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# On conics, rational functions and the Teager-Kaiser operator

Alfredo Restrepo Palacios
Universidad de los Andes, Bogotá

Abstract. Given a rational transformation of a multi-pinched plane (i.e. the plane minus a finite set of points) into the plane, a family of conics invariant under the transformation is found. In phase space, the solutions of a related linear difference equation of the third order live on such conics. This connection between rational functions, families of algebraic curves and linear difference equations is introduced with the help of a few definitions and several examples. The connection results from a generalization of the analysis of the fixed points of the Teager-Kaiser operator, of relevance in signal and image processing, in engineering.

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Resumen. Dada una transformación racional del plano, desprovisto de un número finito de puntos, se encuentra una familia de cónicas invariante por la transformación. En el espacio de fase las soluciones de la correspondiente ecuación lineal en diferencias de tercer orden viven en tales cónicas. Esta conexión entre funciones racionales, familias de curvas algebraicas y ecuaciones lineales en diferencias se introduce con la ayuda de unas pocas definiciones y varios ejemplos. La conexión resulta de una generalización del análisis de los puntos fijos del operador de Teager-Kaiser el cual tiene relevancia en la ingeniería en el procesamiento de señales e imágenes.

## A. **Definitions**

Let a *connecting equation* in $n$ variables be a polynomial equation of the form $f(a_1, a_2, \ldots, a_n) = 0$. Let an *invariant for the connecting equation* be a rational function $\alpha : \mathrm{R}^{n-1} \setminus \mathrm{P}' \to \mathrm{R}^1$, with $\mathrm{P}'$ the set of poles, such that the equality $\alpha(a_1, a_2, \ldots, a_{n-1}) = \alpha(a_2, a_3, \ldots, a_n)$ can be derived from the connecting equation.

In the case of three variables, which will be the only case considered from this point on, we assume that the connecting equation $f(a, b, c) = 0$ can be further taken to the form $c = g(a, b)$, with $g : \mathrm{R}^2 \setminus \mathrm{P} \to \mathrm{R}^1$ rational and P its set of poles. Thus, if $f(a, b, c) = f(c, b, a)$ then $f(a, b, c) = 0$ can also be taken to the form $a = g(c, b)$, and we say that the connecting equation has *no sense of direction*. On the other hand, if for each pair $(x, y)$ in the domain of $\alpha$, $\alpha(x, y) = \alpha(y, x)$, we say that $\alpha$ is *symmetric*. Note that $\alpha(a, b) = \alpha(c, b)$, and symmetric $\alpha$, imply $\alpha$ invariant: $\alpha(a, b) = \alpha(b, c)$.

An often successful method to get an invariant for an $f$ with no sense of direction is to start from

$$a + g(a, b) = g(c, b) + c$$

where there are only $a$´s and $b$´s on the left, and on the right only $b$´s and $c$´s, and to add, subtract, multiply and/or divide by polynomials in the variable $b$ until an expression of the form $\alpha(a, b) = \alpha(c, b)$, with $\alpha$ symmetric, is obtained; then, $\alpha$ is also an invariant: $\alpha(a, b) = \alpha(b, c)$.

In several cases, from the expression $\alpha = \alpha(b, c)$, and from the connecting equation $f(a, b, c) = 0$, it is possible to get a linear equation $\lambda(a, b, c) = 0$, with $\alpha$ a "coefficient" in the equation.

Likewise, from the expression $\alpha = \alpha(a, b)$ it is sometimes possible to get a second degree equation $C(a, b) = 0$, with $\alpha$ a coefficient. In such a case, since $\alpha(a, b) = \alpha(b, c)$, we also have $C(b, c) = 0$, thus $C(a, b) = C(b, c)$; therefore, $C$ is also an invariant.

## B. **Examples**

The two following examples deal with the root and the pre-constant signals of the *Teager-Kaiser operator* [**?**]. The examples of this section will make a discussion of the operator in Section C shorter.

**B1.** Let $ac - b(b - 1) = 0$, be the connecting equation. We get an invariant as follows. From the connecting equation we have $a = b(b-1)/c$ and $c = b(b-1)/a$, then

$$c + b(b - 1)/c = a + b(b - 1)/a \tag{1}$$

as a function of $a$ and $b$, $a+b(b-1)/a$ is neither an invariant for $f=0$ neither symmetric; dividing by $b$, we get

$$c/b+b/c-1/c=a/b+b/a-1/a$$

finally, adding $-1/b$ on each side, we get

$$c/b+b/c-1/c-1/b=a/b+b/a-1/a-1/b$$

now $a/b+b/a+1/a+1/b$ is symmetric in the variables $a$ and $b$; thus,

$$\alpha(x,y)=x/y+y/x-1/x-1/y$$

is an invariant for $f=0$.

A linear equation $\lambda(a,b,c)=0$ is obtained as follows. Write

$$\begin{aligned} b\alpha &= b\,\alpha(a,b) \\ &= 1-c+b(1-b)/c \end{aligned}$$

and, from the connecting equation, we get

$$= 1-c-a$$

the nonhomogeneous

$$a+\alpha b+c=1$$

the validity of this approach will be made clear below, where the variables $a$, $b$ and $c$ are interpreted as shifted versions $x_{n-1}$, $x_n$ and $x_{n+1}$ of a sequence $\{x_n\}$.

Finally we get a (quadratic) invariant, this time involving $\alpha$. From

$$\alpha(a,b)=a/b+b/a-1/a-1/b$$

and writing

$$\begin{aligned} ab\alpha &= ab\,\alpha(a,b) \\ &= a^2+b^2-a-b \end{aligned}$$

we get,

$$a^2+b^2-ab\alpha-a-b=0.$$

Likewise, from

$$\alpha(b,c)=c/b+b/c-1/c-1/b$$

we get

$$b^2+c^2-bc\alpha-b-c=0$$

thus, defining

$$C(x,y)=x^2+y^2-xy\,\alpha(x,y)-x-y$$

since $\alpha$ is an invariant, we have that $C$ is also an invariant: $C(a,b)=C(b,c)$.

**B2.** Let $f(a,b,c) = ac - b^2 + \kappa$ be the connecting equation. From the connecting equation we have $a = (b^2 - \kappa)/c$ and $c = (b^2 - \kappa)/a$. Thus, we write

$$a + (b^2 - \kappa)/a = c + (b^2 - \kappa)/c$$

to get symmetry we divide by b getting

$$a/b + (b^2 - \kappa)/ab = c/b + (b^2 - \kappa)/cb$$

or

$$a/b + b/a - \kappa/ab = c/b + b/c - \kappa/cb$$

thus

$$\alpha(x,y) = x/y + y/x - \kappa/xy$$

is a (symmetric) invariant.

Now, to get a linear equation $\lambda(a,b,c) = 0$, note that

$$\begin{aligned} b\,\alpha(a,b) &= a + b^2/a - \kappa/a \\ &= a + (b^2 - \kappa)/a \\ &= a + c \end{aligned}$$

thus we have,

$$a - \alpha b + c = 0$$

or, with $\alpha'(x,y) := -\alpha(x,y)$, also a symmetric invariant,

$$a + \alpha' b + c = 0.$$

Finally we get a conic equation. From

$$\alpha(a,b) = a/b + b/a - \kappa/ab$$

we get

$$ab\alpha(a,b) = a^2 + b^2 - \kappa$$

then,

$$a^2 + b^2 - ab\,\alpha(a,b) - \kappa = 0.$$

Likewise, from

$$\alpha(b,c) = c/b + b/c - \kappa/bc$$

we get

$$b^2 + c^2 - bc\,\alpha(b,c) - \kappa = 0$$

Therefore, defining

$$C(x,y) = x^2 + y^2 - xy\,\alpha(x,y) - \kappa$$

we have $C(a,b) = C(b,c)$.

