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Constructive error analysis for linear differential and integral equations

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Abstract. This paper deals with a conceptual simple approach for an effective a posteriori error analysis for linear problems ranging from Volterra equations to initial value problems for ordinary and partial differential equations. The theoretical basis is described with an approach using integral equations. It is then demonstrated that this concept leads to computable and safe error estimates for a wide class of problems.

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Resumen. Este artículo trata una aproximación conceptual simple para un efectivo análisis de error posterior de problemas lineales que se extienden desde la ecuación de Voltera hasta problemas con valores iniciales para ecuaciones diferenciales ordinarias y parciales. La teoría básica se describe aproximadamente usando ecuaciones integrales. Entonces se demostró un concepto guiado al cálculo y estimación de errores para una clase amplia de problemas.

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1. Motivation

A severe drawback in contemporary scientific computing is the lack of computable and safe error bounds. In this paper we derive a concept to reach this goal for equations of the type

$$L(x) = f, \tag{1}$$

where L is a linear operator from a normed space X_1 into a normed space X_2. Let $\tilde{x}$ be an approximation for the solution x of (1). Then an a posteriori error bound is given by

$$\|x - \tilde{x}\| \leq \|L^{-1}\| \, \|L(\tilde{x}) - f\|. \tag{2}$$

The local behaviour of the error is estimated by the quantity $L(\tilde{x})-f$, whereas the factor $\|L^{-1}\|$ seizes the influence of the global error. Whereas a calculation of a bound on the residual $L(\tilde{x}) - f$ is feasible in the framework of a precise computer arithmetic, the computation of a bound on $\|L^{-1}\|$ poses in general, difficult theoretical questions, hence as a matter of fact a common practice is to ignore this approach in numerical analysis, because it is held that this is either impossible or at least very hard to do. Thus the accuracy question is left to the users of numerical softwarepackages and is therefore often ignored. To overcome this difficulty we develop a method to compute reliable error estimates for numerical solutions $\tilde{x}$ obtained by traditional schemes. A great advantage of the method described here is its conceptual simplicity and its independence from specific methods.

The paper is organized as follows. In the following section we summarize some facts from interval analysis, which form the basis of the validation step. Then we derive a method for computing bounds on $\|L^{-1}\|$, which is applied to a wide class of linear problems in the subsequent paragraphs, whereas in the last section we discuss aspects of realization and report about numerical tests.

2. Foundations of interval analysis

For readers who are not familiar with interval arithmetic we sketch the basic concepts and facts from interval analysis.

2.1. **Real interval arithmetic and basic properties.** A real interval is a closed and bounded subset of the real numbers $\mathbb{R}$. We write the notation

$$[a] := [\underline{a}, \overline{a}] = \{x \in \mathbb{R} \mid \underline{a} \leq x \leq \overline{a}\},$$

where $\underline{a}$ and $\overline{a}$ denote the lower and upper bounds of the interval $[a]$, respectively. The set of real intervals is denoted by $I\mathbb{R}$. An interval is called point interval if $\underline{a} = \overline{a}$. We simply write a in this case. The elementary real operations $* \in \{+, -, \cdot, /\}$ are extended to $I\mathbb{R}$ by $[a] * [b] := \{a * b | a \in [a], b \in [b]\}$. In the case of division $0 \notin [b]$ is assumed. Since the function $f(a, b) = a * b$, $a \in [a]$, $b \in [b]$, $* \in \{+, -, \cdot, /\}$ is continuous, $[a] * [b]$ is contained in $I\mathbb{R}$. By using monotonicity properties we obtain the following rules for the four operations.

$$[a] + [b] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$

$$[a] - [b] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}],$$

$$[a] \cdot [b] = [\min\{\overline{a}\underline{b}, \overline{a}\overline{b}, \underline{a}\overline{b}, \underline{a}\underline{b}\}, \max\{\overline{a}\underline{b}, \overline{a}\overline{b}, \underline{a}\overline{b}, \underline{a}\underline{b}\}],$$

$$[a]/[b] = [a] \cdot \left[\frac{1}{\overline{b}}, \frac{1}{\underline{b}}\right], 0 \notin [b].$$

