z| \le |x-z|$. We define $T(f)(x) = ||F_t(f)(x)||$.

Note that when $m = 0$, $T^{A_1,\cdots,A_l}$ is just the commutator of T and $A_1,\cdots,A_l$ (see [1], [10]–[13], [18]). While, when $m > 0$, it is a non-trivial generalization of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3]–[7]). The purpose of this paper is to prove the good λ inequalities for the multilinear operators $T_\star^{A_1,\cdots,A_l}$; using this result, the boundedness for the multilinear operators $T^{A_1,\cdots,A_l}$ on Lebesgue spaces are obtained. In Section 4, some applications of Theorems in this paper are given.

Now we state our results as follows:

Theorem 1. *Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$.*

(a) *Suppose* $1 < r < p < \infty$, *then there exist* $\gamma_0 > 0$ *such that, for any* $0 < \gamma < \gamma_0$ *and* $\lambda > 0$,

$$\Bigg|\Bigg\{x \in R^n : T_\star^{A_1,\cdots,A_l}(f)(x) > 3\lambda, \prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_p(f)(x) \le \gamma\lambda\Bigg\}\Bigg| \le C\gamma^r|\{x \in R^n : T_\star^{A_1,\cdots,A_l}(f)(x) > \lambda\}|;$$

(b) $T^{A_1,\cdots,A_l}$ *is bounded on* $L^p(R^n)$ *for* $1 < p < \infty$.

Theorem 2. *Let* $0 < \beta < 1$ *and* $D^\alpha A_j \in \dot{\Lambda}_\beta$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \cdots, l$.

(1) *Suppose* $1 < r < p < \infty$, *then there exist* $\gamma_0 > 0$ *such that, for any* $0 < \gamma < \gamma_0$ *and* $\lambda > 0$,

$$\Bigg|\Bigg\{x \in R^n : T_\star^{A_1,\cdots,A_l}(f)(x) > 3\lambda, \prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{l\beta,p}(f)(x) \le \gamma\lambda\Bigg\}\Bigg| \le C\gamma^r|\{x \in R^n : T_\star^{A_1,\cdots,A_l}(f)(x) > \lambda\}|;$$

(2) $T^{A_1,\cdots,A_l}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1 < p < n/l\beta$ *and* $1/p - 1/q = l\beta/n$.

3. Proofs of Theorems

To prove the theorem, we need the following lemmas.

Lemma 1 (see [15]). *Let* $0 < \beta < 1, 1 \le p \le \infty$, *then*

$$||b||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x) - b_Q|dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^p dx\right)^{1/p}.$$

Lemma 2 (see [6]). Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $Q(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 3 (see [2]). Let $0 \le \mu < n$, $1 \le r < p < n/\mu$ and $1/q = 1/p - \mu/n$, then

$$||M_{\mu,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Proof of Theorem 1(a). Without loss of generality, we may assume $l = 2$. By the Whitney decomposition, $\{x \in R^n : T_\star^{A_1,A_2}(f)(x) > \lambda\}$ may be written as a union of cubes $\{Q_k\}$ with mutually disjoint interiors and with distance from each to $R^n \setminus \bigcup_k Q_k$ comparable to the diameter of Q_k. It suffices to prove the good λ estimate for each Q_k. There exists a constant $C = C(n)$ such that for each k, the cube $\tilde{Q}_k$ intersects $R^n \setminus \bigcup_k Q_k$, where $\tilde{Q}_k$ denotes the cube with the same center as Q_k and with the diam $\tilde{Q}_k = C$ diam Q_k. Then, for each k, there exists a point $x_0 = x_0(k) \in \tilde{Q}_k$ such that

$$T_\star^{A_1,A_2}(f)(x_0) \le \lambda.$$

Now, we fix a cube Q_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}\right) M_p(f)(z) \le \gamma\lambda.$$