## C. The Teager-Kaiser operator

The Teager-Kaiser operator TK : $\mathrm{R}^{\mathrm{Z}} \to \mathrm{R}^{\mathrm{Z}}$ is a map from the space of bilateral real sequences to itself, given by $\mathrm{TK}(x)_n = x_n^2 - x_{n-1}x_{n+1}$.

A sequence $x$ is a *fixed point* of the operator if $\mathrm{TK}(x) = x$. Also, if $k$ is the constant sequence of value $\kappa$ and $\mathrm{TK}(x) = \kappa$, $x$ is said to be a *pre-constant* sequence for the constant $\kappa$.

**C1. The determinate fixed points of TK live on conics in phase space.** Let $x$ be a fixed point i.e., such that for each integer $n$,

$$x_n = x_n^2 - x_{n-1}x_{n+1}.$$

Then, so long as $x_{n-1} \neq 0$,

$$x_{n+1} = (x_n^2 - x_n)/x_{n-1} \tag{2}$$

If a sequence is solution of Equation (**??**), it is called a *determinate fixed point* of TK, in order to differentiate it from fixed points that take the value 0.

Let $x$ be a determinate fixed point and note that, for each $n$, with $a = x_{n-1}$, $b = x_n$, and $c = x_{n+1}$, (**??**) is a connecting equation with no sense of direction. Therefore, according to Example B.1, $\alpha$ being an invariant,

$$\alpha(x_{n-1}, x_n) = x_{n-1}/x_n + x_n/x_{n-1} - 1/x_{n-1} - 1/x_n$$

is a constant, independent of $n$. Then $x$ is a solution of the linear equation

$$x_{n-1} + \alpha x_n + x_{n+1} = 1 \tag{3}$$

and also a solution of the conic equation

$$C(x_{n-1}, x_n) = 0$$

where

$$C(x_{n-1}, x_n) = x_n^2 + x_{n-1}^2 + \alpha x_n x_{n-1} - x_n - x_{n-1}$$

is a constant, independent of $n$ and in particular, if $C(x_{n-1}, x_n) = 0$, also $C(x_n, x_{n+1}) = 0$; thus, in phase space, each pair of consecutive components $(x_{n-1}, x_n)$ of $x$, belongs to the same conic.

Thus, a point in a conic $x^2 + y^2 - \alpha xy - x - y = 0$ remains on the same conic under the transformation $g : \mathrm{R}^2 - S \to \mathrm{R}^2$, with $S = \{(0, y) : y \in \mathrm{R}^1\}$, given by $g(x, y) = (y, y(y-1)/x)$. Also, the solutions of $x_{n-1} + \alpha x_n + x_{n+1} = 1$ live on the corresponding (i.e. that with the same $\alpha$) conic.

**C2. Determinate pre-constant sequences live on conics in phase space.** Let $x$ be a pre-constant signal for the constant $\kappa$. Then, for each integer $n$, $\kappa = x_n^2 - x_{n-1}x_{n+1}$. Then, so long as $x_{n-1} \neq 0$,

$$x_{n+1} = (x_n^2 - \kappa)/x_{n-1} \tag{4}$$

If a sequence is solution of Equation (**??**), it is called a *determinate pre-constant sequence*.

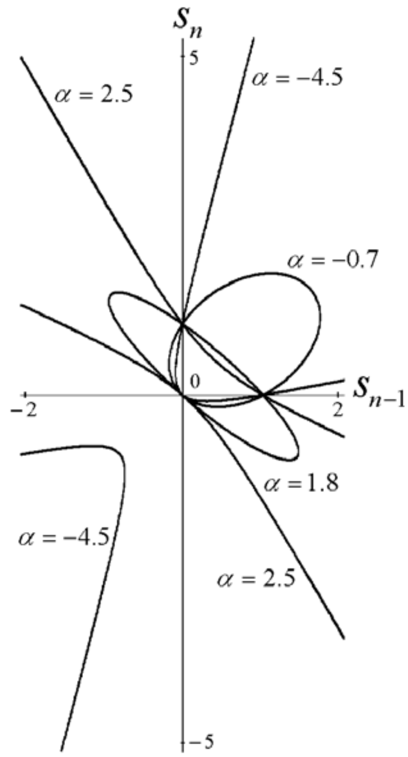


FIGURE 1. A family of conics.

