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On the assessment of probabilities when new information is at glance: A Bayesian review of the Rapoport's and Marinoff's treatment of the Ace-deuce game

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Abstract. This paper provides a solution to the Ace-deuce game as described by Rapoport (1996) and Marinoff (1996). The solution follows the elucidation of the random phenomenon, its associated sample space, and the involvement of Bayesian Analysis in a fundamental way. It confronts Rapoport and Marinoff's views and argues the non-existence of paradoxical aspects in the game.

Key words and phrases. Ace-deuce Game, Bayesian Analysis, Sample Space, Relevant Space, Probability Paradox.

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Resumen. Se presenta una solución del juego del "as" y el "dos", en términos de los planteamientos controversiales de Rapoport (1996) y Marinoff (1996). La solución procede de la especificación del fenómeno aleatorio, el espacio muestral asociado, y el uso del enfoque Bayesiano de manera fundamental. Se examinan los detalles de la divergencia entre Rapoport y Marinoff y particularmente se argumenta la no existencia de aspectos paradójicos en este juego.

1. Introduction

This document is closely related to Rapoport's [4] and Marinoff's [3] papers concerning probabilistic assessment when new information is at glance. The Ace-deuce game, which constitutes the issue of discussion by the cited authors is brought up as the case study. Therefore, as in the above-mentioned papers, the conclusions derived from the present discussion will be mainly based on the analysis of this case. In the first part, a solution to the Ace-deuce game is exposed and some relevant aspects on probabilistic treatment for random phenomena are raised and emphasized. Further, some criticism to RAPOPORT's and MARINOFF's arguments is intended, which is useful and unavoidable in order to expose potential inadequacies of the subject of probabilistic assessment in the light of new information. From the discussion, major advantages in the explicit formulation of the sample space for a well-defined random experiment, as well as clear revenues from Bayesian analysis, are recognized.

2. **Discerning the Ace-deuce Game**

The Ace-deuce game, quoting RAPOPORT [4], is stated as follows:

> *Each of two players draws two cards from a deck consisting of four aces and four deuces. Aces are high, suits don't count. Player 1 wins only if his hand beats Player 2's. In case of a tie, Player 2 wins. Suppose Player 1 holds an ace and a deuce. Marinoff and I agree that Player 1 wins with probability* $1/5$*, since of the 15 possible hands held by Player 2, all equiprobable, three, consisting of a pair of deuces are beaten. Next suppose a Kibitzer, who sees Player 2's hand, informs Player 1 that Player 2 has a deuce. I maintain that now Player 1 wins with probability* $1/4$*. Marinoff disagrees".*

Indeed, MARINOFF's [3] assessment of probability to this problem is $2/5$. Now, to establish the context for discussion, a solution to the problem is proposed as follows: For the experiment of drawing four cards from a deck consisting of four aces and four deuces, the amount of different hands that can be obtained is:

$$\binom{8}{4} = \frac{8!}{4!(8-4)!}$$

So, there are 70 different hands of four cards that can be drawn without regard to the order in which they appear. Certainly, while the cards are taken at random, these hands compose a uniform probability space; in other words, a space in which all the points assume the same value of probability. Now, since the outcomes for the players are defined merely by the number of aces and deuces they get, this space of probability might be considered as appropriate whenever a distinction for the cards belonging to each player is made. However, this uniform probability- space reveals some limitations for the analysis that is intended here. (The reason of this statement is assumed to be evident in the sequel.) In contrast, it is preferable that the cards belonging to each player be associated to specific positions. Moreover, suit distinction will be considered for constructing events in the sample space.

Let x, y, z, w represent the outcomes of the experiment. The meaning of the letters is settled by:

x: 'Player 1's first card'

y: 'Player 1's second card'
z: 'Player 2's first card', and
w: 'Player 2's second card'.

The variables just defined are permitted to take values in the set

$$\{1, 2 \mid 1 : \text{ace}, 2 : \text{deuce}\}\,.$$

The total number of ways to distribute eight cards in four places (regarding the order in which they appear) is

$$(8)_4 = 8(8-1)(8-2)(8-3) = 1680$$

(e.g. HOEL [1] p. 29). The above mentioned ways compose a uniform probability space and stand, obviously, for the most explicit description of the experiment. All of the 1680 possibilities are grouped in events; these are summarized in Table 1.

The events are defined in accordance with the possible outcomes for both players; moreover, the events compose a partition of the sample space. In Table 1, the headings assume the following connotations: 'Hand' column describes the partition of the sample space; 'Outcomes' column shows the implication of each event for Player 1 (W denotes 'wins' and L means 'loses'); the column headed *Number of permutations* shows the calculations for the number of different hands defining each event with regard to 'Wins' and 'Loses'.

From the information provided in Table 1, the probability for Player 1 to win, denoted by $P(W1)$, is calculated by

$$P(W1) = P(w_2 \cup w_3 \cup w_6 \cup w_{12} \cup w_{13})\,,$$

where all of the w's are mutually disjoint and, consequently, the additive property of probability applies. This is,

$$P(W1) = P(w_2) + P(w_3) + P(w_6) + P(w_{12}) + P(w_{13})\,.$$

Substituting numerical values it yields

$$\begin{aligned} P(W1) &= \frac{96}{1860} + \frac{96}{1860} + \frac{144}{1860} + \frac{96}{1860} + \frac{96}{1860} \\ &= \frac{528}{1860} \approx 0.284\,. \end{aligned}$$

This value may be called the *a priori* estimate of player 1's probability to win. Now, if Player 1 is known to hold an ace and a deuce, then it is necessary to restrict the sample space to be consistent with the new player 1's condition (that is, new information at glance). Though, it should be noted that the cards in each player's hand come from a well-defined random experiment in which there are not stages or any restraint with respect to the process of picking cards. (Indeed, the experiment is taking four cards at random from a deck consisting of four aces and four deuces) Therefore, 'given Player 1 got an ace and a deuce', then there is a set of specific events assuming probability values above zero. In other words, the sample space claims to be restricted to the events shown in Table 2. This is indeed what RAPOPORT refers as 'Relevant Space'. In the 'restricted' sample space, as shown in Table 2, the calculation for Player 1's probability to win is:

$$P(W1) = P(w_{12} \cup +w_{13} = P(w_{12}) + P(w_{13}) = \frac{96}{960} + \frac{96}{960} = \frac{192}{960} = \frac{1}{5}\,.$$

As it has been asserted, the same probability value can be obtained by means of a conditional probability assignment or Bayesian type evaluation in which the values involved are taken directly from Table 1. This computation is performed as follows:

$$\begin{aligned} P(W1) &= P(w_{12} \cup w_{13} \mid w_4 \cup w_5 \cup w_7 \cup w_8 \cup w_9 \cup w_{10} \cup w_{12} \cup w_{13}) \\ &= \frac{P(w_{12}) + P(w_{13})}{P(w_4) + P(w_5) + P(w_7) + P(w_8) + P(w_9) + P(w_{10}) + P(w_{12}) + P(w_{13})} \\ &= \frac{192}{960} = \frac{1}{5}\,. \end{aligned}$$

Certainly, this value agrees with the assignment by MARINOFF and RAPOPORT; nonetheless, it must be noted that there is not a set of *equiprobable* points involved in the calculations, this contrasting RAPOPORT's claim. In fact, the argument provided by RAPOPORT to guaranteeing the value of 1/5 exhibits a drawback. RAPOPORT addresses that given Player 1 holds an ace and a deuce, then there are 15 equiprobable hands consisting of two cards that Player 2 might hold, three of these containing two deuces, thus 3/15 quotient yields 1/5. I argue that this assertion is incorrect. Incidentally, that would be the case if Player 1 took deliberately an ace and a deuce whatever the suit of these cards were leaving three aces and three deuces for Player 2 to be chosen randomly. Still, this is not the case: As depicted in the textual reference included at the beginning of this paper, "... each of two players draws two cards from the deck consisting of four aces and four deuces..." So, the experiment is undoubtedly different.

In regard to this sort of mistakes, my claim is similar to HOEL *et al.* [1] and LINDLEY [2] (p.13), who favour definitely the use of Bayes' rule as the proper procedure to override them. HOEL *et al.* [1] illustrate the topic by means of a (classical) example, which is conceptually similar to the one considered here. The example is as follows: "Suppose there are three chests each having two drawers. The first chest has a gold coin in each drawer, the second chest has a gold coin in one drawer and a silver coin in the other drawer, and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer is opened. If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin? We ask the reader to pause and guess what the answer is before reading the solution. Often in this problem the erroneous answer of 1/2 is given." (HOEL *et al.* [1] p.17)