Set $\overline{Q}_k = \tilde{\tilde{Q}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{Q}_k}$ and $f_2 = f\chi_{R^n\setminus\overline{Q}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1**.** Choose $\varphi \in C^\infty$ such that $\varphi(x) \equiv 1$ for $x \in \overline{Q}_k$, $\varphi(x) \equiv 0$ for $x \notin \overline{\overline{Q}}_k$, $|\varphi(x)| \le 1$ for all x, and $|\varphi(x)| \le C(\text{diam}\overline{\overline{Q}}_k)^{-|\alpha|}$ for any multiindex α with $|\alpha| \le m$. Define

$$A_1^\varphi(y) = R_{m_1}\left(A_1(\cdot) - \sum_{|\alpha|=m_1} \frac{1}{\alpha!}(D^\alpha A_1)_{Q_k}(\cdot)^\alpha; y, z\right) \cdot \varphi(y)$$

and

$$A_2^\varphi(y) = R_{m_2}\left(A_2(\cdot) - \sum_{|\alpha|=m_2} \frac{1}{\alpha!}(D^\alpha A_2)_{Q_k}(\cdot)^\alpha; y, z\right) \cdot \varphi(y).$$

Then, for $x \in Q_k$,

$$T_\star^{A_1,A_2}(f_1)(x) = T_\star^{A_1^\varphi,A_2^\varphi}(f_1)(x).$$

Similar to the proof in [6], we obtain

$$||D^\alpha A_1^\varphi||_{L^q} \le C \sum_{|\alpha|=m_1} ||D^\alpha A_1||_{BMO}|\overline{\overline{Q}}_k|^{1/q} \text{ for } |\alpha| = m_1,$$

$$||D^\alpha A_2^\varphi||_{L^q} \le C \sum_{|\alpha|=m_2} ||D^\alpha A_2||_{BMO}|\overline{\overline{Q}}_k|^{1/q} \text{ for } |\alpha| = m_2$$

and

$$||T_\star^{A_1,A_2}(f)||_{L^r} \le \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j^\varphi||_{L^q}\right)||f||_{L^p}$$

for $1/r = 1/p + 2/q < 1$, thus, for $\eta > 0$,

$$|\{x \in R^n : T_\star^{A_1,A_2}(f_1)(x) > \eta\lambda\}| = |\{x \in R^n : T_\star^{A_1^\varphi,A_2^\varphi}(f_1)(x) > \eta\lambda\}|$$

$$\begin{aligned} &\le (\eta\lambda)^{-r}\left|\left|T_\star^{A_1^\varphi,A_2^\varphi}(f_1)\right|\right|_{L^r}^r \le C(\eta\lambda)^{-r}\left[\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j^\varphi||_{L^q}\right)||f_1||_{L^p}\right]^r \\ &\le C(\eta\lambda)^{-r}\left[\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_p(f)(z)\right]^r|\overline{\overline{Q}}_k|^{r(1/p+2/q)} \\ &\le C(\eta\lambda)^{-r}(\gamma\lambda)^r|\overline{\overline{Q}}_k| \le C(\gamma/\eta)^r|Q_k|. \end{aligned}$$

The estimates on f_2. Let $H = H(n)$ be a large positive integer depending only on n. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{Q}_k) \le \varepsilon \le H\mathrm{diam}(\tilde{Q}_k)$. Let

$$A_1^k(x) = A_1(x) - \sum_{|\alpha|=m_1}\frac{1}{\alpha!}(D^\alpha A_1)_{Q_k}\cdot x^\alpha$$

and

$$A_2^k(x) = A_2(x) - \sum_{|\alpha|=m_2}\frac{1}{\alpha!}(D^\alpha A_2)_{Q_k}\cdot x^\alpha,$$

then $F_{t,\varepsilon}^{A_1,A_2}(f_2)(x) = F_{t,\varepsilon}^{A_1^k,A_2^k}(f_2)(x)$. Set