Heindl [9] has shown that for $*$ the number of multiplications in the preceding formula can be reduced to three. We can reduce this number to the multiplication of two real numbers in the case in which not simultaneously $0 \in [a]$ and $0 \in [b]$. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a real-valued elementary function, which is continuous on every closed interval in its domain A. Then for $[a] \subseteq A$ we define the so-called unary operation in $I\mathbb{R}$

$$f([a]) = \{f(a) | a \in [a]\}.$$

Example 1. For $[a]$ restricted to the domain of f holds for example

$$e^{[a]} = \exp([a]) = [e^{\underline{a}}, e^{\overline{a}}],$$

$$\sqrt{[a]} = \sqrt{}([a]) = [\sqrt{\underline{a}}, \sqrt{\overline{a}}],$$

$$[a]^n = \begin{cases} [\underline{a}^n, \overline{a}^n] & \text{if } \underline{a} > 0 \text{ or } n \text{ is odd;} \\ [\overline{a}^n, \underline{a}^n] & \text{if } \overline{a} < 0 \text{ or } n \text{ is even;} \\ [0, \max\{|\underline{a}|^n, |\overline{a}|^n\}] & \text{if } 0 \in [a] \text{ and } n \text{ is even.} \end{cases}$$

$$f([a]) = [f(\underline{a}), f(\overline{a})], \quad f \in \{\arctan, \operatorname{arcsinh}, \ln, \sinh\}.$$

Interval addition and multiplication are associative and commutative. But interval arithmetic does not follow the same rules as the real arithmetic (cf. Claudio et al [2]): For $[a], [b], [c] \in I\mathbb{R}$ we have

$$[a] \cdot ([b] + [c]) \subseteq [a] \cdot [b] + [a] \cdot [b],$$

We call this property subdistributivity. In certain special cases, distributivity holds:

$$[a]([b] + [c]) = [a][b] + [a][c] \quad \text{for} \quad [a] \in \mathbb{R},$$

$$[a]([b] + [c]) = [a][b] + [a][c] \quad \text{if} \quad [b][c] > 0.$$

Radius, absolute value, midpoint and diameter (or width) of an interval $[a]$ are defined as

$$r([a]) := \frac{\overline{a} - \underline{a}}{2},$$

$$|[a]| := \max\{|\underline{a}|, |\overline{a}|\},$$

$$m([a]) := \frac{\underline{a} + \overline{a}}{2},$$

$$d([a]) := \overline{a} - \underline{a}.$$

The distance of two intervals $[a]$ and $[b]$ is defined as the real number

$$q([a], [b]) := \max\{|\underline{b} - \underline{a}|, |\overline{b} - \overline{a}|\}.$$

2.2. **Interval arithmetic evaluation.** A fundamental problem of interval arithmetic is to compute an enclosure of the range of a function $f(a, b, \dots, z)$. The interval arithmetic evaluation or interval extension of f is defined by $f([a], [b], \dots, [z])$. The following important rules hold

Inclusion monotonicity

1. If $[a] \subseteq [a'], [b] \subseteq [b'], \dots, [z] \subseteq [z']$ then

$$f([a], [b], \dots [z] \subseteq f([a'], [b'], \dots, [z']).$$

2. Inclusion property
   If $a \in [a]$, $b \in [b], \dots, z \in [z]$ then

$$f(a, b, \dots, z) \in f([a], [b], \dots, [z]).$$

From the second rule we can see that the interval arithmetic evaluation

$$f([a], [b], \dots, [z])$$

always contains the range $R(f; [a], [b], \dots, [z])$ of the real function f defined on the Cartesian Product $[a] \times [b] \times \dots \times [z]$

$$\begin{aligned} R(f; [a], [b], \dots, [z]) &= \{f(a, b, \dots, z) | a \in [a],\ b \in [b], \dots, z \in [z]\} \\ &\subseteq f([a], [b], \dots, [z]). \end{aligned}$$

