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Khovanov type homologies and Frobenius extensions

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Abstract. It is reviewed how the Khovanov homology is defined from a TQFT-functor between the category of $(1+1)-$cobordisms and the category of $R-$modules. Then, we indicate how it is possible to obtain Khovanov type homologies from Frobenius systems.

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Resumen. Se realiza una revisión de cómo la homología de Khovanov se define a partir de un functor TQFT entre la categoría de los $(1+1)-$cobordismos y la categoría de los $R-$módulos. Luego se indica de qué manera es posible obtener homologías del tipo Khovanov a partir de sistemas de Frobenius

1. Introduction

The purpose of this paper is to define the special class of invariants of *links* that we have called in the title *Khovanov type homologies*. Khovanov [10] constructed an invariant of links which has attracted a lot of attention [15]. Bar–Natan in [3] showed how to compute it and found that it is a stronger invariant than the Jones Polynomial. Lee [14] developed another homology theory, which also can be computed explicitly, and has analogous underlying algebraic structures to Khovanov's. Lee's aim was to prove several conjectures formulated by Khovanov [10], Garoufalidis [7] and Bar–Natan [3]. Later the work of Lee was used by Rasmussen [17] to define a knot invariant with values in $\mathbb{Z}$, which provided a lower bound for the slice genus of knots. Subsequently, Bar–Natan [4] presented different homology theories and Khovanov himself discussed the relations between them, and showed how they were produced by Frobenius extensions.

Actually, any Frobenius extension gives rise to a Frobenius system, and this to a chain complex whose homotopy type is the invariant we are going to exhibit. Obviously, the homology of this chain complex is also an invariant of links and this is the one we are referring to in the title. We explain this process here and prove that there exist special conditions where a Frobenius system produces a homology theory.

The work is organized as follows. In section 2, we present the restriction, induction and coinduction functors, and use these concepts to introduce Frobenius extensions. The reader can find an alternative way to do this in [9]. In this section, we also prove that a necessary and sufficient condition to have such an extension is the existence of two homomorphisms which lead to the definition of a Frobenius system. Section 3 reviews the concepts of 2-dimensional cobordism and $(1+1)-$TQFT and shows that there exists a one to one correspondence between $(1+1)-$TQFT and Frobenius systems. We refer the reader to [2] and [1] to study these concepts in depth. Section 4 is devoted to present the chain complex used by Bar–Natan [4] and prove that this is an invariant of tangles, and hence an invariant for links and knots. Finally, section 5 contains the main theorem, where we apply a $(1+1)-$TQFT coming from a Frobenius system to the cobordisms of section 3, and prove that, under special conditions, we obtain a homology theory.

2. **Frobenius systems**

Given a ring R we use $\mathcal{M}_R$ to denote the category of $R-$modules and $\mathcal{M}_R(M,N)$ to denote the set of $R-$module homomorphisms between M and N. For the $R-$modules A and M, $A\otimes_R M$ denotes the tensor product of A and M with scalars in the ring R, which is again an $R-$module. $A\otimes_R M$ also has a structure of $A-$module by defining $a(b\otimes_R m) = ab\otimes_R m$, where $a,b\in A$ and $m\in M$.

Definition 2.1. Let $i : R\to A$ be an injective commutative ring homomorphism such that $i(1_R) = 1_A$. The restriction functor, $\text{Res} : \mathcal{M}_A \to \mathcal{M}_R$ assigns to every $A-$module M, the $R-$module M_R which has the same underlying abelian group M, but with the scalar multiplication $(\cdot)$ defined as follows: if $x\in M$ and $r\in R$, then $r\cdot x = i(r)\,x$.

The proof that M with this operation has an $R-$module structure is an easy exercise.

Proposition 2.2. *The induction functor* Ind : $\mathcal{M}_R \to \mathcal{M}_A$ *defined by*

$$\mathrm{Ind}\,(M) = A \otimes_R M$$

is left adjoint to the restriction functor Res .

Proof. The proof is based on the definition of left adjoint functor –see for example [16]. Given $M \in \mathcal{M}_R$ and $N \in \mathcal{M}_A$, we have to show that

a: there exists a bijection

$$\eta_{_{MN}} : \mathcal{M}_R\,(M, \mathrm{Res}\,N) \to \mathcal{M}_A\,(A \otimes_R M, N)$$

b: for all $f \in \mathcal{M}_R\,(M, M')$, the diagram

$$\begin{array}{ccc} \mathcal{M}_R\,(M, \mathrm{Res}\,N) & \xrightarrow{\eta_{_{MN}}} & \mathcal{M}_A\,(A \otimes_R M, N) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (Id \otimes f)^*} \\ \mathcal{M}_R\,(M', \mathrm{Res}\,N) & \xrightarrow{\eta_{_{M'N}}} & \mathcal{M}_A\,(A \otimes_R M', N) \end{array} \tag{2.1}$$

is commutative.

c: for all $g \in \mathcal{M}_A\,(N, N')$ the diagram

$$\begin{array}{ccc} \mathcal{M}_R\,(M, \mathrm{Res}\,N) & \xrightarrow{\eta_{_{MN}}} & \mathcal{M}_A\,(A \otimes_R M, N) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \mathcal{M}_R\,(M, \mathrm{Res}\,N') & \xrightarrow{\eta_{_{MN'}}} & \mathcal{M}_A\,(A \otimes_R M, N') \end{array} \tag{2.2}$$

is commutative.

To prove (a), let us define the following functions

$$\eta_{_{MN}} : \mathcal{M}_R\,(M, \mathrm{Res}\,N) \to \mathcal{M}_A\,(\mathrm{Ind}\,M, N)$$
$$\rho_{_{MN}} : \mathcal{M}_A\,(\mathrm{Ind}\,M, N) \to \mathcal{M}_R\,(M, \mathrm{Res}\,N)$$

by the formulas

$$\eta_{_{MN}}\,(\varphi)\left(\sum_i a_i \otimes_R w_i\right) = \sum_i a_i \varphi(w_i).$$

and

$$\rho_{_{MN}}\,(\phi)\,(x) = \phi\,(1 \otimes_R x)\,.$$

It is clear that $\eta_{_{MN}}\,(\varphi) \in \mathcal{M}_A\,(\mathrm{Ind}\,M, N)$. Furthermore, if $\phi \in \mathcal{M}_A\,(\mathrm{Ind}\,M, N)$, then $\rho_{_{MN}}(\varphi) \in \mathcal{M}_R\,(M, \mathrm{Res}\,N)$. Indeed, given $x, y \in M$

$$\begin{aligned} \rho_{_{MN}}(\varphi)(x+y) &= \phi(1 \otimes_{_R} (x+y)) = \phi((1 \otimes_{_R} x) + (1 \otimes_{_R} y)) \\ &= \phi(1 \otimes_{_R} x) + \phi(1 \otimes_{_R} y) = \rho_{_{MN}}(\varphi)(x) + \rho_{_{MN}}(\varphi)(y) \end{aligned}$$

and for any $r \in R$

$$\rho_{MN}(\varphi)(rx) = \phi(1 \otimes_R rx) = r\rho_{MN}(\varphi)(x).$$

Now, suppose that $\varphi \in \mathcal{M}_R(M, \operatorname{Res} N)$. We claim that $\rho_{MN}(\eta_{MN}(\varphi)) = \varphi$, which implies $\rho_{MN}\eta_{MN} = \operatorname{Id}_{\mathcal{M}_R(M,\operatorname{Res} N)}$. In fact

$$\rho_{MN}(\eta_{MN}(\varphi))(x) = \eta_{MN}(\varphi)(1 \otimes_R x) = 1\varphi(x) = \varphi(x).$$

On the other hand, if $\phi \in \mathcal{M}_A(\operatorname{Ind} M, N)$, then $\rho_{MN}(\phi) \in \mathcal{M}_R(M, \operatorname{Res} N)$. Hence it is possible to define $\eta_{MN}(\rho_{MN}(\phi))$. Now we can claim that if $a \otimes_R x \in$ Ind(M) $= A \otimes_R M$, then

$$\eta_{MN}(\rho_{MN}(\phi))(a \otimes_R x) = a\rho_{MN}(\phi)(x) = a\phi(1 \otimes_R x) = \phi(a \otimes_R x).$$

We have already proved that $\eta_{MN}(\rho_{MN}(\phi)) = \phi$, and $\eta_{MN}\rho_{MN} = \operatorname{Id}_{\mathcal{M}_A(\operatorname{Ind} M,N)}$. Finally, we can conclude that η_{MN} is a bijection.

To prove (b), regard M and M' as $R-$modules and $f \in \mathcal{M}_R(M, M')$. If $\varphi \in \mathcal{M}_R(M, \operatorname{Res} N)$, then $\eta_{MN} f^*\varphi = \eta_{MN}\varphi f$. Hence for all $a \otimes_R x \in A \otimes_R M$ we have

$$\eta_{MN}((f^*\varphi))(a \otimes x) = a(f^*\varphi)(x) = a\varphi(f(x)).$$

Furthermore, $(\operatorname{Id} \otimes_R f)^*\eta_{M'N}(\varphi) = \eta_{M'N}(\varphi)(\operatorname{Id} \otimes_R f)$. Consequently, every $a \otimes_R x \in A \otimes_R M$ satisfies

$$\begin{aligned}(\operatorname{Id} \otimes_R f)^*\eta_{M'N}(\varphi)(a \otimes_R x) &= \eta_{M'N}(\varphi)(\operatorname{Id} \otimes_R f)(a \otimes_R x)\\ &= \eta_{M'N}(\varphi)(a \otimes_R f(x))\\ &= a\varphi(f(x))\end{aligned}$$

This shows that

$$\eta_{MN} f^* = (\operatorname{Id} \otimes_R f)^*\eta_{M'N}$$

and so the diagram 2.1 commutes.

Finally, to prove the assertion (c), let us choose arbitrary $A-$modules N and N', and let $g \in \mathcal{M}_A(N, N')$. Thus for all $\phi \in \mathcal{M}_R(M, \operatorname{Res} N)$, $\eta_{MN'}(g_*(\phi)) = \eta_{MN'} g\phi$, and hence for all $a \otimes_R x \in A \otimes_R M$ we obtain

$$\eta_{MN'}(g_*(\phi))(a \otimes_R x) = a(g\phi)(x) = ag(\phi(x)),$$

Moreover,

$$g_*(\eta_{MN}(\phi))(a \otimes_R x) = g(\eta_{MN'}(\phi)(a \otimes_R x)) = g(a\phi(x)) = ag(\phi(x))$$

This proves that the diagram 2.2 commutes. ☑

Proposition 2.3. *The functor coinduction* CoInd : $\mathcal{M}_R \to \mathcal{M}_A$ *given by*

$$\operatorname{CoInd}(M) = \operatorname{Hom}_R(A, M)$$

is right adjoint to the functor Res.

Proof. As before, we must define a bijection $\rho_{MN} : \mathcal{M}_A (N, \mathrm{Hom}_R(A, M)) \to \mathcal{M}_R (\mathrm{Res}\, N, M)$, such that for all $R-$module homomorphisms $f : N \to N'$ and $g : M \to M'$ the following diagrams commute:

$$\begin{array}{ccc} \mathcal{M}_A (N, \mathrm{Hom}_R(A, M)) & \xrightarrow{\rho_{MN}} & \mathcal{M}_R (\mathrm{Res}\, N, M) \\ \uparrow {\scriptstyle f^*} & & \uparrow {\scriptstyle f^*} \\ \mathcal{M}_A (N', \mathrm{Hom}_R(A, M)) & \xrightarrow{\rho_{MN'}} & \mathcal{M}_R (\mathrm{Res}\, N', M) \end{array} \tag{2.3}$$

$$\begin{array}{ccc} \mathcal{M}_A (N, \mathrm{Hom}_R(A, M)) & \xrightarrow{\rho_{MN}} & \mathcal{M}_R (\mathrm{Res}\, N, M) \\ \downarrow {\scriptstyle (g_*)_*} & & \downarrow {\scriptstyle g_*} \\ \mathcal{M}_A (N, \mathrm{Hom}_R(A, M')) & \xrightarrow{\rho_{M'N}} & \mathcal{M}_R (\mathrm{Res}\, N, M') \end{array} . \tag{2.4}$$

This is achieved by defining

$$[\rho_{MN} (\phi)] (x) = [\phi (x)] (1).$$

It is easy to prove that $\eta_{MN} : \mathcal{M}_R (\mathrm{Res}\, N, M) \to \mathcal{M}_A (N, \mathrm{Hom}_R(A, M))$, defined by $\eta_{MN} (\varphi) (x) (a) = \varphi (ax)$ is the inverse of ρ, and then conclude that ρ is a bijection. The reason to use the symbol $(g_*)_*$ is justified since g_* is already the induced homomorphism between $\mathrm{Hom}_R(A, M)$ and $\mathrm{Hom}_R(A, M')$. ☑

Now we are ready to state the main concept of this section and to prove the principal proposition. In fact, it is interesting to compare the induction and coinduction functors. In particular, we want to see what happens if they are isomorphic.

Definition 2.4. An injective commutative ring homomorphism $i : R \to A$ is called a Frobenius extension if the associated induction and coinduction functors are isomorphic.

Proposition 2.5. *The injective $R-$module homomorphism $i : R \to A$ is a Frobenius extension if and only if there exists an $A-$bimodule homomorphism $\Delta : A \to A \otimes_R A$ and a $R-$module homomorphism $\varepsilon : A \to R$ such that*

$$(\mathrm{Id} \otimes_R \varepsilon)\Delta = \mathrm{Id} = (\varepsilon \otimes_R \mathrm{Id})\Delta \tag{2.5}$$

The morphisms Δ and ε are known respectively as the comultiplication and counit. The last proposition is equivalent to ensure that Δ and ε allow the commutativity of the following diagram.

$$\begin{array}{ccccc} R\otimes_R A & \xleftarrow{\varepsilon\otimes Id} & A\otimes_R A & \xrightarrow{Id\otimes\varepsilon} & A\otimes_R R \\ & \cong\nwarrow & \uparrow\Delta & \nearrow\cong & \\ & & A & & \end{array}$$

Furthermore, it is possible to prove that if Δ and ε satisfy the above proposition, then Δ is also coassociative and cocommutative. That is to say, the commutativity of the following diagrams holds (see [6])

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A\otimes_R A \\ \downarrow\Delta & & \downarrow\Delta\otimes_R Id \\ A\otimes_R A & \xrightarrow[Id\otimes_R\Delta]{} & A\otimes_R A\otimes A \end{array} \qquad \begin{array}{ccc} A\otimes_R A & \xrightarrow{\tau_{A,A}} & A\otimes_R A \\ \Delta\nwarrow & & \nearrow\Delta \\ & A & \end{array}$$

where τ denotes the transposition operation.

Proof. If the map $i : R \to A$ is a Frobenius extension, by definition, it follows that there exists a natural isomorphism $\eta : \text{Ind} \to \text{CoInd}$, which means that given a $R-$module homomorphism $\varphi : M \to M'$ the following diagram is commutative

$$\begin{array}{ccc} A\otimes_R M & \xrightarrow{\eta_M} & \text{Hom}_R(A, M) \\ \downarrow Id_A\otimes_R\varphi & & \downarrow\varphi_* \\ A\otimes_R M' & \xrightarrow{\eta'_M} & \text{Hom}_R(A, M') \end{array} \tag{2.6}$$

Obviously, $\eta_M : A\otimes_R M \to \text{Hom}_R(A, M)$ is an $A-$module homomorphism and $\eta_M^{-1} : \text{Hom}_R(A, M) \to A\otimes_R M$ always exists.

First of all, we have to define Δ and ε, and then prove the equality (2.5). For each $a \in A$, define $\varphi_a : A \to A$ by $\varphi_a(x) = ax$. φ_a is an $A-$bimodule endomorphism. Similarly, we have the same for the map $f : A \to \text{Hom}_R(A, A)$ defined by $f(a) = \varphi_a$. Both M and M' in the diagram (2.6) can be replaced by A, and then we define $\Delta : A \to A\otimes_R A$ by the formula $\Delta = \eta_A^{-1} f$. Since Δ is the composition of two $A-$bimodule homomorphisms, it is an $A-$bimodule homomorphism itself.

In the same way, we can replace in (2.6) M by R and M' by A, then we obtain that the following diagram is commutative for all $b \in A$,

$$\begin{array}{ccc} A\otimes_R R & \xrightarrow{\eta_R} & \text{Hom}_R(A, R) \\ \downarrow Id_A\otimes_R\phi_b & & \downarrow\phi_{b_*} \\ A\otimes_R A & \xrightarrow{\eta_A} & \text{Hom}_R(A, A) \end{array} \tag{2.7}$$

where the homomorphism $\phi_b : R \to A$ is defined by $\phi_b(x) = bx$. Since $R \otimes_R A \simeq A \simeq A \otimes_R R$, then there exist isomorphisms $g : A \to A \otimes_R R$ and $g' : A \to R \otimes_R A$ such that $g(1_A) = 1_A \otimes_R 1_R$ and $g'(1_A) = 1_R \otimes_R 1_A$. Thus, we can define $\varepsilon : A \to R$ by $\varepsilon = \eta_R(g(1_A)) = \eta_R(1_A \otimes_R 1_R)$. To prove (2.5) it is enough to verify the following equalities

$$1_R \otimes_R 1_A = (\varepsilon \otimes_R \mathrm{Id})\Delta(1_A)\,, \tag{2.8}$$

$$1_A \otimes_R 1_R = (\mathrm{Id} \otimes_R \varepsilon)\Delta(1_A)\,. \tag{2.9}$$

¿From the commutativity of the diagram (2.7) it follows that

$$\eta_A(\mathrm{Id} \otimes_R \phi_b)(1_A \otimes_R 1_R) = \phi_b \eta_R(1_A \otimes_R 1_R)\,.$$

And by the definition of ε, this last assertion implies that $\eta_A(1_A \otimes_R b) = \phi_b \varepsilon$, and hence for all $a, b, c \in A$

$$[\eta_A(a \otimes_R b)](c) = \varepsilon(ac)b\,. \tag{2.10}$$

In fact,

$$[\eta_A(a \otimes_R b)](c) = [a\eta_A(1_A \otimes_R b)](c) = [\eta_A(1_A \otimes_R b)](ac) = \phi_b \varepsilon(ac)\,.$$

Since $\Delta(1) \in A \otimes_R A$, we have $\Delta(1_A) = \sum_i e_{1i} \otimes_R e_{2i}$, where each element e_{ij}: $j \in \{1, 2\}$ is an element of A. Applying (2.10) we conclude that for all i, $\eta_A(e_{1i} \otimes_R e_{2i})(1_A) = \varepsilon(e_{1i})e_{2i}$. Therefore $\eta_A(\Delta(1_A))(1_A) = \sum_i \varepsilon(e_{1i})e_{2i}$.

Now using the definition of Δ we obtain

$$\eta_A(\Delta(1_A))(1_A) = \eta_A \eta_A^{-1}(f(1_A))(1_A) = 1_A\,.$$

So we have proved that

$$\sum_i \varepsilon(e_{1i})e_{2i} = 1_A\,.$$

This last equality allows us to compute the following

$$\begin{aligned}(\varepsilon \otimes_R \mathrm{Id})\Delta(1_A) &= (\varepsilon \otimes_R \mathrm{Id})(\sum_i e_{1i} \otimes_R e_{2i}) = \sum_i (\varepsilon \otimes_R \mathrm{Id})(e_{1i} \otimes_R e_{2i}) \\ &= \sum_i \varepsilon(e_{1i}) \otimes_R e_{2i} = g'(\sum_i \varepsilon(e_{1i})e_{2i}) = g'(1_A) = 1_R \otimes_R 1_A\,,\end{aligned}$$

which proves (2.8).

To prove (2.9), we take again the diagram (2.6), and use the fact that η is a natural isomorphism so that η^{-1} exists, and then replace M by A, M' by R, and φ by ε, to obtain

$$\begin{array}{ccc} \mathrm{Hom}_R(A, A) & \xrightarrow{\eta_A^{-1}} & A \otimes_R A \\ \Big\downarrow{\scriptstyle \varepsilon_*} & & \Big\downarrow{\scriptstyle Id_A \otimes_R \varepsilon} \\ \mathrm{Hom}_R(A, R) & \xrightarrow{\eta_R^{-1}} & A \otimes_R R \end{array}$$

Applying this to $\mathrm{Id}_A = f(1_A) \in \mathrm{Hom}_R(A, A)$ we obtain

$$(\mathrm{Id}_A \otimes_R \varepsilon)\eta_A^{-1}(f(1_A)) = \eta_R^{-1}\varepsilon\,\mathrm{Id}_A\ .$$

Therefore

$$(\mathrm{Id}_A \otimes_R \varepsilon)\Delta(1_A) = \eta_R^{-1}\varepsilon = g(1_A) = 1_A \otimes_R 1_R\,.$$

Conversely, let us assume that there exists an $A-$bimodule homomorphism $\Delta : A \to A \otimes_R A$ and a $R-$module homomorphism $\varepsilon : A \to R$ satisfying 2.5. We have to show that there exists a natural isomorphism $\eta : \mathrm{Ind} \to \mathrm{CoInd}$.

For all $N \in \mathcal{M}_R$, we define $\eta_N : A \otimes_R N \to \mathrm{Hom}_R(A, N)$ by

$$\eta_N\,(a \otimes_R n)\,(b) = \varepsilon(ba)n\,.$$

Clearly $\eta_N\,(a \otimes_R n)\,(b) \in \mathrm{Hom}_R(A, N)$. Thus, we will begin proving that η is a natural transformation. Let $h : N \to N'$ be a $R-$module homomorphism, then

$$h[\eta_N(a \otimes_R \eta)(b)] = h[\varepsilon(ba)n] = \varepsilon(ba)h(n)$$

and

$$\eta_{N'}(\mathrm{Id}_A \otimes_R h)(a \otimes_R n)(b) = \eta_{N'}(a \otimes_R h(n))(b) = \varepsilon(ba)h(n)\,.$$

This proves that the diagram

$$\begin{array}{ccc} A \otimes_R N & \xrightarrow{\eta_N} & \mathrm{Hom}_R(A, N) \\ {\scriptstyle Id_A \otimes_R h}\downarrow & & \downarrow{\scriptstyle h_*} \\ A \otimes_R N' & \xrightarrow{\eta_{N'}} & \mathrm{Hom}_R(A, N') \end{array}$$

is commutative and so η defines a natural transformation.

To prove that η is a natural isomorphism, we only need to show that η has an inverse. We define $\rho_N : \mathrm{Hom}_R(A, N) \to A \otimes_R N$ in the following way

$$\rho_N(\varphi) = \sum_i e_{1i} \otimes_R \varphi\,(e_{2i})\ .$$

Remember that e_{1i} and e_{2i} are defined so that $\Delta(1_A) = \sum_i e_{1i} \otimes e_{2i}$. It is easily noticed that $\rho_N(\varphi) \in A \otimes_R N$. Moreover the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(A, N) & \xrightarrow{\rho_N} & A \otimes_R N \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle Id_A \otimes_R h} \\ \mathrm{Hom}_R(A, N') & \xrightarrow{\rho_{N'}} & A \otimes_R N' \end{array}$$

commutes, and consequently ρ is a natural transformation. ