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Open 3-manifolds and branched coverings: a quick exposition

3-variedades abiertas y cubiertas ramificadas

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Dedicado a María Teresa Lozano Imízcoz tras 27 años de fructífera colaboración

Abstract. Branched coverings relate closed, orientable 3-manifolds to links in S^3, and open, orientable 3-manifolds to strings in $S^3 \smallsetminus T$, where T is a compact, totally disconnected tamely embedded subset of S^3. Here we give the foundations of this last relationship. We introduce Fox theory of branched coverings and state the main theorems. We give examples to illustrate the theorems.

Key words and phrases. Knot, link, manifold, string, wild, tame, locally tame, Cantor set, tangle, branched covering, colored knot, Smith conjecture.

2000 Mathematics Subject Classification. 57M12, 57M30,57N10, 54B15.

Resumen. Las cubiertas ramificadas relacionan las 3-variedades orientables cerradas con los enlaces en S^3 y las 3-variedades abiertas con las cuerdas en $S^3 \smallsetminus T$, donde T es un subconjunto compacto, totalmente desconectado y dócilmente encajado en S^3. Aquí exponemos los fundamentos básicos de esta última relación. Introducimos la teoría de Fox de las cubiertas ramificadas y enunciamos los principales teoremas. Damos ejemplos que ilustran los teoremas.

Palabras y frases clave. Nudo, enlace, variedad, cuerda, salvaje, dócil, localmente dócil, conjunto de Cantor, ovillo, conjetura de Smith.

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1. Introduction

Branched coverings (to be defined later) relate closed 3-manifolds to (tame) knots and links (see [3], [16], [17], [27]).

From this relationship "knot theory" obtains invariants of knots, and "3-manifold theory" reduces its study to knot theory. As we see, this is a fruitful relationship.

This paper is devoted to give an exposition of this relationship in the case of open 3-manifolds and wild knots and strings in S^3 via branched coverings. Details are in the papers [18], [22], [19], [21], [20]. We give new examples and some new theorems.

2. Some mixed preliminaries

2.1. **Topological preliminaries.** In Fox´s terminology [7] *X is locally connected in Y* iff Y has a base whose members intersect X in connected sets.

Every connected, locally connected, locally compact, metrizable and separable space X is contained in a compact space Y with the same properties in such a way that X is dense, open and locally connected in Y, and the *end space* $E(X) := Y \setminus X$ is (compact and) totally disconnected. Moreover, this compactification Y of X (*ideal* or *Freudenthal compactification*) is determined by these properties [9] (see also [7] and [23]).

We say that a compact, totally disconnected subset T of S^3 is *tamely embedded* if there exists a homeomorphism of S^3 placing T on a rectilinear segment of $S^3 = \mathbb{R}^3 + \infty$. Otherwise, T is *wildly embedded*. Accordingly, we will consider tamely (or wildly) embedded Cantor subsets of S^3.

A 3-*cell* will be a space homeomorphic to the closed ball of radius 1 in $\mathbb{R}^3$; an *open* 3-*cell* will be a space homeomorphic to $\mathbb{R}^3$.

A 3-*manifold* M is a connected, metrizable topological space such that every point of M has an open 3-cell neighbourhood. By the theorem of invariance of domain, that open 3-cell neighbourhood is an open subset of M. Therefore, every point of M has also a 3-cell neighbourhood. A $3-$*manifold-with-boundary* M is a connected, metrizable topological space such that every point of M has a 3-cell neighbourhood. Therefore, every 3-manifold is a 3-manifold-with-boundary but the converse is false: a 3-cell is a 3-manifold-with-boundary but it is not a 3 -manifold.

A 3-manifold will be called *closed* (resp. *open*) iff it is compact (resp. noncompact). Of course, an open 3-cell is an open 3 -manifold. The 3-sphere S^3 is a closed 3-manifold.

2.2. **Combinatorial preliminaires.** A *polyhedron* is the underlying space of a locally finite simplicial complex. Moise proved [12] that all 3-manifolds-with-boundary are *triangulated* by polyhedra.