Since $(I\mathbb{R}, q)$ is a metric space, the concepts of convergence and continuity may be introduced in the usual manner (see Alefeld et al [1]). Moore [13] has shown that under reasonable assumptions the following inequality holds:

$$q(R(f; [x]), f([x])) \leq \gamma d([x]), \quad \gamma \geq 0,$$

where $[x]$ is contained in some fixed interval $[x]^0$. This means that the overestimation of $R(f; [x])$ by $f([x])$ goes linearly to zero with the diameter of $[x]$. This holds also for functions of several variables.

Example 2. *Let*

$$f(x) = x - x^2, \quad x \in [x]^0 = [0, 1]$$

and

$$[x] = \left[\frac{1}{2} - n, \frac{1}{2} + n\right].$$

We have exact range of values:

$$R(f;[x]) = \left[\frac{1}{4} - n^2, \frac{1}{4}\right].$$

Nested form:

$$f_1([x]) = [x](1-[x]) = \begin{cases} \left[\left(\frac{1}{2}-n\right)^2, \left(\frac{1}{2}+n\right)^2\right] & \text{if} \quad n \le \frac{1}{2} \\ \left[\frac{1}{4}-n^2, \left(\frac{1}{2}+n\right)^2\right] & \text{if} \quad n > \frac{1}{2} \end{cases},$$

interval evaluation:

$$f_2([x]) = [x] - [x]^2 = \left[\frac{1}{4} - 2n - n^2, \frac{1}{4} + 2n - n^2\right],$$

$$q(R(f;[x]), f_2([x]))) = \max(2n, 2n - n^2) = 2n = \gamma \cdot d([x]), \gamma = 1,$$

as predicted by Moore's result.

It is possible to rearrange the variables of the given function in such a manner that the interval arithmetic evaluation gives higher than linear convergence to the range of values.

Theorem 3. *(The centered form)* Let the function $f : \mathbb{R} \to \mathbb{R}$ be represented in the so-called centered form

$$f(x) = f(x) + (x - z) \cdot h(x),$$

for some $z \in [x]$. If $h(x)$ has an interval arithmetic evaluation $h([x])$ then, under weak conditions on the arithmetic evaluation $h([x])$ for $f([x])$ defined by,

$$f([x]) := f(z) + ([x] - z) \cdot h([x]),$$

it holds that

$$a) \; R(f;[x]) \subseteq f([x])$$

and

$$b)\ q(R(f;[x]);f([x])) \leq \gamma(d([x]))^2.$$

Property b) is called *quadratic approximation property* of the centered form. The centered form was introduced by Moore in [13] where he conjectured that the quadratic approximation property holds. The conjecture has been proved by Hansen in [8]. The question whether for a given (rational) function there exists a representation $\tilde{f}$ such that

$$q(R(f;[x]),\tilde{f}([x])) \leq \gamma(d([x]))^m, \quad \gamma \geq 0,$$

with $m > 2$ is open. Up to now such representations are only known under special assumptions.

Example 4. The function $f(x) = x - x^2$, $x \in [0,1]$ of Example 2 can be written as

$$f(x) = x - x^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right), \quad x \in [0,1],$$

$$f([x]) = \left[\frac{1}{4} - n^2, \frac{1}{4} + n^2\right] \quad \text{for} \quad [x] = \left[\frac{1}{2} - n, \frac{1}{2} + n\right],$$

and therefore

$$q(R(f;[x]),f([x])) = n^2 = \frac{1}{4}(d([x]))^2,$$

which means that the distance goes quadratically to zero with $d([x])$.

Another interval extension of practical value is the mean value form.

Suppose f is differentiable on its domain D, then

$$f(x) = f(c) + f'(\xi)(x - c)$$

with some fixed $c \in D$ and ξ between x and c. Let $c, x \in [x]$, therefore

$$\begin{aligned} f(x) &= f(c) + f'(\xi)(x-c) \in f(c) + f'([x])(x-c) \\ &\subseteq f(c) + f'([x])([x]) - c) \\ &=: f([x]) \end{aligned}$$

this evaluation does not depend on the expression of f, but it depends on the interval evaluation of f'. That means that different expressions of f' lead to different values of $f([x])$.

