A MEAN VALUE ITERATION FOR A HÖLDER MAP

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ABSTRACT. We give a convergence result in an arbitrary Banach space for the Mann iteration when applied to a Hölder map.

2000 AMS Mathematics Subject Classification: 47H10
Key words: Mann-iteration, Hölder map.

1. INTRODUCTION

Let X be a real Banach space, B be a nonempty convex subset of X and $T : B \to B$ be a map. In [3], the following iteration, known as the Mann iteration, is introduced:

$$\begin{aligned} & x_1 \in B, \\ & x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{aligned} \tag{1}$$

where the sequence $(\alpha_n)_{n\geq 1} \subset (0,1)$, which we maintain fixed throughout the paper, is such that $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Observe that $(x_n)_{n\geq 1}$ is a sequence of B.

Definition 1. The map $T : B \to B$ verifies the Hölder condition if the following relation is true for a fixed $k \in (0,1)$ and all $x.y \in B$:

$$\|Tx - Ty\| \leq k \ \|x-y\|^2 . \tag{2}$$

Let us remark that T satisfying (2) can be a nonconstant map. For B a nonempty, convex, bounded set and $T : B \to B$ satisfying (2), there exists $M > 0$ such that:

$$\|x-y\| \leq M, \tag{3}$$

$$\|Tx - Ty\| \leq k \ \|x-y\|^2 \leq k\ M \ \|x-y\| \ , \tag{4}$$

for all $x, y \in B$. Our aim is to prove convergence results for iteration (1) and for T a map which satisfies (2).

For this purpose we need the following lemma from [7].

Lemma 1. [7] *Let* $(\beta_n)_{n\geq 1}$ *be a sequence such that* $\beta_n \in (0,1]$, for all $n \in \mathbb{N}$. *If* $\sum_{n=1}^{\infty} \beta_n = \infty$, *then* $\prod_{n=1}^{\infty}(1-\beta_n) = 0$.

2. Main result

We are now able to establish the following result:

Theorem 1. *Let X be a Banach space, B be a nonempty convex subset of X, let $T : B \to B$ be a Hölder map with constant $k \in (0,1)$ and $F(T) := \{x^* \in B : Tx^* = x^*\}$ be the set of fixed points of T. Then for all $x^* \in F(T)$ and all $x_1 \in B$ such that*

$$\|x_1 - x^*\| < \frac{1}{k}, \tag{5}$$

the sequence $(x_n)_{n\geq 1}$ given by (1) *strongly converges to x^*. Thus, for all $x_1 \in B$ there is at most one fixed point x^* of T for which* (5) *holds.*

Proof. Because of (1) and (2), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - Tx^*)\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|Tx_n - Tx^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\, k\, \|x_n - x^*\|^2 \\ &= \|x_n - x^*\| \left[1 - \alpha_n (1 - k\|x_n - x^*\|)\right]. \end{aligned}$$

Let $a_n := \|x_n - x^*\|, \forall n \in N$. Then

$$a_{n+1} \leq a_n \left[1 - \alpha_n (1 - k a_n)\right], \forall n \in N. \tag{6}$$

We intend to prove that $\lim_{n\to\infty} a_n = 0$. From (6) we obtain

$$\begin{aligned} a_n &\leq a_{n-1}\ \left[1 - \alpha_{n-1}(1 - k a_{n-1})\right] \\ &\ldots \\ a_2 &\leq a_1\ \left[1 - \alpha_1 (1 - k a_1)\right] \end{aligned}$$

Hence

$$a_{n+1} \leq a_1 \prod_{i=1}^{n} \left[1 - \alpha_i (1 - k a_i)\right]. \tag{7}$$

now, from (5) we have $a_1 < \dfrac{1}{k}$. We then suppose that $a_n < \dfrac{1}{k}$ and prove that $a_{n+1} < \dfrac{1}{k}$. But, from (6), we have

$$\begin{aligned} a_{n+1} &\leq a_n\left[1-\alpha_n\left(1-k a_n\right)\right] < \frac{1}{k}\left[1-\alpha_n+\alpha_n\, k\, a_n\right] \\ &< \frac{1}{k}\left[1-\alpha_n+\alpha_n\, k\, \frac{1}{k}\right] = \frac{1}{k}. \end{aligned}$$

and the assertion follows. Thus $0 < 1 - ka_n \leq 1$ for all $n \in \mathbb{N}$, and it follows from (6) that $a_{n-1} \leq a_n$, $n \in \mathbb{N}$. Hence $q = \inf_{n\in\mathbb{N}}(1-ka_n) > 0$ and we have

$$1-\alpha_n(1-ka_n) \leq 1-\alpha_n q,\ \ n \in \mathbb{N}. \tag{8}$$

From (7) and (8) we get

$$a_{n+1} \leq a_1 \prod_{i=1}^{n}(1-\alpha_n q), \tag{9}$$

and fom Lemma 1 we have,since $\displaystyle\sum_{n=1}^{\infty} q\alpha_n = \infty$, that $\Rightarrow \displaystyle\prod_{n=1}^{\infty}(1-q\alpha_n) = 0$. Thus, (9) ensures that $\lim\limits_{n\to\infty} a_n = 0$, i.e., $\lim\limits_{n\to\infty} x_n = x^*$, and the proof is complete. □

Corollary. *Let X be a Banach space, let B be a nonempty convex bounded subset of X and $T : B \to B$ be a Hölder map with constant $k \in (0,1)$. Let $F(T) := \{x^* \in B : Tx^* = x^*\}$ be the set of fixed points of T and assume that*

$$\operatorname{diam}(B) = d < \frac{1}{k}. \tag{10}$$

Then, if $x^* \in F(T)$, the sequence $(x_n)_{n\geq 1}$ given by (1) strongly converges to x^* for all $x_1 \in B$, and x^* is the unique fixed point of T.

Proof. Convergence of (x_n) to x^* follows from Theorem 1 above by observing that (5) holds for x^* and x_1. Uniqueness of x^* follows from this or from the fact under the assumptions T is a contraction map. □

Remark. If B is also closed, the Banach contraction principle ensures the existence of $x^* \in F(T)$.

In papers dealing with Mann's iteration, the space X is usually a uniformly convex or a smooth real Banach space. In our results, X is an arbitrary Banach space.

3. Acknowledgements

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