2.3. **Tame and wild sets.** A closed set F in a *3*-manifold-with-boundary M is *tame* if there is a homeomorphism of M in itself sending F onto a subcomplex

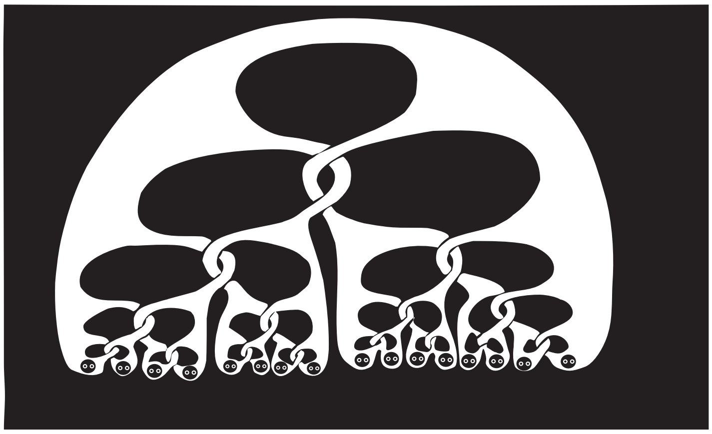


Figure 1. A wild 2-cell.

of some locally finite simplicial complex triangulating M. If there is no such homeomorphism, we say that F is *wild*. For example the 2-cell in S^3 shown in Figure 1 is wild.

The set X is *locally tame* at a point x of X if there exist a neighbourhood U of x in M and a homeomorphism of U into M that takes $U \cap X$ onto a tame set. Otherwise we say that X is *locally wild* at x.

Bing [1] showed that a closed set is tame in S^3 if it is locally tame at each of its points. The set of points of X at which it is locally tame is open in X and is called the *tame subset* of X, while the subset at which it is locally wild is closed, and is called the *wild subset* of X.

A subset N of a 3-manifold is a *knot* if N is homeomorphic to the 1-sphere $\mathbb{S}^1$. A locally finite disjoint union of knots is a *link*. A *wild knot or link* has a non empty wild subset. Otherwise it is a *tame knot* or *link*.

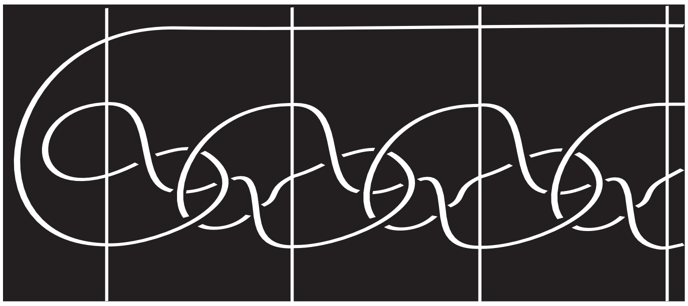


Figure 2. A string.

For instance, the wild 2-cell of Figure 1 bounds a wild knot whose wild subset is a tamely embedded Cantor set.

A closed subset of a 3-manifold is a *string* (see Figure 2) if it is homeomorphic to the real line $\mathbb{R}^1$. A locally finite disjoint union of strings is a *string-link*. A *wild string* or *string-link* has a non empty wild subset. Otherwise it is a *tame string* or *string-link*.

A knot N in S^3 is the *unknot* if N bounds a tame disk in S^3. A string in $\mathbb{R}^3$ is the *unstring* if it bounds a tame half-plane in $\mathbb{R}^3$.

3. Combinatorial branched coverings

Let $f : Y \to Z$ be a (continuous) map. An open neighbourhood W of $z \in Z$ is called *elementary* if f maps each (connected) component of $f^{-1}(W)$ homeomorphically onto W. A point of Z that admits an elementary neighbourhood is an *ordinary point*. A non-ordinary point is *singular*. The set of ordinary points of Z is an open subset Z_o of Z. Its complement Z_s is the closed set of singular points. If Y is connected and locally connected, and $Z = Z_o$, f is called *covering*.

The definition of branched covering is relatively easy if the spaces involved are polyhedra.

Thus, if M and N are 3-manifolds and B is a subset of N, then a continuous map

$$f : M \to N$$

is a *(combinatorial) covering of N branched over B* if there are triangulations of M and N such that

(i) B is a subpolyhedron of the 1-skeletton of N;
(ii) f is a non-degenerate simplicial map;
(iii) the restriction
$$f|(M \smallsetminus f^{-1}(B)) : M \smallsetminus f^{-1}(B) \to N \smallsetminus B$$
is a covering called the *associated covering*. (The *associated unbranched covering* is the composition of the covering $f|(M \smallsetminus f^{-1}(B))$ with the natural inclusion $N \smallsetminus B \subset N$).
(iv) the set Z_s of singular points of f is B. The set B is called the *branching set of f*.

Then $f^{-1}(B)$ is a subgraph of M which is the disjoint union of the subgraph $\widetilde{P}$ (*pseudo-branching cover)* of points at which f is a local homeomorphism and the subgraph $\widetilde{B}$ (*branching cover*) of points at which f is not a local homeomorphism.

The importance of this combinatorial definition of branched covering lies in the fact that, given the monodromy of the associated unbranched covering, it is possible to construct the branched covering (the *monodromy* is a representation of $\pi_1(N \smallsetminus B)$ into the group of bijections of the fiber [25]). We explain what we understand by this with an example.

Let (L, ω) be a *represented knot or link* in $X = S^3$ (resp. a *represented string or string-link* in $X = \mathbb{R}^3$), where ω is a simple representation (homomorphism) of $\pi_1(X \setminus L)$ onto the symmetric group S_3 of the indices $\{1, 2, 3\}$. Thus ω sends meridians of L to transpositions $(1, 2), (1, 3)$, or $(2, 3)$ of S_3, which, following a beautiful idea of Fox, will be represented by colors Red ($R = (1, 2)$), Green ($G = (1, 3)$) and Blue ($B = (2, 3)$). If the representation exists we can endow each overpass of a normal projection of L with one of the three colors R, G, B in such a way that the colors meeting in a crossing are all equal or all are different. Moreover, at least two colors are used. A knot or link L (resp. string or string-link L) in $X = S^3$ (resp. $X = \mathbb{R}^3$) with a coloration corresponding to some ω is a *colored knot or link* (resp. *colored string or string-link*).

A colored knot or link (resp. string or string-link) (L, ω) in $X = S^3$ (resp. $X = \mathbb{R}^3$) defines a complete conjugation class of subgroups of $\pi_1(X \setminus L)$. Namely the set $\{\omega^{-1}(Stab_i) : i \in \{1, 2, 3\}\}$ where $Stab_i$ is the subgroup of elements of S_3 fixing the index i. This class of subgroups determines an ordinary covering of three sheets

$$f' : Y \to X \setminus L$$

and an unbranched covering $f'' = j \circ f'$ where

$$j : X \setminus L \to X$$

is the inclusion map. A *construction* described by Neuwirth in [24] gives an extension of $f''$to a combinatorial covering $f : M(L, \omega) \to X$ branched over L such that the associated unbranched covering of f turns out to be f''. Due to a theorem of Fox (see the next section), this f is *unique*. We emphasize that f is uniquely determined by f'', not in general by f' [11]. The space $M(L, \omega)$ is a closed (resp. open), orientable 3-manifold.

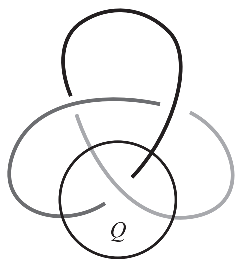


FIGURE 3. A colored knot.

Example 1. *Consider the colored knot (L, ω) of Figure 3. This colored knot was considered by R. H. Fox in* [8]. *Neuwirth construction shows that the closed manifold $M(L, \omega)$ is S^3 (we do not give details; see* [16], [17] *and* [27]*).*

On top of the 3-cell Q (resp. $S^3 \setminus IntQ$) of Figure 3 lies a 3-cell.

An easy consequence of this is the following result.

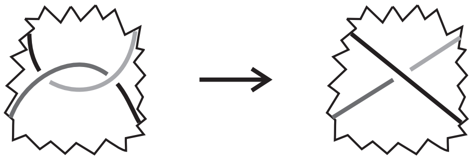


Figure 4. The move.

The move of Figure 4, [13], [14], [16], [17], has the following property: *if this move is applied to a portion of a colored knot or link (resp. string or string-link), we obtain a new colored knot or link (resp. string or string-link) whose corresponding 3-fold branched covering spaces are homeomorphic.*

It was proved in [10] and [15], independently, that *every closed orientable* 3*-manifold is of the form* $M(L,\omega)$ *for countably many mutually inequivalent knots* L. In other words, *every closed orientable* 3*-manifold is a simple* 3 *fold covering of* S^3 *branched over a knot in many different ways.*

Example 2. *Consider the colored string* (L,ω) *in* $\mathbb{R}^3$ *of Figure 5. This colored string was first considered by R. H. Fox in the paper* [6]*; entitled* "A remarkable simple closed curve". *We will call* L Fox string. *We will prove that the space* $M(L,\omega)$ *is homeomorphic to* $\mathbb{R}^3$*. Thus* there exist a 3-fold simple covering $\widehat{p}:\mathbb{R}^3 \to \mathbb{R}^3$ branched over the Fox string L.