$$K_t(x,y) = \frac{F_t(x-y)}{|x-y|^m}\prod_{j=1}^{2}R_{m_j+1}(A_j^k;x,y).$$

Choose $x_0 \in \tilde{Q}_k$ with $x_0 \in R^n \setminus \bigcup_k Q_k$. Following [6], we have, for $x \in Q_k$,

$$\begin{aligned} &F_{t,\varepsilon}^{A_1^k,A_2^k}(f)(x)| = \int_{|x-y|>\varepsilon}(K_t(x,y) - K_t(x_0,y))f(y)dy \\ &+\int_{R(x)}K_t(x_0,y)f(y)dy + \int_{R(x_0)}K_t(x_0,y)f(y)dy + F_{t,\varepsilon}^{A_1,A_2}(f)(x_0) \\ = \;& I + II + III + IV. \end{aligned}$$

where $R(u) = \{y \in R^n : \text{diam}(\tilde{Q}_k) < |u-y| \le h\text{diam}(\tilde{Q}_k)\}$. Now, let us treat I, II and III, respectively. For I, we write

$$\begin{aligned}
I = & \int_{|x-y|>\varepsilon} \left(\frac{F_t(x-y)}{|x-y|^m} - \frac{F_t(x_0-y)}{|x_0-y|^m} \right) \prod_{i=1}^{2} R_{m_j}(A_j^k; x, y) f(y) dy \\
& + \int_{|x-y|>\varepsilon} \left(R_{m_1}(A_1^k; x, y) - R_{m_1}(A_1^k; x_0, y) \right) \frac{F_t(x_0-y)}{|x_0-y|^m} R_{m_2}(A_2^k; x, y) f(y) dy \\
& + \int_{|x-y|>\varepsilon} \left(R_{m_2}(A_2^k; x, y) - R_{m_2}(A_2^k; x_0, y) \right) \frac{F_t(x_0-y)}{|x_0-y|^m} R_{m_1}(A_1^k; x_0, y) f(y) dy \\
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{|x-y|>\varepsilon} \left[\frac{R_{m_2}(A_2^k; x, y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x-y) \right. \\
& \qquad \left. - \frac{R_{m_2}(A_2^k; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} F_t(x_0-y) \right] \times D^{\alpha_1} A_1^k(y) f(y) dy \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{|x-y|>\varepsilon} \left[\frac{R_{m_1}(A_1^k; x, y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x-y) \right. \\
& \qquad \left. - \frac{R_{m_1}(A_1^k; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} F_t(x_0-y) \right] \times D^{\alpha_2} A_2^k(y) f(y) dy \\
& + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{|x-y|>\varepsilon} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} F_t(x-y) \right. \\
& \qquad \left. - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} F_t(x_0-y) \right] \times D^{\alpha_1} A_1^j(y) D^{\alpha_2} A_2^k(y) f(y) dy \\
& \qquad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

By Lemma 2 and the following inequality, for $b \in BMO$,

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|) ||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we get, for $\nu \ge 1$,

$$\begin{aligned}
|R_{m_j}(\tilde{A}_j; x, y)| & \le C|x-y|^{m_j} \sum_{|\alpha|=m_j} (||D^\alpha A_j||_{BMO} + |(D^\alpha A_j)_{Q(x,y)} - (D^\alpha A_j)_Q|) \\
& \le C\nu |x-y|^{m_j} \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}.
\end{aligned}$$