Let $x$ be a determinate pre-constant signal and note that, for each $n$, with $a = x_{n-1}$, $b = x_n$, and $c = x_{n+1}$, (**??**) is a connecting equation with no sense of direction. Therefore, according to Example B.2,

$$\alpha(x_{n-1}, x_n) = x_{n-1}/x_n + x_n/x_{n-1} - \kappa/x_{n-1}x_n$$

is constant, independent of $n$. With reference to Example B.2 and letting $\beta = -\alpha$, $x$ is a solution of

$$x_{n-1} + \beta x_n + x_{n+1} = 0 \tag{5}$$

The solutions, in phase space , live on conics since (**??**) implies

$$(x_{n-1} + \alpha x_n + x_{n+1} - 1)(x_{n+1} - x_{n-1}) = 0$$

which in turn implies that

$$C(a, b) = a^2 + b^2 + \beta ab - \kappa$$

is another invariant, that is,

$$C(x_{n-1}, x_n) = C(x_n, x_{n+1})$$

then if $C(x_{n-1}, x_n) = 0$, also $C(x_n, x_{n+1}) = 0$ then, in phase space, each pair of consecutive points $(x_{n-1}, x_n)$ belongs to the same conic. Again, we have obtained a family of conics from a connecting rational equation.

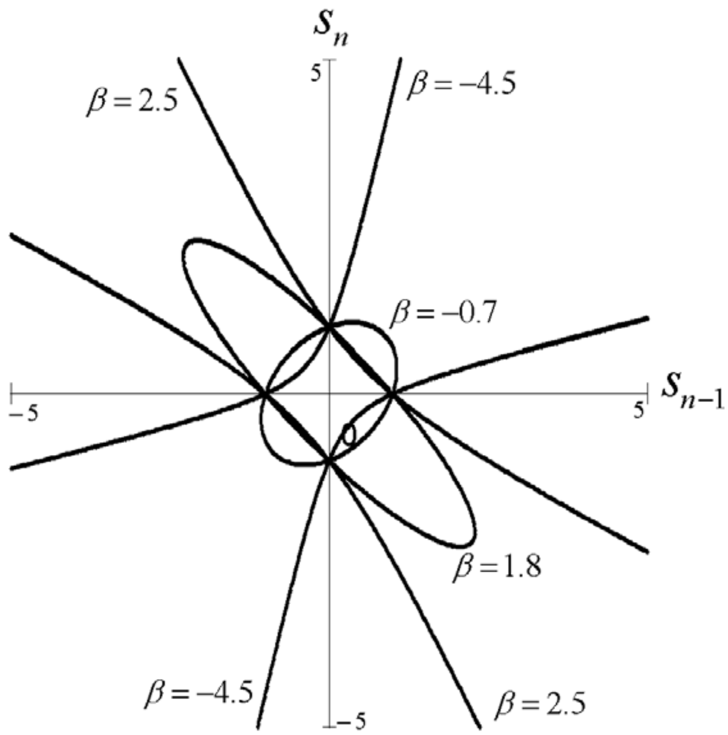


Figure 2. A family of conics.

**C3. Further examples based on the TK operator.** We present more examples, based on the same operator, to illustrate the generality of the approach.

**C31.** If we want to know the signals that get doubled by the operator, we proceed as follows. Start from $\mathrm{TK}(x) = 2x$; that is, $2x_n = x_n^2 - x_{n-1}x_{n+1}$ with $a = x_n$, $b = x_n$ and $c = x_{n+1}$ we have

$$2b = b^2 - ac,$$

or

$$ac = -2b + b^2\,;$$

then

$$a + b(b-2)/a = b(b-2)/c + c\,,$$

and then

$$a/b + (b-2)/a = (b-2)/c + c/b\,,$$

or

$$a/b - 2/a + b/a = -2/c + b/c + c/b\,,$$

and

$$a/b - 2/a - 2/b + b/a = -2/c - 2/b + b/c + c/b\,;$$

and we have the invariant

$$\alpha(x,y) = x/y + y/x - 2/x - 2/y\,,$$

from which we get the invariant family of conics

$$x^2 + y^2 - \alpha xy - 2x - 2y = 0\,,$$

and the linear equation

$$x_{n-1} - \alpha x_n + x_{n+1} = 2\,.$$

**C32.** In order to find the signals that get "added to a constant" by the operator, we proceed as follows. Start from $TK(x) = x + \kappa$; that is,

$$x_n + \kappa = x_n^2 - x_{n-1}x_{n+1}\,;$$

then, with $a = x_n$, $b = x_n$ and $c = x_{n+1}$ we have

$$b + \kappa = b^2 - ac,$$

or

$$ac = b^2 - b - \kappa\,;$$

then

$$a + (b^2 - b - \kappa)/a = (b^2 - b - \kappa)/c + c\,,$$

and then

$$a + b^2/a - b/a - \kappa/a = c + b^2/c - b/c - \kappa/c\,,$$

or

$$a/b + b/a - 1/a - \kappa/ab = c/b + b/c - 1/c - \kappa/bc\,,$$

and

$$a/b + b/a - 1/a - 1/b - \kappa/ab = c/b + b/c - 1/c - 1/b - \kappa/bc\,;$$

and we have the invariant

$$\alpha(x,y) = x/y + y/x - 1/x - 1/y - \kappa/xy\,;$$

we also have

$$\alpha(b,c) = b/c + c/b - 1/b - 1/c - \kappa/bc\,,$$

or

$$\begin{aligned} b\alpha &= b^2/c + c - 1 - b/c - \kappa/c \\ &= (b^2 - b - \kappa)/c + c - 1 \\ &= a + c - 1\,, \end{aligned}$$

which corresponds to the linear difference equation

$$x_{n-1} - \alpha x_n + x_{n+1} = 1\,,$$

which is obeyed by the signals that get added to a constant.

Also, we have

$$ab\alpha = a^2 + b^2 - a - b - \kappa\,,$$

which corresponds to the conic

$$0 = x_n^2 + x_{n-1}^2 - \alpha x_n x_{n-1} - x_n - x_{n-1} - \kappa\,,$$

which is where, in phase space, the signals that get added to a constant live.

**C33.** Likewise, if we want to know the signals that get "double squared" by the operator, we proceed as follows. Starting from TK(x) $= 2x^2$, that is,

$$2x_n^2 = x_n^2 - x_{n-1}x_{n+1}\,.$$

Then, with $a = x_{n-1}$, $b = x_n$ and $c = x_{n+1}$, we have

$$2b^2 = b^2 - ac\,,$$

or

$$ac = -b^2\,;$$

then

$$a - b^2/a = -b^2/c + c\,,$$

or

$$a/b - b/a = -b/c + c/b\,,$$

and we have the invariant

$$\alpha(x, y) = x/y - y/x\,.$$

Also, from

$$ac = -b^2\,,$$

we get

$$(a/b)(c/b) = -1\,,$$

or

$$a/b = -b/c\,,$$

and

$$(a/b)^2 = (b/c)^2\,,$$

and we have the invariant

$$\alpha(x, y) = (x/y)^2\,.$$

From the expression of the invariant, we get the conic

$$\alpha xy = x^2 - y^2\,,$$

and the linear equation

$$b\alpha(b, c) = b^2/c - c\,;$$

that is

$$x_{n-1} + \alpha x_n + x_{n+1} = 0\,.$$

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DEPARTAMENTO DE INGENIERÍA ELÉCTRICA Y ELECTRÓNICA
UNIVERSIDAD DE LOS ANDES
BOGOTÁ, COLOMBIA
*e-mail:* arestrep@uniandes.edu.co