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On the semilocal convergence of a fast two-step Newton method

Convergencia semilocal de un método de Newton de dos pasos

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Abstract. We provide a semilocal convergence analysis for a cubically convergent two-step Newton method (2) recently introduced by H. Homeier [8], [9], and also studied by A. Özban [13]. In contrast to the above works we examine the semilocal convergence of the method in a Banach space setting, instead of the local in the real or complex number case. A comparison is given with a two step Newton–like method using the same information.

Key words and phrases. Two-step Newton method, Newton method, Banach space, majorizing sequence, Newton–Kantorovich hypothesis, semilocal convergence, Fréchet-derivative.

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Resumen. Proporcionamos un análisis de convergencia semilocal para un método de Newton de dos pasos, cúbicamente convergente, recientemente introducido por H. Homeier [8], [9], también estudiado por A. Özban [13]. En contraste con esto, examinamos la convergencia local del método en espacios de Banach en lugar del local, en el caso real y complejo. Damos una comparación con el método de Newton de dos pasos usando la misma información.

Palabras y frases clave. Método de Newton de dos pasos, método de Newton, espacio de Banach, secuencia mayorante, hipótesis de Newton–Kantorovich, convergencia semilocal, derivada de Fréchet.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on the closure $\overline{U}(x_0, R)$ $(R > 0)$ of a ball $U(x_0, R) = \{x \mid x \in X \mid \|x - x_0\| < R\}$ in a Banach space X with values in a Banach space Y.

Many problems in applied mathematics, and also in engineering, can be formulated as in equation (1) for a suitable choice of the operator F [4], [10], [12].

Recently H. Homeier [8], [9] and A. Özban [13] studied the local convergence of the cubically convergent two-step Newton method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0), \qquad (x_0 \in D), \\ x_{n+1} &= x_n - F'(z_n)^{-1}F(x_n), \quad z_n = \frac{x_n + y_n}{2} \end{aligned} \tag{2}$$

for all $n \geq 0$ in the special case when $X = Y = \mathbb{R}$ or $\mathbb{C}$. In [7], [10] it was already demonstrated experimentally that method (2) can compete in efficiency with other methods using the same information.

Method (2) was originally studied in [11], [5], where the cubic convergence was established under hypotheses on the second Fréchet–derivative of operator F.

Semilocal and local convergence theorems on Newton-like methods under various conditions can be found in [1], [14], and the references there. Therefore one can immediately obtain sufficient convergence conditions for the local as well as the semilocal case by simply referring to those results (see, in particular [3], [4]).

Results on other fast methods can be found in [1], [6], [7]. However here we decided to study the semilocal convergence of method (2) on a Banach space setting motivated by the efficiency of the method when $X = Y = \mathbb{R}$ or $\mathbb{C}$ using a direct approach and precise majorizing sequences along the lines of our works in [3], [4].

We assume that for some $x_0 \in D$, $F'(x_0)^{-1} \in L(Y, X)$ and for all $x, y \in U(x_0, R)$

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq w_0(\|x - x_0\|), \tag{3}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq w(\|x - y\|) \tag{4}$$

for some monotonically increasing functions w_0, w defined on $[0, R]$ and satisfying

$$\lim_{r \to 0} w_0(r) = \lim_{r \to 0} w(r) = 0. \tag{5}$$

Conditions of the form (3) - (5) were inaugurated in the elegant work in [2] (see also [3], [4]) in connection with the study of Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n) \quad (n \geq 0), \qquad (x_0 \in D), \tag{6}$$

in the special case when $w_0(r) = w(r)$ for all $r \in [0, R]$.

The advantages of introducing function w_0 in the study of Newton-like methods have been explained in [3], [4]. In fact this way under the same or even weaker hypotheses finer error bounds on the distances $\|y_n - x_n\|$, $\|x_{n+1} - x_n\|$,

$\|y_n - x^*\|$, $\|x_n - x^*\|$ $(n \geq 0)$ can be obtained and an at least as precise information on the location of the solution x^*.

A comparison with the two–step Newton-like method

$$\begin{array}{rcl} y_n & = & x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0), \qquad (x_0 \in D) \\ x_{n+1} & = & x_n - F'(y_n)^{-1}F(x_n) \end{array} \tag{7}$$

is given. Note that both methods (2) and (7) use two inverses and one function evaluation at every step. Numerical examples can also be found in [8], [13].

2. Semilocal convergence analysis of Newton-like method

Let $\eta \geq 0$. It is convenient for us to define scalar sequences $\{s_n\}$, $\{t_n\}$ $(n \geq 0)$ for $t_0 = 0$, $s_0 = \eta$, $t_1 = s_0 + \frac{s_0}{1-w_0\left(\frac{s_0+t_0}{2}\right)}$ by

$$s_{n+1} = t_{n+1} + \frac{\int_0^1 w\left(t\left(t_{n+1} - t_n\right)\right)(s_n - t_n)dt + [1 + w_0(t_n)](t_{n+1} - s_n)}{1 - w_0(t_{n+1})}, \tag{8}$$

and

$$t_{n+2} = t_{n+1} + \frac{\int_0^1 w\left[\frac{1}{2}(s_n - t_n) + t(t_{n+1} - t_n)\right](t_{n+1} - t_n)dt}{1 - w_0\left(\frac{t_{n+1}+s_{n+1}}{2}\right)}, \tag{9}$$

for all $n \geq 0$.

It follows by the definition of sequences $\{s_n\}$, $\{t_n\}$ that if there exists $\alpha \in [0, R]$ such that

$$s_n \leq t_{n+1} \leq \alpha < w_0^{-1}(1) \text{ for all } n \geq 0, \tag{10}$$

then both sequences are monotonically increasing, bounded above by α, and as such they converge to a common limit t^* such that

$$t_n \leq s_n \leq t_{n+1} \quad (n \geq 0), \tag{11}$$

and

$$t^* \leq \alpha. \tag{12}$$

We can show the following semilocal convergence theorem for Newton-like method (2) using majorizing sequences $\{t_n\}$ and $\{s_n\}$.

Theorem 2.1. *Under conditions (3), (4) and (8) for* $\left|F'(x_0)^{-1}F(x_0)\right| \leq \eta$, $\|F'(z_0)^{-1}F(x_0)\| \leq t_1$ *sequence* $\{x_n\}$ $(n \geq 0)$ *generated by Newton-like method (2) is well defined, remains in* $\overline{U}(x_0, t^*)$ *for all* $n \geq 0$, *and converges to a unique solution* x^* *of equation* $F(x) = 0$ *in* $\overline{U}(x_0, t^*)$.

Moreover the following estimates hold for all $n \geq 0$*:*

$$\|y_n - x_n\| \leq s_n - t_n, \tag{13}$$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{14}$$

$$\|y_n - x^*\| \leq t^* - s_n, \tag{15}$$

and

$$\|x_n - x^*\| \leq t^* - t_n. \tag{16}$$

Furthermore if there exists $R_0 \in (t^*, R]$ such that

$$\int_0^1 w[tt^* + (1-t)R_0]dt < 1, \tag{17}$$

then the solution x^* is unique in $U(x_0, R_0)$.

Proof. We shall show:

$$\begin{aligned} \|y_k - x_k\| &\leq s_k - t_k, &(18)\\ \|x_{k+1} - x_k\| &\leq t_{k+1} - t_k, &(19)\\ \overline{U}(y_k, t^* - s_k) &\subseteq \overline{U}(x_k, t^* - t_k), &(20) \end{aligned}$$

and

$$\overline{U}(x_{k+1}, t^* - t_{k+1}) \subseteq \overline{U}(x_k, t^* - t_k). \tag{21}$$

For every $z \in \overline{U}(y_0, t^* - s_0)$,

$$\|z - y_0\| \leq \|z - y_0\| + \|y_0 - x_0\| \leq t^* - s_0 + s_0 = t^* = t^* - t_0$$

implies $z \in \overline{U}(y_0, t^* - t_0)$. Similarly, for every $w \in \overline{U}(x_1, t^* - t_1)$

$$\|w - x_0\| \leq \|w - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 = t^*$$

implies $w \in \overline{U}(x_0, t^* - t_0)$.