On the other hand, by the formula (see [6]):

$$R_{m_j}(A_j^k; x, y) - R_{m_j}(A_j^k; x_0, y) = \sum_{|\theta|<m_j} \frac{1}{\theta!} R_{m_j - |\theta|}(D^\theta A_j^k; x, x_0)(x-y)^\theta$$

and Lemma 2, we obtain, similar to the proof of [6],

$$||I_1|| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)$$
$$\times\sum_{\nu=1}^{\infty}\nu^2\int_{2^\nu\varepsilon<|x-y|\le 2^{\nu+1}\varepsilon}\left(\frac{|x-x_0|}{|x-y|^{n+1}}+\frac{|x-x_0|^\delta}{|x-y|^{n+\delta}}\right)|f(y)|dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)$$
$$\times\sum_{\nu=1}^{\infty}\nu^2(2^{-\nu}+2^{-\nu\delta})\left(\frac{1}{(2^{\nu+1}\varepsilon)^n}\int_{|x-y|\le 2^{\nu+1}\varepsilon}|f(y)|dy\right)$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_p(f)(z)\le C\gamma\lambda;$$

$$||I_2|| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{\nu=1}^{\infty}\nu^2\int_{2^\nu\varepsilon<|x-y|\le 2^{\nu+1}\varepsilon}\frac{|x-x_0|}{|x-y|^{n+1}}|f(y)|dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_p(f)(z)\le C\gamma\lambda; ||I_3|| \le C\gamma\lambda;$$

$$||I_4|| \le C\sum_{|\alpha_1|=m_1}\int_{|x-y|>\varepsilon}\left|\left|\frac{(x-y)^{\alpha_1}F_t(x-y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0-y)}{|x_0-y|^m}\right|\right|$$
$$\times|R_{m_2}(A_2^k;x,y)||D^{\alpha_1}A_1^k(y)||f(y)|dy$$
$$+C\sum_{|\alpha_1|=m_1}\int_{|x-y|>\varepsilon}|R_{m_2}(A_2^k;x,y)-R_{m_2}(A_2^k;x_0,y)|\frac{||(x_0-y)^{\alpha_1}F_t(x_0-y)||}{|x_0-y|^m}|$$
$$\times D^{\alpha_1}A_1^k(y)||f(y)|dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_p(f)(z)\le C\gamma\lambda;$$

$$||I_5|| \le C\gamma\lambda;$$

$$||I_6|| \le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{|x-y|>\varepsilon}\left|\left|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x-y)}{|x-y|^m}\right.\right.$$
$$\left.\left.-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0-y)}{|x_0-y|^m}\right|\right||D^{\alpha_1}A_1^k(y)||D^{\alpha_2}A_2^k(y)||f(y)|dy\le C\gamma\lambda,$$

thus, $||I|| \le C\gamma\lambda$.

For II and III, note that, for $y \in R(x)$,

$$|x-y| \le H\text{diam}(\tilde{Q}_k),$$

we get, similar to the proof of [6],

$$\begin{aligned}
||II|| \le & \int_{R(x)} \frac{||F_t(x_0-y)||}{|x_0-y|^m} \prod_{j=1}^{2} |R_{m_j}(A_j^k;x,y)||f(y)|dy \\
& + C \sum_{|\alpha_1|=m_1} \int_{R(x)} \frac{|R_{m_2}(A_2^k;x_0,y)|}{|x_0-y|^{m_2}} ||F_t(x_0-y)|||D^{\alpha_1}A_1^k(y)||f(y)|dy \\
& + C \sum_{|\alpha_2|=m_2} \int_{R(x)} \frac{|R_{m_1}(A_1^k;x_0,y)|}{|x_0-y|^{m_1}} ||F_t(x_0-y)|||D^{\alpha_2}A_2^k(y)||f(y)|dy \\
& + C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_{|x-y|>\varepsilon} ||F_t(x_0-y)|||D^{\alpha_1}A_1^k(y)||D^{\alpha_2}A_2^k(y)||f(y)|dy \\
& \le C\gamma\lambda;
\end{aligned}$$

$$||III|| \le C\gamma\lambda.$$

For IV, since $x \notin \bigcup_k Q_k$, we have $||IV|| \le \lambda$. Thus, for $x \in Q_k$,

$$\sup_{\varepsilon \approx \text{diam}(\tilde{Q}_k)} |T_\varepsilon^{A_1,A_2}(f_2)(x)| \le C\gamma\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{Q}_k)$. Let Q_k^ε denote the cube with the same center as Q_k and with the diam $Q_k^\varepsilon = \varepsilon$. Set