Sample space partition	Hand		Outcome (Player 1)	Number of permutations		
	Player 1	Player 2		For the hand	Wins	Loses
w_1:	1 1	1 1	L	$(4)(3)(2)(1) = 24$		24
w_2:	1 1	1 2	W	$(4)(3)(2)(4) = 96$	96	
w_3:	1 1	2 1	L	$(4)(3)(4)(2) = 96$	96	
w_4:	1 2	1 1	L	$(4)(4)(3)(2) = 96$		96
w_5:	2 1	1 1	L	$(4)(4)(3)(2) = 96$		96
w_6:	1 1	2 2	W	$(4)(3)(4)(3) = 144$	144	
w_7:	1 2	1 2	L	$(4)(4)(3)(3) = 144$		144
w_8:	2 1	1 2	L	$(4)(4)(3)(3) = 144$		144
w_9:	1 2	2 1	L	$(4)(4)(3)(3) = 144$		144
w_{10}:	2 1	2 1	L	$(4)(4)(3)(3) = 144$		144
w_{11}:	2 2	1 1	L	$(4)(3)(4)(3) = 144$		144
w_{12}:	1 2	2 2	W	$(4)(4)(3)(2) = 96$	96	
w_{13}:	2 1	2 2	W	$(4)(4)(3)(2) = 96$	96	
w_{14}:	2 2	1 2	L	$(4)(3)(4)(2) = 96$		96
w_{15}:	2 2	2 1	L	$((4)(3)(2)(4) = 96$		96
w_{16}:	2 2	2 2	L	$(4)(3)(2)(1) = 24$		24
			Totals:	1680	528	1152

TABLE 1. Partitioned sample space for the Ace-deuce game, and its implications for both players.

Sample space partition	Hand		Outcome (Player 1)	Number of permutations		
	Player 1	Player 2		For the hand	Wins	Loses
w_4:	1 2	1 1	L	(4)(4)(3)(2) = 96		96
w_5:	2 1	1 1	L	(4)(4)(3)(2) = 96		96
w_7:	1 2	1 2	L	(4)(4)(3)(3) = 144		144
w_8:	2 1	1 2	L	(4)(4)(3)(3) = 144		144
w_9:	1 2	2 1	L	(4)(4)(3)(3) = 144		144
w_{10}:	2 1	2 1	L	(4)(4)(3)(3) = 144		144
w_{12}:	1 2	2 2	W	(4)(4)(3)(2) = 96	96	
w_{13}:	2 1	2 2	W	(4)(4)(3)(2) = 96	96	
			Totals:	960	192	768

TABLE 2. Partitioned relevant sample space 'given Player 1 holds an ace and a deuce'.

Sample space partition	Hand		Outcome	Number of permutations		
	Player 1	Player 2		TotalFor the hand	Wins	Loses
w_7:	1 2	1 2	L	(4)(4)(3)(3) = 144		144
w_8:	2 1	1 2	L	(4)(4)(3)(3) = 144		144
w_9:	1 2	2 1	L	(4)(4)(3)(3) = 144		144
w_{10}:	2 1	2 1	L	(4)(4)(3)(3) = 144		144
w_{12}:	1 2	2 2	W	(4)(4)(3)(2) = 96	96	
w_{13}:	2 1	2 2	W	(4)(4)(3)(2) = 96	96	
			Totals:	768	192	576

TABLE 3. Partitioned relevant sample space for the ace deuce game 'given Player 1 holds and ace and a deuce, and Player 2 holds at least a deuce'.

Sample space partition	Hand		Outcome	Number of permutations		
	Player 1	Player 2		For the hand	Wins	Loses
w_7:	1 2	1 2	L	(4)(4)(3)(3) = 144		144
w_8:	2 1	1 2	L	(4)(4)(3)(3) = 144		144
w_9:	1 2	2 1	L	(4)(4)(3)(3) = 144		144
w_{10}:	2 1	2 1	L	(4)(4)(3)(3) = 144		144
			Totals:	576	0	576

TABLE 4. Sample space restriction for the ace deuce game given Player 1 holds and ace and a deuce and Player 2 holds exactly a deuce.

The similarity between the chest problem and the Ace-deuce game, in the case that Player 1 had already gotten an ace and a deuce, is clear in the context proposed by RAPOPORT with respect to a uniform probability space. I argue that RAPOPORT's assertion implies the improper, but non-paradoxical answer, which HOEL *et al.* [1] prevent. That is, whether a drawer of a randomly chosen chest is opened and a gold coin is found there, then I assert that a reasoning following RAPOPORT's arguments would indicate that a uniform probability space should stand for the event of getting a gold coin form the remaining drawer. Of course, this would represent a quite different experiment, one whose sample space would be different in essence and not necessarily uniform.

In HOEL *et al.* [1] view: "this problem is easily and correctly solved by using Bayes' rule, once the description is deciphered" (HOEL *et al.*; p.17). Let us bring up some of the words of these authors. Define:

A_1: 'First chest being selected'
A_2: 'Second chest being selected'
A_3: 'Third chest being selected'

Since only one chest is selected, then these events are disjoint and their union is the sample space. Further, if the chest is chosen randomly, and there is not additional information concerning this action, then it is correctly presumed that the probability for each of the three chests to be chosen satisfies $P(A_i) = 1/3$, $i = 1, 2, 3$.

Hence, if G is defined to be the event of the coin observed was gold, then:

$$P(A_1 \mid G) = 1\,, \quad P(G \mid A_2) = 1/2\,, \quad \text{and} \quad P(G \mid A_3) = 0\,.$$

The probability of getting another gold coin from the chosen chest, given the first one was gold, is equivalent to finding $P(G \mid A_1)$, due to a second gold coin can be selected only if the chest selected was the first. Applying Bayes' rule,

$$P(A_1 \mid G) = \frac{P(G \mid A_1)}{P(G \mid A_1) + P(G \mid A_2) + P(G \mid A_3)} = \frac{1}{1 + 1/2} = \frac{2}{3}\,.$$

Presumably, it could be arguable if the choice was proposed as being made at random in RAPOPORT's dissertation. Of course, this shortcoming might also be used to incorporate some subjective yet defensive interpretation. For the major model of the experiment, however, the card taking for the two players demands to be considered as a random choice. Thus, according to the information in Table 1, the probability for player 2 of getting two aces is

$$P(w_{12} \cup w_{13}) = \frac{1}{5}\,.$$

The probability to get an ace and a deuce is

$$P(w_7 \cup w_8 \cup w_9) = \frac{3}{5}\,.$$

And the probability to get two deuces is

$$P(w_4 \cup w_5) = \frac{1}{5}\,.$$

Definitely, in contrast to the referred author's statements, this is not a uniform space. Hence, RAPOPORT's affirmation is unsupported.

On the other hand, MARINOFF's [3] analysis of the Ace-deuce game includes the following assertions: "Suppose I hold an ace and a deuce, and RAPOPORT draws two cards from the deck, which contains the remaining three aces and three deuces. My hand defeats his just in case he holds two deuces. The probability of that, RAPOPORT and I agree, is 3/15 or 1/5" (MARINOFF, [3] p.163).

It is interesting to recognize that the experiment as defined by MARINOFF [3] is fairly similar to that considered by RAPOPORT [4], explained so far. Notice that whereas a second couple of cards is taken after a specific couple had been taken, it is almost obvious that all the remaining couples are equiprobable for a second player. Thus, despite the fact that MARINOFF [3] does not establish a sheer relation between his definition of the experiment and its probability assignments, his conclusion is correct. Notwithstanding, since their respective definitions of random experiments are different, my conclusion is that the agreement between MARINOFF and RAPOPORT, regarding 1/5, is entirely casual.

3. Discrepancies between Rapoport and Marinoff

The gist of the discussion between RAPOPORT [4] and MARINOFF [3] is the following statement, paraphrasing RAPOPORT:

"Suppose a kibitzer, who sees Player 2's hand, informs Player 1 that Player 2 has a deuce. I maintain that now Player 1 wins with probability 1/4. Marinof Disagrees."

On this respect, let us consider two cases: First, if the kibitzer's information is regarded as 'Player 2 has at least a deuce', then the sample space, given Player 1 has an ace and a deuce, demands to be shrunk to the points in Table 3.

In whose context, the probability of Player 1 to win is:

$$P(W1) = P(w_{12} \cup w_{13}) = P(w_{12}) + P(w_{13}) = \frac{96}{768} + \frac{96}{768} = \frac{192}{768} = \frac{1}{4}\,,$$

which is the value that RAPOPORT bears.

Otherwise, if Player 2 has 'exactly' a deuce, then the sample space restriction provides no possibility for Player 1 to win; such conclusion is obtained straightforward from the information in Table 4.

The restriction excludes the points enabling Player 1 to win, and consequently his probability to win vanishes.