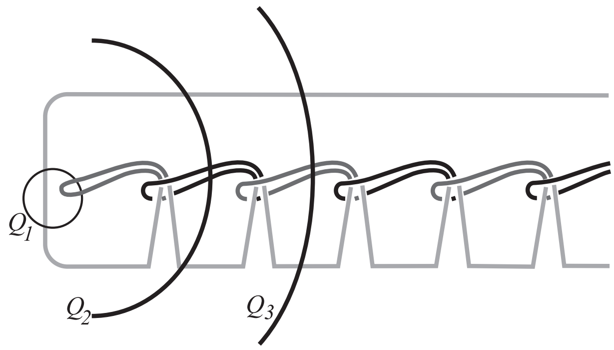


Figure 5. Fox string colored.

In fact, select a sequence of 3-cells $\{Q_i\}_{i=1}^{\infty}$ such that $Q_i \subset Int(Q_{i+1})$ and

$$\cup_{i=1}^{\infty} Q_i = \mathbb{R}^3 = S^3 \setminus \{\infty\},$$

as indicated in Figure 5. Let

$$p : M(L,\omega) \to \mathbb{R}^3$$

be the simple branched covering given by the representation ω. Then, for $i \geq 1$, $p^{-1}(Q_i)$ is a 3-cell Q'_i. In fact,

$$p \mid p^{-1}(Q_i) : p^{-1}(Q_i) \to Q_i$$

is a 3-fold simple covering of the 3-cell Q_i, branched over two properly embedded arcs; these arcs are embedded exactly as in case $i = 1$ (see Figure 5). By Example 1, $p^{-1}(Q_i)$ is a 3-cell Q'_i. Then

$$M(L, \omega) = \cup_{i=1}^{\infty} Q'_i.$$

And from this follows that $M(L, \omega)$ is homeomorphic to $\mathbb{R}^3$ [2] as we wanted to prove.

Exercise. Find an alternative proof using the move of Figure 4.

4. **Topological branched coverings**

We will see in this paper that open 3-manifolds are naturally related to coverings branched over wild knots and strings. Therefore the combinatorial definition of branched covering is not sufficient. We need a topological definition of branched covering.

This definition was provided by R. H. Fox in his celebrated paper [7] (a fresh exposition of it is in [23]). We now explain this definition.

If Z is a topological space denote by $\mathcal{E}(z)$ the set of open neighbourhoods of $z \in Z$. If $f : Y \to Z$ is a map, a *thread* y_z *over* z is a function $W \longmapsto y_z(W)$ where $W \in \mathcal{E}(z)$ and $y_z(W)$ is a component of $f^{-1}(W)$ such that

$$y_z(W_2) \subset y_z(W_1) \text{ if } W_2 \subset W_1.$$

A *branched covering* is a continuous map $g : Y \to Z$ between connected, locally connected T_1 spaces such that

(i) the connected components of the inverse images of the open sets of Z form a base for the topology of Y;
(ii) the set Z_o of ordinary points is connected, dense and locally connected in Z;
(iii) the set $g^{-1}(Z_o)$ is connected, dense and locally connected in Y; and
(iv) g is *complete*, that is, for every thread y_z over z, and for every z, the intersection

$$\bigcap_{W \in \mathcal{E}(z)} y_z(W)$$

is non-empty (and consists of just one point).

The map

$$g \mid g^{-1}(Z_o) : g^{-1}(Z_o) \to Z;$$

is called the *unbranched covering associated to* g, and the map

$$g \mid g^{-1}(Z_o) : g^{-1}(Z_o) \to Z_o$$

is called the *covering associated to* g. The degree of the covering associated to g is, by definition, the *degree of* g.

In a branched covering $g : Y \to Z$, the set $Z_s = Z \setminus Z_o$ will be called the *branching set* and we will say that g *is a covering of* Z *branched over* Z_s.

Let z_s be a point of Z_s. The *branching index* $b(y)$ of $y \in g^{-1}(z_s)$ is the infimum of the degrees of the coverings associated with the branched coverings

$$g \mid C : C \to W,$$

where C is the y-component of $g^{-1}(W)$, for every open, connected neighbourhood W of z_s. If all these degrees are infinite, we will say that the branching index is infinite.

The subset of $g^{-1}(Z_s)$ of points with branching index 1 is called the *pseudo-branching cover*, and the subset of $g^{-1}(Z_s)$ of points with branching index ≥ 1 is called the *branching cover*.

The main result in [7] implies the following

Theorem 4.1. *Let $f : Y \to Z$ be a continuous map between connected, locally connected T_1 spaces that satisfies conditions (i), (ii), (iii) above. Then there exists a unique branched covering $g : X \to Z$ extending f.*

The branched covering g granted by this theorem will be called the *Fox completion* of f.

A useful method to construct branched coverings is to complete unbranched coverings by means of Theorem 4.1. This is the content of the next corollary.

Corollary 4.1. *Let $g : Y \to Z$ be a branched covering. Then g is the Fox completion of its associated unbranched covering*

$$g \mid g^{-1}(Z_o) : g^{-1}(Z_o) \to Z.$$

Thus, g is determined by the unbranched covering

$$g \mid g^{-1}(Z_o) : g^{-1}(Z_o) \to Z;$$

or by the associated covering

$$g \mid g^{-1}(Z_o) : g^{-1}(Z_o) \to Z_o$$

and the inclusion $Z_o \subset Z$.

Of course, this construction can be brought about when Z_s is tame (see last section). But for particularly nice wild branching sets we can also bring about the construction. We explain this in the next section.

5. Constructing branched coverings for nice branching sets

Let T be a compact, totally disconnected subset of a manifold M. A T-tangle will be a closed subset F of M containing T such that (i) $M \setminus F$ is dense and locally connected in M and, (ii) $F \setminus T$ is tame in $M \setminus T$. The interest of this definition lies in the following

Theorem 5.1. *Let M be a closed manifold. Let $g : Y \to M$ be a finite sheeted covering branched over a T-tangle B. Then g is uniquely determined by the (combinatorial) covering*

$$g|\left(Y \setminus g^{-1}(T)\right) : Y \setminus g^{-1}(T) \to M \setminus T$$

branched over the tame set $F \setminus T$. Moreover, Y is the ideal compactification of $Y \setminus g^{-1}(T)$.

Proof. It follows from the Jordan separation theorem that *if T is a compact, totally disconnected subset of a closed n-manifold M, then M is the ideal compactification of $M \setminus T$.* Since $M \setminus B$ is dense and locally connected in M it follows that $M \setminus T$ is dense and locally connected in M. By the uniqueness of the ideal compatification, it follows that the inclusion of $M \setminus T$ in M is unique up to homeomorphism. Therefore the Fox completion of

$$g|\left(Y \setminus g^{-1}(T)\right) : Y \setminus g^{-1}(T) \to M$$

is uniquely determined by

$$g|\left(Y \setminus g^{-1}(T)\right) : Y \setminus g^{-1}(T) \to M \setminus T.$$

That Y is the ideal compactification of $Y \setminus g^{-1}(T)$ follows from the following theorem (see [7] and [23]). ☑

Theorem 5.2. *Let X and Z_1 be metrizable and separable spaces. Assume also that they are connected, locally connected, locally compact, but not compact. Let*

$$f : X \to Z_1$$

be a surjective branched covering. Let Z be the ideal compactification of Z_1 and let

$$j : Z_1 \to Z$$

be the canonical inclusion. If

$$g : Y \to Z$$

denotes the Fox completion of

$$j \circ f : X \to Z,$$

then Y is the ideal compactification of X if Z has a base such that, for each member W of it, $f^{-1}(W)$ has a finite number of components.

Example 1. A tamely embedded or wildly embedded Cantor set C in S^3 is a C-tangle.

Example 2. A wild knot in S^3 with one-point wild subset x is an $\{x\}$-tangle. The wild knot in $S^3 = \mathbb{R}^3 + \{\infty\}$ with wild subset $\{\infty\}$ whose associated tame string in $\mathbb{R}^3$ is Fox string L of Figure 5 will be called *Fox remarkable wild-knot.*

Example 3. More generally, let L be a tame string-link in $\mathbb{R}^3$. Then $\widehat{L} := L \cup \{\infty\}$ is a $\{\infty\}$-tangle in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The string-link in $\mathbb{R}^3$ of Figure 6 gives rise to a very complicated $\{\infty\}$-tangle in S^3.