$$A_1^\varepsilon(x) = A_1(x) - \sum_{|\alpha|=m_1} \frac{1}{\alpha!}(D^\alpha A_1)_{Q_k^\varepsilon} \cdot x^\alpha$$

and

$$A_2^\varepsilon(x) = A_2(x) - \sum_{|\alpha|=m_2} \frac{1}{\alpha!}(D^\alpha A_2)_{Q_k^\varepsilon} \cdot x^\alpha,$$

then, similar to the proof of **Case 1**, to get

$$\sup_{\varepsilon > H\text{diam}(\tilde{Q}_j)} |T_\varepsilon^{A_1,A_2}(f_2)(x)| \le C\gamma\lambda + \lambda.$$

Thus, we have shown that for $x \in Q_k$,

$$T_\star^{A_1,A_2}(f_2)(x) \le C\gamma\lambda + \lambda.$$

Now, choose γ_0 such that $C\gamma_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, to get

$$\begin{aligned}\Bigg|\Bigg\{x \in Q_k : T_\star^{A_1,A_2}(f)(x) > 3\lambda,\\ \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) M_p(f)(x) \le \gamma\lambda\Bigg\}\Bigg|\\ \le |\{x \in Q_k : T_\star^{A_1,A_2}(f_1)(x) > 2\lambda - C\gamma\lambda\}|\\ + |\{x \in Q_k : T_\star^{A_1,A_2}(f_2)(x) > \lambda + C\gamma\lambda\}|\\ \le C\gamma^r|\{x \in Q_k : T_\star^{A_1,A_2}(f_1)(x) > \lambda\}| \le C\gamma^r|Q_k|.\end{aligned}$$

(b) follows from **(a)** and Lemma 3. This completes the proof of Theorem 1.

Proof of Theorem 2(1). Without loss of generality, we may assume $l = 2$. By the Whitney decomposition, $\{x \in R^n : T_\star^{A_1,A_2}(f)(x) > \lambda\}$ may be written as a union of cubes $\{Q_k\}$ with mutually disjoint interiors and with distance from each to $R^n \setminus \bigcup_k Q_k$ comparable to the diameter of Q_k. It suffices to prove the good λ estimate for each Q_k. There exists a constant $C = C(n)$ such that for each k, the cube $\tilde{Q}_k$ intersects $R^n \setminus \bigcup_k Q_k$, where $\tilde{Q}_k$ denotes the cube with the same center as Q_k and with the diam $\tilde{Q}_k = C$ diam Q_k. Then, for each k, there exists a point $x_0 = x_0(k) \in \tilde{Q}_k$ such that

$$T_\star^{A_1,A_2}(f)(x_0) \le \lambda.$$

Now, we fix a cube Q_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right) M_{2\beta,p}(f)(z) \le \gamma\lambda.$$

Set $\overline{Q}_k = \tilde{\tilde{Q}}_j$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{Q}_j}$ and $f_2 = f\chi_{R^n\setminus\overline{Q}_j}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1. Choose $\varphi \in C^\infty$ such that $\varphi(x) \equiv 1$ for $x \in \overline{Q}_j$, $\varphi(x) \equiv 0$ for $x \notin \overline{\overline{Q}}_j$, $|\varphi(x)| \le 1$ for all x, and $|\varphi(x)| \le C(\mathrm{diam}\overline{\overline{Q}}_j)^{-|\alpha|}$ for any multiindex α with $|\alpha| \le m$. Define

$$A_1^\varphi(y) = R_{m_1}\left(A_1(\cdot) - \sum_{|\alpha|=m_1}\frac{1}{\alpha!}(D^\alpha A_1)_{Q_k}(\cdot)^\alpha; y, z\right)\cdot\varphi(y)$$

and

$$A_2^\varphi(y) = R_{m_2}\left(A_2(\cdot) - \sum_{|\alpha|=m_2}\frac{1}{\alpha!}(D^\alpha A_2)_{Q_k}(\cdot)^\alpha; y, z\right)\cdot\varphi(y).$$