Remark 5. Interval arithmetic has been implemented in hardware and in software on many different platforms and it is supported by powerful programming languages. The XSC (eXtended Scientific Computation) library provides powerful tools necessary for achieving high accuracy and reliability. It provides a large number of predefined numerical data types and operators to deal with uncertain data (see Hammer et al [7]).

3. Bounds on $\|L^{-1}\|$

Here we look at an approach using integral equations, which can be applied to a variety of problems. In (1) we choose especially $L = I - K$, where I denotes the identity and K is a linear Volterra integral operator, so problem (1) becomes

$$(I - K)(x) = f, \tag{3}$$

if $k(s,t)$ is the kernel function of K, then (3) is written in detail

$$x(s) - \int_0^s k(s,t)x(t)dt = f(s), \quad 0 \leq s \leq a, \tag{4}$$

for notational simplicity, we choose, without loss of generality, the range of the independent variable so that the lower integration bound is zero. We assume that f and k to be continuous on their domains of definition. Here the spaces X_1 and X_2 coincide and (4) is dealt within the framework of the space $X = C[0,a]$, which is, equipped with the maximum norm $\|.\|_\infty$, a Banach space.

Theorem 6. We consider the operator $L = I - K$, where K is a Volterra operator with a continuous kernel $k(s,t)$, $0 \leq s,t \leq a$. Let M be a constant satisfying

$$|k(s,t)| \leq M, \quad 0 \leq s,t \leq a. \tag{5}$$

Then a bound on the inverse L^{-1} is given by

$$\|L^{-1}\| \leq e^{Ma}. \tag{6}$$

The solution x can be represented through a Neumann series

$$x(s) = \sum_{\nu=0}^{\infty} K^{\nu}(f) = L^{-1}(f), \tag{7}$$

where $K^0 = I$ and K^{ν} is for $\nu = 1, 2, \ldots$, an integral operator with the ν-th iterated kernel $k_{\nu}(s,t)$ as kernel function:

$$k_1(s,t) = k(s,t),$$

$$k_{\nu}(s,t) = \int_t^s k(s,\tau) k_{\nu-1}(\tau, t) d\tau, \quad \nu = 2, 3, \ldots \quad .$$

¿From (5) we derive by induction

$$\|K^{\nu}\| \leq \frac{M^{\nu} a^{\nu}}{\nu!}, \quad \nu = 0, 1, 2, \ldots\ , \tag{8}$$

and $\|K^{\nu}\|^{1/\nu}$ tends to zero as ν goes to infinity, so a bound on $\|L^{-1}\|$ is deduced from (7)

$$\|L^{-1}\| = \|\sum_{\nu=0}^{\infty} K^{\nu}\| \leq \sum_{\nu=0}^{\infty} \frac{M^{\nu} a^{\nu}}{\nu!} \tag{9}$$

and the proof is complete.

We want to stress that the foregoing results for scalar equations can be extended without any difficulties to the case of vectorvalued functions. We can also drop the restriction that s, t are scalar variables. Let

$$[0, a] = ([0, a_1], \ldots, [0, a_n]) \subseteq \mathbb{R}^n$$

be an interval, the notation $t = (t_1, \ldots, t_n) \in [0, a]$ means $0 \leq t_i \leq a_i$, $i = 1, \ldots, n$. Then the Volterra equation has the form

$$x(s_1, \ldots, s_n) - \int_0^{s_1} \ldots \int_0^{s_n} k(s_1, \ldots, s_n, t_1, \ldots, t_n) x(t_1, \ldots, t_n) dt_1, \ldots, dt_n$$
$$= f(s_1, \ldots, s_n)$$

or in short notation

$$x(s) - \int_{\Delta} k(s,t)x(t)dt = f(s),$$

when denoting the domain $\{(s,t) : 0 \leq t_i \leq s_i,\ i = 1, \dots, n\} \subseteq \mathbb{R}^n$ with Δ.