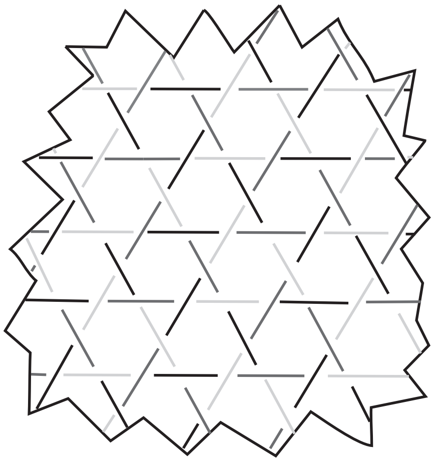


Figure 6. Colored string-link.

Example 4. The boundary L of the wild disk of Figure 1 is a wild knot in S^3 with wild subset L_w a tamely embedded Cantor set. Moreover L is a L_w-tangle because $L \smallsetminus L_w$ is a tame string-link in $S^3 \smallsetminus L_w$ (with countably many components).

Example 5. In $S^3 = \mathbb{R}^3 \cup \{\infty\}$ consider (see Figure 7) the union L' of the circles of diameter $1/3^{(n+1)}$, lying in the xy-plane and centered at $(m/(2\cdot 3^n), 0, 0)$ for every integer $n \geq 1$ and every integer m such that $0 < m < 2 \cdot 3^n$ and $gcd(m, 6) = 1$, together with the circle centered at $(1/2, 0, 0)$ of radius 1. The adherence L of this set L' is a C-tangle. Here C is the standard tamely embedded Cantor set and $L \smallsetminus C$ is the union of half-circles $L' \smallsetminus ([0, 1] \times \{0\} \times \{0\})$.

A *represented* T*-tangle* (L, ω) is a T-tangle L in S^3 together with a transitive representation ω of $\pi_1(S^3 \smallsetminus L)$ into the symmetric group S_n of n indices. The representation is *simple* if it represents meridians of $L \smallsetminus T$ by transpositions.

A represented T-tangle (L, ω) in S^3 defines a complete conjugation class of subgroups of $\pi_1(S^3 \smallsetminus L)$. Namely the set $\{\omega^{-1}(Stab_i) : i \in \{1, 2, ..., n\}\}$ where $Stab_i$ is the subgroup of elements of S_n fixing the index i. This class of subgroups determines a combinatorial covering

$$f' : M(L, \omega) \to S^3 \smallsetminus T$$

branched over the tame set $L \smallsetminus T$, where $M(L, \omega)$ is an open 3-manifold, oriented and connected that can be constructed by Neuwirth method.

Let f be the composition $j \circ f'$, where

$$j : S^3 \smallsetminus T \to S^3$$

is the inclusion map. Since f satisfies the conditions listed in Theorem 4.1 there exists the Fox completion

$$\widehat{f} : \widehat{M}(L, \omega) \to S^3$$

of f. This map is a covering of S^3 branched over L, extending f'.

The space $\widehat{M}(L, \omega)$ is the ideal compactification of the manifold $M(L, \omega)$ (Theorem 5.1).

In general $\widehat{M}(L, \omega)$ is not a manifold at the points belonging to the end space

$$E(M(L, \omega)) := \widehat{M}(L, \omega) \smallsetminus M(L, \omega).$$

Of course, $\widehat{f}(E(M(L, \omega))) = T$.

The covering

$$\widehat{p} : \widehat{M}(L, \omega) \to S^3$$

will be called *simple* if ω is simple.

Example 6. Fox remarkable wild knot L whose tame part $L \smallsetminus \{\infty\}$ is depicted in Figure 5 is a colored $\{\infty\}$-tangle (L, ω). The manifold $M(L, \omega)$ is $\mathbb{R}^3$ and $\widehat{M}(L, \omega)$ is the one-point compactification of $\mathbb{R}^3$. Thus there is a 3-fold simple branched covering $\widehat{p} : S^3 \to S^3$ branched over the Fox remarkable wild knot L [19].

Trading what is outside the 3-cell Q_1 of Figure 5 with what is inside it, one gets the move indicated in Figure 8.

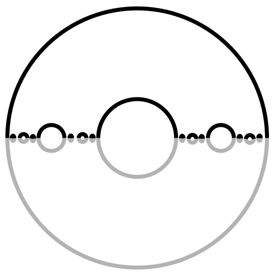


FIGURE 7. A C-tangle.

Applying this move to colored (tame) knots one gets wild knots having the same branched coverings. Therefore, exactly as with S^3, *every closed, oriented 3-manifold is a 3-fold covering of S^3 branched over a colored wild knot.*

Fox remarkable wild knot L admits a representation η onto S_2. The manifold $M(L, \eta)$ is the 2-fold cyclic covering of $\mathbb{R}^3$ branched over the Fox string of Figure 5. Then $M(L, \eta)$ is the union of an ascending sequence of solid tori $\widetilde{Q}_i$, namely, the solid tori covering the 3-cells Q_i of Figure 5. It is not hard to see that the core of $\widetilde{Q}_i$ goes twice around the core of $\widetilde{Q}_{i+1}$ (see Figure 9). An appropriate

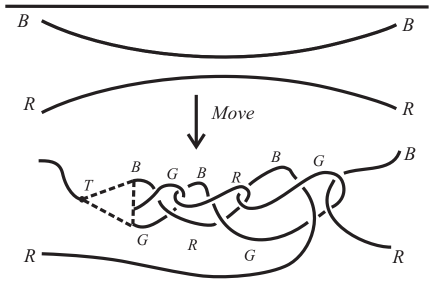


FIGURE 8. A move.

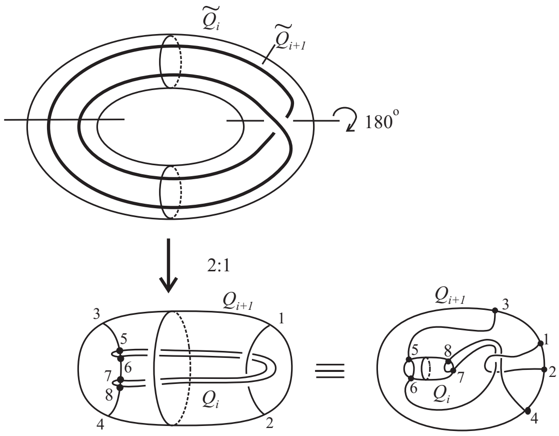


FIGURE 9. A 2-fold branched covering.

name for this open 3-manifold could be *ascending solenoidal manifold* (see [20] and [19]).

Example 7. Consider the C-tangle L of Example 5. Here C is the standard tamely embedded Cantor set so that $\pi_1(S^3 \smallsetminus L_w)$ is trivial. Then $\pi_1(S^3 \smallsetminus L)$ is freely generated by the meridians of the upper half circles of $L' \smallsetminus L' \smallsetminus ([0,1] \times \{0\} \times \{0\})$ and the meridians of the lower half circles of $L' \smallsetminus L' \smallsetminus ([0,1] \times \{0\} \times \{0\})$. We define a colored C-tangle (L, ω) coloring the upper (resp. lower) meridians red (resp. green) as depicted in Figure 7.

This defines a 3-fold simple covering

$$\widehat{p} : \widehat{M}(L, \omega) \to S^3$$

branched over L.

The space $\widehat{M}(L, \omega)$ is S^3. In fact, applying the move shown in Figure 10, which does not affect the covering space (see [14]), simultaneously in all possible places in Figure 7 we see that $\widehat{M}(L, \omega)$ is homeomorphic to the 3 -fold cyclic covering of S^3 branched over the unknot. Thus $\widehat{M}(L,\omega)$ is the 3-sphere. (To help the reader, in Figure 5 a sequence of the moves of Figures 10 and 11 (and their inverses) is used to produce the move of Figure 4, [14]).

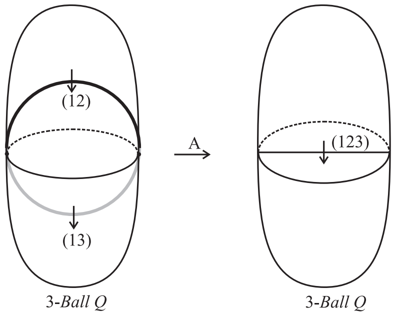


Figure 10. A move.

As in the previous example, we can apply the move of Figure 8 countably many times to any colored tangle and we easily obtain that *every closed, oriented 3-manifold is a 3-fold simple covering of S^3 branched over a colored C-tangle L such that C is a tamely embedded Cantor set and $L \smallsetminus C$ is a string-link.*

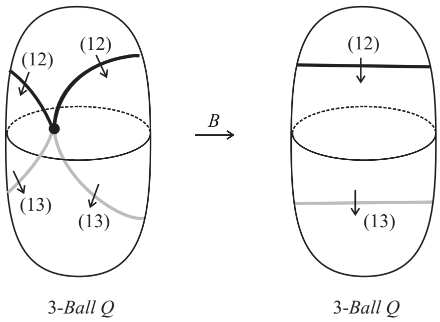
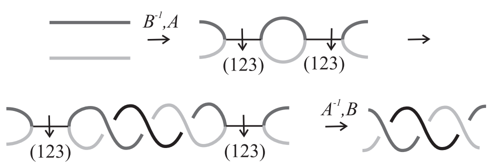


Figure 11. A move.



In [18] we have proved the following theorem.

Theorem 5.3. *Let M be an orientable 3-manifold. Let $E(M)$ be the end space of M. Then there is a colored T-tangle (L,ω) whose associated 3-fold simple covering*

$$\widehat{p} : \widehat{M}(L,\omega) \to S^3$$

branched over L satisfies:

(i) *$\widehat{M}(L,\omega)$ is the ideal compactification of M;*
(ii) *$\widehat{p}|E(M)$ maps $E(M)$ homeomorphically onto T;*
(iii) *T is tamely embedded in S^3;*
(iv) *if T is not empty, $L\setminus T$ is a tame string-link in $S^3\setminus T$ with $\overline{L\setminus T} = L$;*
(v) *$M(L,\omega)$ is M and $p := \widehat{p}\,|\,M$ is a combinatorial 3-fold simple covering $p : M \to S^3 \setminus T$ branched over the tame string-link $L \setminus T$.*

This theorem generalizes [10] and [15]. In these two papers the above theorem is proved for closed, orientable 3-manifolds with the additional property that L is a (tame) knot and T is empty.

A corollary of the above theorem is that *if C is a wildly embedded Cantor set in S^3 there exists a 3-fold simple branched covering $p : S^3 \to S^3$ sending C homeomorphically onto a tamely embedded Cantor set.* This has interesting consequences (see [21]).

Example 8. Consider the C-tangle of Example 4, which is the boundary L of the wild disk of Figure 1. We have proved [22] that the n-fold cyclic coverings of S^3 branched over L are all 3-spheres for every $n \geq 2$. Moreover the preimages of C are wildly embedded. There are uncountably many different disks, like the one of Figure 1, enjoying the same property.

Corollary 5.1. *Let M be a closed, orientable 3-manifold which is an n-fold cyclic covering of S^3 branched over a tame knot or link. Then M is also an n-fold cyclic covering of S^3 branched over a C-tangle, where C is a Cantor set in uncountably many different ways. Moreover C can be made to be tamely embedded or wildly embedded.*

Proof. We see [22] that there is an n-fold cyclic covering $S^3 \to S^3$ branched over a wild knot L with L_w a tamely embedded Cantor set. If $n = 2m$, this covering factors through an m-fold cyclic covering $S^3 \to S^3$ branched over a wild knot L' with L'_w a wildly embedded Cantor set, and a 2-fold cyclic covering $S^3 \to S^3$ branched over a wild knot L with L_w a tamely embedded Cantor set. We use these branching sets to create moves. These moves applied to links in S^3 give the required branching sets. ☑

These theorems and corollaries can be rephrased in terms of exotic cyclic actions on closed, orientable 3-manifolds. We leave this to the reader.

Another consequence of the moves in the proof of the above corollary is that *every closed, oriented 3-manifold is a 3-fold simple covering of S^3 branched over a colored wild knot N such that N_w is a Cantor set. Moreover C can be made to be tamely embedded or wildly embedded.*

6. Open problems

To finish, here are some open problems.

Question 1 (Allan Edmonds). *Let L be a wild knot in S^3 such that, for some $n \geq 2$ the n-fold cyclic covering of S^3 branched over L is the 3-sphere. Then L is the boundary of an embedded 2-cell with locally tame interior (see MR2031886).*

If the answer to this question is yes, the following conjecture follows.

Conjeture 1 (Smith Conjecture for R^3). *Let L be a tame string in $\mathbb{R}^3$ such that for some $n \geq 2$, the n-fold cyclic covering of $\mathbb{R}^3$ branched over L is $\mathbb{R}^3$. Then L is the unstring.*

Question 2. *Is there a 3-manifold M and a covering $p : M \to S^3$ branched over a (wildly embedded) Cantor set? I conjecture in the negative. (Note that there are Cantor sets wildly embedded in S^3 with no simply connected complement.)*

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