Then, for $x \in Q_k$,

$$T_\star^{A_1,A_2}(f_1)(x) = T^{A_1^\varphi,A_2^\varphi}(f_1)(x).$$

Similar to the proof in [3][6], we obtain

$$||D^\alpha A_1^\varphi||_{L^q} \le C \sum_{|\alpha|=m_1} ||D^\alpha A_1||_{\dot\wedge_\beta} |\overline{\overline{Q}}_k|^{\beta/n+1/q} \text{ for } |\alpha| = m_1$$

and

$$||D^\alpha A_2^\varphi||_{L^q} \le C \sum_{|\alpha|=m_2} ||D^\alpha A_2||_{\dot\wedge_\beta} |\overline{\overline{Q}}_k|^{\beta/n+1/q} \text{ for } |\alpha| = m_2.$$

Taking $1/r = 1/p + 2/q < 1$, we get, for $\eta > 0$,

$$\begin{aligned}
&|\{x \in R^n : T_\star^{A_1,A_2}(f_1)(x) > \eta\lambda\}| = |\{x \in R^n : T_\star^{A_1^\varphi,A_2^\varphi}(f_1)(x) > \eta\lambda\}| \\
\le\ & C(\eta\lambda)^{-r} \left|\left|T_\star^{A_1^\varphi,A_2^\varphi}(f_1)\right|\right|_{L^r}^r \le C(\eta\lambda)^{-r} \left[\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j^\varphi||_{L^q}\right) ||f_1||_{L^p}\right]^r \\
\le\ & C(\eta\lambda)^{-r} \left[\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{\dot\wedge_\beta}\right) M_{2\beta,p}(f)(z)\right]^r |\overline{\overline{Q}}_k|^{r(1/p+2/q)} \\
\le\ & C(\eta\lambda)^{-r}(\gamma\lambda)^r |\overline{\overline{Q}}_k| \le C(\gamma/\eta)^r |Q_k|.
\end{aligned}$$

The estimates on f_2. Let $H = H(n)$ be a large positive integer depending only on n. We consider the following two cases:

Case 1. $\operatorname{diam}(\tilde{Q}_k) \le \varepsilon \le H\operatorname{diam}(\tilde{Q}_k)$. Let

$$A_1^k(x) = A_1(x) - \sum_{|\alpha|=m_1} \frac{1}{\alpha!}(D^\alpha A_1)_{Q_k} \cdot x^\alpha$$

and

$$A_2^k(x) = A_2(x) - \sum_{|\alpha|=m_2} \frac{1}{\alpha!}(D^\alpha A_2)_{Q_k} \cdot x^\alpha,$$

then $T_\varepsilon^{A_1,A_2}(f_2)(x) = T_\varepsilon^{A_1^k,A_2^k}(f_2)(x)$. Set

$$K_t(x,y) = \frac{F_t(x,y)}{|x-y|^m} \prod_{j=1}^2 R_{m_j+1}(A_j^k; x, y).$$

Choose $x_0 \in \tilde{Q}_k$ with $x_0 \in R^n \setminus \bigcup_k Q_k$. Following [6], we write, for $x \in Q_k$,

$$\begin{aligned}
& F_{t,\varepsilon}^{A_1^k,A_2^k}(f)(x)| = \int_{|x-y|>\varepsilon} (K_t(x,y) - K_t(x_0,y))f(y)dy \\
& + \int_{R(x)} K_t(x_0,y)f(y)dy + \int_{R(x_0)} K_t(x_0,y)f(y)dy + F_{t,\varepsilon}^{A_1,A_2}(f)(x_0) \\
=\ & J + JJ + JJJ + JJJJ,
\end{aligned}$$

where $R(u) = \{y \in R^n : \text{diam}(\tilde{Q}_k) < |u - y| \le h\text{diam}(\tilde{Q}_k)\}$. Now, let us treat J, JJ and JJJ, respectively. By Lemma 2 and the following inequality, for $b \in \dot{\Lambda}_\beta$ and the cube $Q = Q(x_0, d)$,

$$|b(x) - b_Q| \le \frac{1}{|Q|}\int_Q ||b||_{\dot{\Lambda}_\beta}|x-y|^\beta dy \le ||b||_{\dot{\Lambda}_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_{m_j}(A_j^k; x, y)| \le \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{\dot{\Lambda}_\beta}(|x-y|+d)^{m_j+\beta}.$$

On the other hand, by the formula (see [6]):

$$R_{m_j}(A_j^k; x, y) - R_{m_j}(A_j^k; x_0, y) = \sum_{|\theta|<m_j} \frac{1}{\theta!} R_{m_j-|\theta|}(D^\theta A_j^k; x, x_0)(x-y)^\theta$$

and Lemma 2, we obtain, similar to the proof of [3][6] and Theorem 1,

$$\begin{aligned}
||J|| \le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{\dot{\Lambda}_\beta}\right) |\tilde{Q}_k|^{2\beta/n} \\
\times \sum_{\nu=1}^\infty \nu^2 \int_{2^\nu\varepsilon<|x-y|\le 2^{\nu+1}\varepsilon} \left(\frac{|x-x_0|}{|x-y|^{n+1}} + \frac{|x-x_0|^\delta}{|x-y|^{n+\delta}}\right)|f(y)|dy \\
\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{\dot{\Lambda}_\beta}\right) \\
\times \sum_{\nu=1}^\infty \nu^2 (2^{-(1+2\beta)\nu} + 2^{-(\delta+2\beta)\nu}) \left(\frac{1}{(2^{\nu+1}\varepsilon)^{n-2\beta}} \int_{|x-y|\le 2^{\nu+1}\varepsilon} |f(y)|dy\right) \\
\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{\dot{\Lambda}_\beta}\right) M_{2\beta,p}(f)(z) \le C\gamma\lambda.
\end{aligned}$$

For JJ and JJJ, note that, for $y \in R(x)$,

$$|x-y| \le H\text{diam}(\tilde{Q}_j),$$

we get, similar to the proof of [3], [6],

$$\begin{aligned}
||JJJ|| &\leq \int_{R(x)} \frac{||F_t(x_0-y)||}{|x_0-y|^m} \prod_{j=1}^{2} |R_{m_j}(A_j^k; x, y)||f(y)|dy \\
&\quad + C \sum_{|\alpha_1|=m_1} \int_{R(x)} \frac{|R_{m_2}(A_2^k; x_0, y)|}{|x_0-y|^{m_2}} ||F_t(x_0-y)|||D^{\alpha_1} A_1^k(y)||f(y)|dy \\
&\quad + C \sum_{|\alpha_2|=m_2} \int_{R(x)} \frac{|R_{m_1}(A_1^k; x_0, y)|}{|x_0-y|^{m_1}} ||F_t(x_0-y)|||D^{\alpha_2} A_2^k(y)||f(y)|dy \\
&\quad + C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_{|x-y|>\varepsilon} ||F_t(x_0-y)|||D^{\alpha_1} A_1^k(y)||D^{\alpha_2} A_2^k(y)||f(y)|dy \\
&\leq C\gamma\lambda; \\
||JJJ|| &\leq C\gamma\lambda.
\end{aligned}$$

For $JJJJ$, since $x \notin \bigcup_k Q_k$, we have $||JJJJ|| \leq \lambda$. Then, for $x \in Q_k$,

$$\sup_{\varepsilon \approx \text{diam}(\tilde{Q}_k)} |T_\varepsilon^{A_1,A_2}(f_2)(x)| \leq C\gamma\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{Q}_k)$. Let Q_k^ε denote the cube with the same center as Q_k and with the diam $Q_k^\varepsilon = \varepsilon$. Set

$$A_1^\varepsilon(x) = A_1(x) - \sum_{|\alpha|=m_1} \frac{1}{\alpha!}(D^\alpha A_1)_{Q_k^\varepsilon} \cdot x^\alpha$$

and

$$A_2^\varepsilon(x) = A_2(x) - \sum_{|\alpha|=m_2} \frac{1}{\alpha!}(D^\alpha A_2)_{Q_k^\varepsilon} \cdot x^\alpha,$$

then, similar to the proof of **Case 1**, we get

$$\sup_{\varepsilon > H\text{diam}(\tilde{Q}_j)} |T_\varepsilon^{A_1,A_2}(f_2)(x)| \leq C\gamma\lambda + \lambda.$$

Thus, we have shown that for $x \in Q_k$,

$$T_\star^{A_1,A_2}(f_2)(x) \leq C\gamma\lambda + \lambda.$$

Now, choose γ_0 such that $C\gamma_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned}
&\left|\left\{x \in Q_k : T_\star^{A_1,A_2}(f)(x) > 3\lambda,\ \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta}\right) M_{2\beta,p}(f)(x) \leq \gamma\lambda\right\}\right| \\
\leq\ & |\{x \in Q_k : T_\star^{A_1,A_2}(f_1)(x) > 2\lambda - C\gamma\lambda\}| + |\{x \in Q_k : T_\star^{A_1,A_2}(f_2)(x) > \lambda + C\gamma\lambda\}| \\
\leq\ & C\gamma^r |\{x \in Q_k : T_\star^{A_1,A_2}(f_1)(x) > \lambda\}| \leq C\gamma^r |Q_k|.
\end{aligned}$$

(2) follows from **(c)** and Lemma 3. This completes the proof of Theorem 2.

4. Applications

Now we give some applications of Theorems in this paper.

Application 1. Littlewood-Paley operators. Fixed $\delta > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^\delta (1+|x|)^{-(n+1+\delta)}$ when $2|y| < |x|$;

We denote that $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y) f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [17]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad S_\psi(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$g_\mu^A(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^A(f)(x,y) \right|\right|,$$

$$g_\mu(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t(f)(y) \right|\right|.$$

It is easily to see that g_ψ^A, S_ψ^A and g_μ^A satisfy the conditions of Theorem 1 and 2 (see [10][12-13]), thus the conclusions of Theorem 1 and 2 hold for g_ψ^A, S_ψ^A and g_μ^A.

Application 2. *Marcinkiewicz operators.* Fixed Fix $\lambda > \text{máx}(1, 2n/(n+2))$ and $0 < \delta \leq 1$. Let Ω be homogeneous of degree zero on R^n with

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

Assume that $\Omega \in Lip_\delta(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators (see [18]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$\mu_\lambda^A(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x,y) \right|\right|,$$

$$\mu_\lambda(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right|\right|.$$

It is easily to see that μ_Ω^A, μ_S^A and μ_λ^A satisfy the conditions of Theorem 1 and 2 (see [10][12]), thus Theorem 1 and 2 hold for μ_Ω^A, μ_S^A and μ_λ^A.

Application 3. Bochner-Riesz operator. Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y)dy,$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator (see [14]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^A$ satisfies the conditions of Theorem 1 and 2, thus Theorem 1 and 2 hold for $B_{\delta,*}^A$.

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(Recibido en marzo de 2011. Aceptado para publicación en mayo de 2011)

LANZHE Z. LIU
COLLEGE OF MATHEMATICS
CHANGSHA UNIVERSITY OF SCIENCE AND TECHNOLOGY
CHANGSHA 410077, P.R. OF CHINA
e-mail: lanzheliu@163.com