From the last conclusion, it is almost clear that the random experiment to which Rapoport applies his analysis consists on the event that the information obtained from the kibitzer is 'Player 2 has at least two deuces', no matter what he had been asked. Rapoport sustains, however, that "whether information changes assessed probabilities and by how much depends on the question to which the information provided the answer" I disagree, my claim on this issue is that no matters what he had been asked, unless the response is evidently meaningless in the absence of the question. This, of course, comes to pass mainly in cases when the possible answers are twofold, i.e.'yes' or 'no'. In any other case, any relation answer-question taken for granted introduces a subjective view or assumption (e.g. Schaffer).

At this stage of the discussion, it is meaningful to explore some possible reasons of the discrepancies between Marinoff and Rapoport; even though their conclusions seen finally to converge, it is imperative to review their arguments. In place, Rapoport admits to have made an error of omission in failing to specify the question to which the kibitzer's second report was the answer. He asserts that the question was "Does the opponent hold the deuce of spades?" Here, it must be clear that in response to this enquiry the only way to admit a 'wrong' response occurs in the case of Kibitzer's answer was clearly unrelated to the question. Moreover, from the point of view of Probability Theory, the effect of unrelated events on any previous probabilistic assignment is that the previous values ought to remain unchanged. Probability Theory doesn't involve concepts concerning a 'wrong' response characterization by itself; indeed, it just provides a conceptual structure for the logical treatment of random phenomena or uncertainty.

On this respect, Marinoff [4] argues "... the issue that serves either to determine the probability in question, or to show it to be indeterminate, is not merely what information we come to possess; rather, is additionally the consideration of the way in which we come to possess it..." However, what should be understood from the previous statement? Certainly, Marinoff offers an example to explain his points; it is as follows, "Suppose an urn contains one red marble and two green marbles, all of identical size, mass and contexture. You draw a marble at random, but do not observe its colour. What is the probability that it is red? Obviously, the answer is 1/3. Now suppose I draw a second marble from the urn, and show it to you, and you observe that it is green. Now what is the probability that your marble is red? The answer is: it depends on how I drew mine."

The last is evidently true, since the way of drawing the marbles specifies the random experiment. Yet, what follows in Marinoff's dicussion deals basically with assumptions concerning the way of getting marbles from the urn and the appropriate probability assignment to each case in accordance with Bayes' theory. Actually, there is no more in his dissertation than accurate aplications of probability rules regarding subjective views; I am unable to find any concept that Probability Theory does not involve aside of the necessity of preventing an improper definition of the Random Experiment and its associated Sample Space. In fact, this agrees with MARINOFF who asserts "The different sample spaces yield different but respectively consistent answers." Consequently, the treatment by MARINOFF is correct since he establishes a clear definition of the random experiments he deals with, their relevant sample spaces, and uses probability rules in accordance with such statements.

4. **Concluding Remarks**

In order to derive some conclusions, a point that is worthwhile to be discussed concerns 'how' the information provided by the Kibitzer has to be embodied into the game structure. Indeed, as it has been asserted, this regard suggests that subjective aspects may, or eventually demand, to be included into the analysis. Questions such as, 'is the Kibitzer truthful?', or, 'at which extend is he truthful?' make sense under any individual treatment. But, accordingly, it demands any assumption to be appropriately accounted in the solving approach.

I sustain that in probabilistic assessment, it is not relevant the 'way' in which information is obtained, but the relevance of the information to the problem framework. Moreover, I argue that it does not matter if a question is made either as type 'yes' or 'no', or it owes a longer answer. Certainly, it is difficult to admit that something different than what is assumed as the information provided by the kibitzer should be used in the model.

As well, what I find surprising in the anyalisis carried up by either MARINOFF or RAPOPORT is nearly a kind of inherent reasoning attributed to Probability Theory. It is advisable that Probability Theory should not be seen as a way of getting the "truth" of anything, such as it was a categorical representation of reality. In my view, Probability Theory must be used purposefully as a structural approach useful to analyzing rather identified or defined experiments, and such identification to be appropriately translated to the probabilistic framework.

I assert that MARINOFF's discussion, in which he argues "The different sample spaces yield different but respectively consistent answers." (MARINOFF [3]) is correct. Positively, treatment by MARINOFF is rather consistent, since he establishes a clear definition for the random experiments he copes, and their sample spaces; furthermore, he uses this framework in accordance with probability theory statements. Though, this is not the case in RAPOPORT's dissertation.

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