Integrodifferential equations, that are problems of the form

$$x'(s) = f(s) + g(s)x(s) + \int_0^s k_1(s,t)x(t)dt + \int_0^s k_2(s,t)x'(t)dt, \quad x(0) = x_0$$

are reduced to a system of Volterra equations, by introducing the quantities $x_1 := x$, $x_2 = x'$ yielding

$$x_1(s) = x_0 + \int_0^s x_2(t)dt$$

$$x_2(s) = f(s) + g(s)x_1(s) + \int_0^s k_1(s,t)x_1(t)dt + \int_0^s k_2(s,t)x_2(t)dt.$$

For equations of higher order, for each derivative a new unknown has to be introduced.

4. **Application to integral equations and initial value problems**

In this sequel we apply the results of the previous section obtaining explicit estimates for various problems. All equations are assumed to be in the canonical form (1) and we deal with several types of L. The basic idea is to rewrite $L(x) = f$, if necessary, into an equivalent Volterra equation of the form $I - Q$.

4.1. **Volterra equations.** The first operator is associated with Volterra integral equations of the second kind. The general form is

$$L(x\,(s)) = x(s) - \int_{\Delta} k(s,t)x(t)dt, \tag{10}$$

the kernel is supposed to be continuous, with $s, t \in \mathbb{R}^n$ and Δ is the domain of integration. Then

$$\|L^{-1}\|_\infty \leq e^{M|\Delta|}, \tag{11}$$

$|\Delta|$ is the area of Δ and M such that

$$\|k(s,t)\|_\infty \leq M, \quad (s,t) \in \Delta. \tag{12}$$

Next we treat the operator

$$L(x\,(s)) = \int_0^s k(s,t)x(t)dt, \quad 0 \leq s \leq a, \tag{13}$$

thus (1) is an ill-posed problem that is L^{-1} is unbounded. In some cases however the first kind equation

$$\int_0^s k(s,t)x(t)dt = f(s), \quad 0 \leq s \leq a, \tag{14}$$

can be recasted as second kind problem. Due to the nature of the problem we can only compute a bound on x and not a bound on L^{-1}.

Theorem 7. Let $k(s,t)$, $0 \leq t \leq s \leq a$, be continuous and continuously differentiable with respect to s. If $k(s,s) \neq 0$, $0 \leq s \leq a$, then for the solution x of (14) holds

$$\|x\| \leq M_1 M_2, \tag{15}$$

where

$$\left|\frac{f'(s)}{k(s,s)}\right| \leq M_1, \quad 0 \leq s \leq a \tag{16}$$

$$\|\sum_{\nu=0}^{\infty} Q^\nu\| \leq M_2, \tag{17}$$

Q is an integral operator with kernel $\frac{1}{k(s,s)}\frac{\partial k(s,t)}{\partial s}$ and Q^ν the integral operator with the $\nu-$iterated kernel q_ν as kernel function.

The first kind problem $L(x) = f$ is recasted as an second kind equation

$$x(s) + \int_0^s \frac{1}{k(s,s)} \frac{\partial k(s,t)}{\partial s} x(t)dt = \frac{f'(s)}{k(s,s)}, \quad 0 \le s \le a, \tag{18}$$

the solution of both equations coincide cf Linz [11]. Aplying (7) to (18) completes the proof.

Remark 8. A bound on the inverse of the integral operator occuring in (18) is attained by (17).

4.2. **Ordinary differential equations.** In this section we turn our attention to initial value problems

$$x'(s) - A(s)x(s) = f(s), \tag{19}$$

$$x(0) = x_0$$

in which the vector valued function $x(s)$ takes values within the space $\mathbb{R}^n$, $A(s)$ is a square matrix of order n with real valued continuous entries and $f(s)$ stands for a vector valued continuous function with values in $\mathbb{R}^n$ defined on an interval $[0, a]$, $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$.

