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An example for the convergence of Mann iteration

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Abstract. Two examples of Mann's iteration for the same map are presented. In one of them, the iteration yields a sequence converging to the unique fixed point of the map. In the other one, the resulting sequence does not converge to the fixed point and, as a matter of fact, can be made divergent.

Key Words and Phrases: Mann's iteration, fixed point.

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Resumen. Se presentan dos ejemplos de iteraciones de Mann para la misma función. En una de ellas, la iteración produce una sucesión que converge a un único punto fijo de la función. En la otra , la sucesión que se encuentra no converge al punto ficjo y, de hecho, puede diverger.

1. Introduction

Let $X = \mathbb{R}^2$ and $B \subset X$ be a nonempty convex, closed and bounded set. Let $T : B \to B$ be a map. We consider the following iterations (see [1]):

$$\begin{aligned} &v_1 \in B - \{(0,0)\}, \\ &v_{n+1} = \frac{n}{n+1} v_n + \frac{1}{n+1} T v_n. \end{aligned} \tag{1}$$

and

$$u_1 \in B - \{(0,0)\}, \quad u_{n+1} = Tu_n. \tag{2}$$

The aim of this note is to give a class of maps T for which:

(i) the iteration (1) converges to the fixed point of T;
(ii) the iteration (2) is not convergent to the fixed point of T.

In [2] we established the following Lemma. Here we give another proof of it.

Lemma 1. [2]. *Let $(\beta_n)_n$ be a sequence such that $\beta_n \in (0,1]$, for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n = \infty$, then $\prod_{n=1}^{\infty}(1-\beta_n) = 0$.*

Proof. We know that $1 - x \leq e^{-x}$ for all $x \in (0,1]$. Thus $(1-\beta_1) \leq e^{-\beta_1}$, and induction readily gives

$$(1-\beta_1)\cdots(1-\beta_n) \leq e^{-(\beta_1+\cdots+\beta_n)}, \ n \geq 1. \tag{3}$$

Letting $n \to \infty$ in (3) yields the conclusion. ☑

2. Main Result

We are able now to state our main result:

Theorem 1. *Let $X = \mathbb{R}^2$, $B = \{v \in \mathbb{R}^2 : |v| \leq 1\}$ and $T : B \to B$ be given by*

$$T(x,y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (x,y) \in B, \tag{4}$$

where

$$a^2 + b^2 = 1, \ \ 0 \leq b, \ \ 0 \leq a < 1. \tag{5}$$

If we have

$$\frac{n^2 + 2an + 1}{n^2 + 2n + 1} < 1, \quad n \in \mathbb{N}\ , \tag{6}$$

then the sequence (1) *converges to the unique fixed point of* T, *while the sequence* (2) *does not.*

Proof. We have $T(0,0) = (0,0)$. This fixed point of T is unique. In fact, if $v = (x^*, y^*)$ is such that $v = Tv$, then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} ax^* + by^* \\ -bx^* + ay^* \end{pmatrix},$$

where

$$\begin{cases} (a-1)x^* + by^* = 0 \\ -bx^* + (a-1)y^* = 0 \end{cases} . \tag{7}$$

But since

$$D := \begin{vmatrix} a-1 & b \\ -b & a-1 \end{vmatrix} = (a-1)^2 + b^2 = a^2 + b^2 - 2a + 1 = 2(1-a)$$

and $0 \le a < 1$, then $D \ne 0$, and the unique solution of (7) is $(x^*, y^*) = (0,0)$.

We first prove that the sequence (2) is not convergent to $(0,0)$. Write $u_n = \binom{p_n}{q_n}$. Then, from

$$u_{n+1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} u_n,$$

it follows that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} a\,p_n + b\,q_n \\ -b\,p_n + a\,q_n \end{pmatrix}$$

which implies that

$$\begin{aligned} |u_{n+1}|^2 &= p_{n+1}^2 + q_{n+1}^2 \\ &= (a\,p_n + b\,q_n)^2 + (-b\,p_n + a\,q_n)^2 = \\ &= a^2p_n^2 + 2abp_nq_n + b^2q_n^2 + a^2p_n^2 - 2abp_nq_n + b^2q_n^2 \\ &= (a^2+b^2)(p_n^2+q_n^2) = p_n^2 + q_n^2 = |u_n|^2 \quad , n \ge 1. \end{aligned}$$

Thus $|u_n| = |u_1| \neq 0$, for all n, and the sequence $(|u_n|)_n$ is not convergent to zero.

Now we prove that the sequence (1) converges to $(0,0)$. Let $v_n = (x_n, y_n)$:. Then

$$\begin{aligned}
v_{n+1} &= \frac{n}{n+1}v_n + \frac{1}{n+1}Tv_n \\
&= \frac{n}{n+1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{1}{n+1}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{nx_n}{n+1} \\ \\ \dfrac{ny_n}{n+1} \end{pmatrix} + \begin{pmatrix} \dfrac{ax_n + by_n}{n+1} \\ \\ \dfrac{-bx_n + ay_n}{n+1} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{(a+n)x_n + by_n}{n+1} \\ \dfrac{-bx_n + (a+n)y_n}{n+1} \end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
|v_{n+1}|^2 &= x_{n+1}^2 + y_{n+1}^2 \\
&= \frac{(a+n)^2x_n^2 + b^2y_n^2 + 2(a+n)b_nx_ny_n}{(n+1)^2} + \\
&\qquad + \frac{(a+n)^2y_n^2 + b^2x_n^2 - 2(a+n)b_nx_ny_n}{(n+1)^2} \\
&= \frac{\left((a+n)^2 + b^2\right)\left(x_n^2 + y_n^2\right)}{(n+1)^2} = \frac{(a+n)^2 + b^2}{(n+1)^2}|v_n|^2 \\
&= \frac{a^2 + b^2 + 2an + n^2}{(n+1)^2}|v_n|^2 = \frac{1 + 2an + n^2}{1 + 2n + n^2}|v_n|^2 < |v_n|^2 \quad , n \geq 1,
\end{aligned}$$

and the sequence $(|v_n|)_n$ is bounded from below and decreasing. Thus, $(|v_n|)_n$ is convergent. Furthermore,

$$|v_{n+1}|^2 = \prod_{k=1}^{n} \frac{1+2ak+k^2}{1+2k+k^2}\ |v_1|^2 = \prod_{k=1}^{n} \frac{1+2k+k^2+2ak-2k}{1+2k+k^2}\ |v_1|^2$$

$$= \prod_{k=1}^{n} \left(1 - \frac{2k\,(1-a)}{(1+k)^2}\right)\ |v_1|^2\ ,$$

and since $\sum_{k=1}^{\infty} \frac{2(1-a)}{(4+k)} \leq \sum_{k=1}^{\infty} \frac{2k(1-a)}{(1+k)^2}$ and $\sum_{k=1}^{\infty} \frac{2(1-a)}{(4+k)} = \infty$, we have that $\sum_{k=1}^{\infty} \frac{2k(1-a)}{(1+k)^2} = \infty$. Then, from Lemma 1 we get

$$\prod_{k=1}^{n} \left(1 - \frac{2k(1-a)}{(1+k)^2}\right) \to 0\ ,$$

and thus $|v_{n+1}| \to 0$. ☑

Remark 1. The case $a = b = \frac{1}{\sqrt{2}}$ yields the example in [1]. The map T is a rotation by an angle of $\pi/4$.

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Referencias

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