Estimates (16) and (17) hold true for $k = 0$ by the initial conditions. Let us assume estimates (16) - (19) hold for $n = 0, 1, \ldots, k$, then

$$\begin{aligned} \|y_k - x_0\| &\leq \|y_k - x_k\| + \sum_{i=1}^{k} \|x_i - x_{i-1}\| \\ &\leq s_k - t_k + t_k - t_0 = s_k - t_0 \leq t^* \\ \|x_{k+1} - x_0\| &\leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) \\ &= t_{k+1} - t_0 \leq t^*, \\ \left\| \frac{y_k + x_k}{2} - x_0 \right\| &\leq \frac{1}{2} \left[\|y_k - x_0\| + \|x_k - x_0\| \right] \\ &\leq \frac{1}{2}(s_k + t_k) \leq \frac{1}{2}(t^* + t^*) = t^*, \end{aligned}$$

and

$$\|x_k + t(x_{k+1} - x_k) - x_0\| \leq t_k + t(t_{k+1} - t_k) \leq t^* \quad \text{for all } t \in [0, 1].$$

Let $u \in \overline{U}(x_0, t^*)$, then using (3) and the induction hypotheses we get

$$\left\| F'(x_0)^{-1} \left[F'(u) - F'(x_0) \right] \right\| \leq w_0(\|u - x_0\|) \leq w_0(t^*) < 1. \tag{22}$$

It follows from (20) and the Banach Lemma on invertible operators [10] that $F'(u)^{-1}$ exists and

$$\left\| F'(u)^{-1} F'(x_0) \right\| \leq \left[1 - w_0(\|u - x_0\|) \right]^{-1}. \tag{23}$$

In view of (2) we obtain the identity

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(y_k - x_k) \\ &= \int_0^1 \left[F'\left(x_k + t\left(x_{k+1} - x_k\right)\right) - F'(x_k)\right](x_{k+1} - x_k)dt \\ &\quad + \left[F'(x_k) - F'(x_0)\right](x_{k+1} - y_k) + F'(x_0)(x_{k+1} - y_k), \end{aligned} \tag{24}$$

and by composing by $F'(x_0)^{-1}$ we get using (4)

$$\begin{aligned} &\left\|F'(x_0)^{-1}F(x_{k+1})\right\| \\ &= \left\|\int_0^1 F'(x_0)^{-1}\left[F'(x_k + t(x_{k+1} - x_k)) - F'(x_k)\right](x_{k+1} - x_k)dt\right\| \\ &\quad + \left\|F'(x_0)^{-1}[F'(x_k) - F'(x_0)](x_{k+1} - y_k)\right\| + \|x_{k+1} - y_k\| \\ &\leq \int_0^1 w\left(\|t(x_{k+1} - x_k)\|\right)\|x_{k+1} - x_k\|dt \\ &\quad + w_0(\|x_k - x_0\|)\|x_{k+1} - y_k\| + \|x_{k+1} - y_k\| \\ &\leq \int_0^1 w(t(t_{k+1} - t_k))(t_{k+1} - t_k)dt + w_0(t_k)(t_{k+1} - s_k) \\ &\quad + (t_{k+1} - s_k). \end{aligned} \tag{25}$$

Similarly from (2) we obtain the identity

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'\left(\frac{x_k + y_k}{2}\right)(x_{k+1} - x_k) \quad (26) \\ &= \int_0^1 \left[F'(x_k + t(x_{k+1} - x_k)) - F'\left(\frac{x_k + y_k}{2}\right)\right](x_{k+1} - x_k)dt. \end{aligned}$$

Therefore again by (24) and (4), we get

$$\begin{aligned} &\left\|F'(x_0)^{-1}F(x_{k+1})\right\| \\ &\quad\leq \int_0^1 w\left[\left\|x_k + t(x_{k+1} - x_k) - \frac{x_k + y_k}{2}\right\|\right]\|x_{k+1} - x_k\|dt \\ &\quad\leq \int_0^1 w\left[\frac{1}{2}\|y_k - x_k\| + t\|x_{k+1} - x_k\|\right]\|x_{k+1} - x_k\|dt \\ &\quad\leq \int_0^1 w\left[\frac{1}{2}(s_k - t_k) + t(t_{k+1} - t_k)\right](t_{k+1} - t_k)dt. \end{aligned} \tag{27}$$