Note that every higher order problem can be turned into a first order system, so (19) covers initial value problems of arbitrary order. The operator corresponding to (19) is

$$L(x) = \frac{dx(s)}{ds} - A(s)x(s). \tag{20}$$

Equation (19) is now converted by integration into a Volterra system

$$x(s) = x_0 + \int_0^s f(t)dt + \int_0^s A(t)x(t)dt \tag{21}$$

so that we can apply Theorem 6.

4.3. **Hyperbolic initial value problems.** We consider the following characteristic initial value problem, also known as Darboux problem:

$$(22)\quad \begin{aligned} L(u) &= u_{st} - (a_1(s,t)u(s,t) + a_2(s,t)u_s(s,t) + a_3(s,t)u_t(s,t)) \\ &= f(s,t), \qquad (s,t) \in \Delta = [0,c_1] \times [0,c_2]. \end{aligned}$$

$$(23)\quad \begin{aligned} u(s,0) &= g(s) \quad , \quad s \in [0,c_1] \\ u(0,t) &= h(t) \quad , \quad t \in [0,c_2] \end{aligned} \qquad g(0) = h(0),$$

We assume that inhomogeneity and coefficient functions are continuous and g and h continuously differentiable. We are looking for classical solutions, that are functions with $u, u_s, u_t, u_{st} \in C(\Delta)$. The initial value problem is formulated first as integrodifferential equation by integrating (22) on both sides, and afterwards rewritten with the techniques of §3 to a Volterra system, setting $u_1 := u, u_2 := u_s, u_3 := u_t$

$$(24)\quad \begin{aligned} u_1(s,t) &= g(s) + h(t) - g(0) + \int_0^s \int_0^t f(\bar{s},\bar{t}) d\bar{s} d\bar{t} \\ &\quad + \int_0^s \int_0^t \sum_{i=1}^3 a_i(\bar{s},\bar{t}) u_i(\bar{s},\bar{t}) d\bar{s} d\bar{t}, \\ u_2(s,t) &= g'(s) + \int_0^t f(s,\bar{t}) d\bar{t} + \int_0^t \sum_{i=1}^3 a_i(s,\bar{t}) u_i(s,\bar{t}) d\bar{t} \\ u_3(s,t) &= h'(t) + \int_0^s f(\bar{s},t) d\bar{s} + \int_0^s \sum_{i=1}^3 a_i(\bar{s},t) u_i(\bar{s},t) d\bar{s}. \end{aligned}$$

Denoting the nonhomogeneous terms on the right hand sides with

$$b_1(s,t), b_2(s,t), b_3(s,t)$$

respectively, the corresponding integral operators with $L_{11}, L_{12}, \dots, L_{33}$ respectively, then (24) is written as

$$(25)\quad u_i(s,t) = b_i(s,t) + \sum_{j=1}^3 L_{ij}(u_i(s,t)), \quad i = 1,2,3,$$

or using vector notation, in short form

$$(I - Q)u = b \tag{26}$$

Applying a slight modification of Theorem 6 yields

$$\|(I - Q)^{-1}\| \leq e^{M|\Delta|}, \tag{27}$$

where

$$\max_{1 \leq i,j \leq 3} \|L_{i,j}\| \leq M, \tag{28}$$

and

$$|\Delta| = c_2 \cdot c_1. \tag{29}$$

Other problems of interest are the Cauchy problem and the Goursat problem leading to integral formulations similar as (24) (cf. Copson [3]).

5. **Aspects of realization and numerical tests**

In this section we discuss first some aspects of implementation. The underlying problem is assumed to be given in the form

$$L(x) = f \tag{30}$$

where L is one of the operators considered previously and B denotes the corresponding function space. In the framework of Theorem 6, the necessary computational steps are:

STEP	OPERATION	ARITHMETIC
1	Recast (30) as equivalent second kind integral equation (31) $\hat{L}(x) = g,$ where $\hat{L} = I - K$, with $k(s,t)$ as kernel of the operator K.	
2	Compute an approximation $\tilde{x}_D$ at discrete points using a suitable numerical method.	floating point
4	Compute $[d(s)] = L(\tilde{x}) - g$.	interval
5	Determine $[r_1, r_2]$ with range $(k(s,t)) \subseteq [r_1, r_2]$ $M := \max\{\|r_1\|, \|r_2\|\}$, $\\|L^{-1}\\| \leq e^{Ma}$.	interval
6	Establish a tolerance limit and check if the criterion is satisfied. If so then, $x(s) \in \tilde{x}(s) + e^{Ma} \cdot [d(s)] =: \tilde{x}(s) + E(s)$, otherwise go to step 2 by diminishing the stepsize or employing another method.	interval

We provide here some simple results to illustrate that the bound obtained from Theorem 6 is reliable.

Example 9. The equation

$$x(s) = e^s - \int_0^s e^{(s-t)} x(t) dt, \quad 0 \leq s \leq 1,$$

has exact solution $x(s) = 1$ (cf. Linz [11]). The errors in the approximate solution by the trapezoidal method are shown in the table for the stepsize $h = 9.7e - 4$:

$\tilde{x}(0.5)$	$E(0.5)$	average error
$9.999999603e - 1$	$8.478e - 7$	$9.849e - 7$

Example 10. Numerical results for the equation

$$x(s) = \frac{1}{\sqrt{1+s}} + \frac{\pi}{8} - \frac{1}{4}\sin\left(\frac{1-s}{1+s}\right) - \frac{1}{4}\int_0^s \frac{x(t)}{\sqrt{1+s+t}}dt, \quad 0 \le s \le 1,$$

obtained by the trapezoid method are displayed in the following table:

$n = 1024$		
s	$\tilde{x}(s)$	$E(s)$
0.25	1.0857	$3.2160e-3$
0.5	1.0238	$1.21178e-2$
0.75	0.9757	$2.1960e-2$
1.0	0.9348	$3.0933e-2$

Example 11.

$$x(s) = \frac{1}{(2+s)^2} - 2\int_0^s \frac{1}{(s-t+2)^2}x(t)dt, \quad 0 \le s \le 5,$$

(cf. Delves/Mohamed [4]). The results computed with a trapezoidal rule are shown in the columns of

$n = 1024$		
s	$\tilde{x}(s)$	$E(s)$
1.25	$5.3444e-2$	$1.0106e-6$
2.5	$1.8918e-2$	$7.0381e-7$
3.75	$9.2349e-3$	$4.7951e-7$
5.0	$5.4700e-3$	$3.4197e-7$

Example 12. The Volterra system

$$x_1(s) + \int_0^s e^{s-t}x_1(t)dt + \int_0^s \cos(s-t)x_2(t)dt = \cosh(s) + s\sin(s)$$

$$x_2(s) + \int_0^s e^{s+t}x_1(t)dt + \int_0^s s\cos(t)x_2(t)dt = 2\sin(s) + s(\sin^2(s) + e^s),$$

($0 \le s \le 1$), has been approximated by a trapezoidal rule with the stepsize $h = 1.95e-3$. With this example we demonstrate that a small defect without knowledge of a bound for the inverse operator is not helpful

s	defect in first equation	defect in second equation
0.25	$3.2168e-7$	$2.1518e-8$
1.0	$1.4989e-6$	$4.3032e-3$

in this case $\|L^{-1}\|$ is bounded by $1.636e-3$.

Example 13. The first kind problem

$$\int_0^s \cos(s-t)x(t)dt = \sin(s), \quad 0 \le s \le 1,$$

has the exact solution $x(s) = 1$, the following table gives the computed result.

h	maximum error
$9.765e-4$	$3.4247e-8$

Example 14. We integrate the initial value problem

$$x'(s) = 2sx(s) + s, \quad x(0) = 1,$$

with Euler's method and get

$n = 512$		
s	$\tilde{x}(s)$	$E(s)$
0.5	1.4260	$1.1650e-5$
1.0	3.5774	$6.9350e-5$

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