In view of (2), (21) (for $u = x_{k+1}$, and $u = \frac{x_{k+1} + y_{k+1}}{2}$ respectively), (23) and (25), we obtain:

$$\|y_{k+1} - x_{k+1}\| \leq \left\|F'(x_{k+1})^{-1}F'(x_0)\right\| \cdot \left\|F'(x_0)^{-1}F(x_{k+1})\right\| \leq s_{k+1} - t_{k+1}, \tag{28}$$

and

$$\|x_{k+2} - x_{k+1}\| \leq t_{k+2} - t_{k+1}, \tag{29}$$

which show (16) and (17) for all $n \geq 0$.

Thus for every $w \in \overline{U}\,(x_{k+2}, t^* - t_{k+2})$, we have

$$\begin{aligned} \|w - x_{k+1}\| &\leq \|w - x_{k+2}\| + \|x_{k+2} - x_{k+1}\| \leq t^* - t_{k+2} + t_{k+2} - t_{k+1} \\ &= t^* - t_{k+1}, \end{aligned} \tag{30}$$

which imply

$$z \in \overline{U}(x_{k+1}, t^* - t_{k+1}). \tag{31}$$

Similarly for every $z \in \overline{U}(y_{k+1}, t^* - s_{k+1})$, we get

$$z \in \overline{U}(y_k, t^* - s_k). \tag{32}$$

The induction for estimates (16) - (19) is now complete.

In view of (8), (9), and (16) - (19), sequences $\{x_n\}$, $\{y_n\}$ are Cauchy in a Banach space X and as such they converge to a common limit $x^* \in \overline{U}(x_0, t^*)$ (since $\overline{U}(x_0, t^*)$ is a closed set). By letting $k \to \infty$ in (26) we get $F(x^*) = 0$. Estimates (13) and (14) follow from (11) and (12) by using standard majorization techniques [4], [10], [12].

To show uniqueness of x^* first in $\overline{U}(x_0, t^*)$, let y^* be a solution of equation $F(x) = 0$ in $\overline{U}(x_0, t^*)$. In view of (3) and (8), we get

$$\begin{aligned} &\left\| F'(x_0)^{-1} \int_0^1 [F'(y^* + t(x^* - y^*)) - F'(x_0)]dt \right\| \\ &\leq \int_0^1 w_0\big[t\,\|x^* - x_0\| + (1-t)\,\|y^* - x_0\|\big]\,dt \leq w_0(t^*) < 1. \end{aligned} \tag{33}$$

It follows from (30) and the Banach Lemma on invertible operators that linear operator L given by

$$L = \int_0^1 F'\,(y^* + t\,(x^* - y^*))\,dt \tag{34}$$

is invertible.

Using the identity

$$0 = F(x^*) - F(y^*) = L(x^* - y^*), \tag{35}$$

we deduce $x^* = y^*$.

Finally to show uniqueness in $U(x_0, R_0)$, again as in (30) we obtain

$$\left\| F'(x_0)^{-1}(L - F'(x_0)) \right\| \leq \int_0^1 w_0\,(tt^* + (1-t)R_0)\,dt < 1, \tag{36}$$

which again together with (33) yields to $x^* = y^*$. That completes the proof of the theorem. ☑

Remark 2.1. *Although stronger but easier to verify conditions implying crucial hypothesis (8) have already been given in* [2]*, when* $w_0(r) = w(r)$ *for all* $r \in [0, R]$*, and us* [3], [4]*, when functions* w_0 *and* w *are not necessarily the same, we decided to leave condition (8) as uncluttered as possible. In order for us* *to find conditions other than (8), let us assume there exists a monotonically increasing function* $\tilde{w}$ *satisfying (5) and for all* $t \geq s$*, with* $s, t \in [0, R]$*:*

$$\int_0^{t-s} w(t)dt \leq \int_s^t [\tilde{w}(t) - w(s)]\, dt. \tag{37}$$

Such an estimate can follow e.g. from

$$\tilde{w}(r) = \sup\{w(u) + w(v) \colon u + v = r\}. \tag{38}$$

This function may be calculated explicitly in some cases. For example, in the Hölder case

$$w(r) = \ell r^{\lambda} \quad (0 < \lambda \leq 1) \tag{39}$$

we have

$$\tilde{w}(r) = 2^{1-\lambda} \ell r^{\lambda}. \tag{40}$$

In general, if w *is a concave function on* $[0, R]$*, we have* $\tilde{w}(r) = 2w\left(\frac{r}{2}\right)$*. Clearly* $\tilde{w}$ *is always increasing, concave, and*

$$w(r) \leq \overline{w}(r) \quad \textit{for all } r \in [0, R]. \tag{41}$$

Conditions of the form (35) - (36) were first given in [2]*. More information on the motivation for the introduction of function* $\tilde{w}$ *can be found in* [2] - [4].

It is convenient for us to define scalar functions f, g on $[0, R]$, and sequences $\{\overline{s}_n\}$, $\{\overline{t}_n\}$, $\{\overline{\overline{s}}_n\}$, $\left\{\overline{\overline{t}}_n\right\}$ $(n \geq 0)$ for all $n \geq 0$ by

$$f(r) = \eta - r + \int_0^r \tilde{w}(t)dt, \tag{42}$$

$$g(r) = \eta - r + \int_0^1 w(t)dt, \tag{43}$$

$$\overline{t}_0 = 0, \quad \overline{s}_0 = \eta, \quad \overline{t}_1 = \overline{s}_0 + \frac{\overline{s}_0}{1 - w\left(\frac{\overline{s}_0 + \overline{t}_0}{2}\right)},$$

$$\overline{s}_{n+1} = \overline{t}_{n+1} + \frac{\int_0^1 w\left(t\left(\overline{t}_{n+1} - \overline{t}_n\right)\right)\left(\overline{s}_n - \overline{t}_n\right) dt}{1 - w(\overline{t}_{n+1})}, \tag{44}$$

$$\overline{t}_{n+2} = \overline{t}_{n+1} + \frac{\int_0^1 w\left[\frac{1}{2}\left(\overline{s}_n - \overline{t}_n\right) + t\left(\overline{t}_{n+1} - \overline{t}_n\right)\right]\left(\overline{t}_{n+1} - \overline{t}_n\right) dt}{1 - w\left(\frac{\overline{t}_{n+1} + \overline{s}_{n+1}}{2}\right)} \tag{45}$$

$$\overline{\overline{t}}_0 = \overline{t}_0, \quad \overline{\overline{s}}_0 = \overline{s}_0, \quad \overline{\overline{t}}_1 = \overline{t}_1,$$

$$\overline{\overline{s}}_{n+1} = \overline{\overline{t}}_{n+1} - \frac{f_1\left(\overline{\overline{t}}_n, \overline{\overline{s}}_n, \overline{\overline{t}}_{n+1}\right)}{g'\left(\overline{\overline{t}}_{n+1}\right)}, \tag{46}$$

$$\overline{\overline{t}}_{n+2} = \overline{\overline{t}}_{n+1} - \frac{f_2\left(\overline{\overline{t}}_n, \overline{\overline{s}}_n, \overline{\overline{t}}_{n+1}\right)}{g'\left(\frac{\overline{\overline{t}}_{n+1} + \overline{\overline{s}}_{n+1}}{2}\right)}, \tag{47}$$

where,

$$f_1(a,b,c) = \int_0^1 \overline{w}[a+t(c-a)](b-a)dt - w(a)(b-a),$$

and

$$f_2(a,b,c) = \int_0^1 \overline{w}[b+t(c-a)](c-a)dt - w\left(\frac{a+b}{2}\right)(c-a).$$

In view of (3) and (4) it follows that

$$w_0(r) \leq w(r) \quad \text{for all } r \in [0,R], \tag{48}$$

and $\frac{w(r)}{w_0(r)}$ can be arbitrarily large [3], [4]. By comparing sequences $\{s_n\}$, $\{t_n\}$ with $\{\overline{s}_n\}$ and $\{\overline{t}_n\}$ and using induction on $n \geq 0$ we deduce

$$s_n \leq \overline{s}_n, \tag{49}$$
$$t_n \leq \overline{t}_n, \tag{50}$$
$$s_n - t_n \leq \overline{s}_n - \overline{t}_n, \tag{51}$$
$$t_{n+1} - t_n \leq \overline{t}_{n+1} - \overline{t}_n, \tag{52}$$
$$t^* - s_n \leq \overline{t}^* - \overline{s}_n, \quad \overline{t}^* = \lim_{n\to\infty} \overline{t}_n, \tag{53}$$
$$t^* - t_{n+1} \leq \overline{t}^* - \overline{t}_{n+1}, \tag{54}$$

and

$$t^* \leq \overline{t}^*. \tag{55}$$

Note also that strict inequality holds in (47) - (50) if (44) also holds as a strict inequality.

Moreover if (35) or (36) hold then

$$\overline{s}_n \leq \overline{\overline{s}}_n, \tag{56}$$
$$\overline{t}_n \leq \overline{\overline{t}}_n, \tag{57}$$
$$\overline{s}_n - \overline{t}_n \leq \overline{\overline{s}}_n - \overline{\overline{t}}_n, \tag{58}$$
$$\overline{t}_{n+1} - \overline{t}_n \leq \overline{\overline{t}}_{n+1} - \overline{\overline{t}}_n, \tag{59}$$
$$\overline{t}^* - \overline{s}_n \leq \overline{\overline{t}}^* - \overline{\overline{s}}_n, \quad \overline{\overline{t}}^* = \lim_{n\to\infty} \overline{\overline{t}}_n, \tag{60}$$
$$\overline{t}^* - \overline{t}_{n+1} \leq \overline{\overline{t}}^* - \overline{\overline{t}}_{n+1}, \tag{61}$$

and

$$\overline{t}^* \leq \overline{\overline{t}}^*. \tag{62}$$

Clearly, if conditions for the convergence of sequences $\{\overline{\overline{s}}_n\}$, $\{\overline{\overline{t}}_n\}$ are imposed, the same conditions will imply the convergence of the finer sequences $\{s_n\}$, $\{t_n\}$, $\{\overline{s}_n\}$, and $\{\overline{t}_n\}$ $(n \geq 0)$. Such a condition is:

(C) Equation

$$f(r) = 0 \tag{63}$$

has a unique solution $\delta \in [0,R]$.

Note that in this case

$$\lim_{n\to\infty} \overline{\overline{s}}_n = \lim_{n\to\infty} \overline{\overline{t}}_n \leq \delta.$$

The proof is omitted since it has essentially been given in Theorem 2 in [2, p. 5].

Remark 2.2. *Concerning related method (7), let us consider the corresponding scalar majorizing sequences* $\{p_n\}$, $\{q_n\}$, $\{\overline{p}_n\}$, $\{\overline{q}_n\}$, $\{\overline{\overline{p}}_n\}$, $\{\overline{\overline{q}}_n\}$, *(*$n \geq 0$*) defined as the s–t–sequences, respectively.*

For example, sequences $\{p_n\}$, $\{q_n\}$ *as defined as* $\{s_n\}$, $\{t_n\}$ *given in (6) and (7) but* s_n, t_n, t_{n+1}, $\frac{t_n + s_n}{2}$ *are now* p_n, q_n, p_{n+1}, p_n, *respectively, etc.*

Clearly, method (7) also converges under condition (C).

Note that a similar proof as in Theorem 2.1 can be given for method (7). We do not known if the s–t–sequences are finer than the p–q–sequences. In practice, we will use both to see which ones provide the more precise estimates on the distances $\| y_n - x_n \|$, $\| x_{n+1} - x_n \|$, $\| y_n - x^* \|$ $(n \geq 0)$.

Finally note that the results obtained here can be extended to the more general method (2) where $z_n = (1-\lambda)x_n + \lambda y_n$, $0 \leq \lambda \leq 1$. However here we decided to examine (2) only in the case $\lambda = \frac{1}{2}$ which although seems to be the most popular [7], [8], [13] we do not know yet if it is always the best choice.

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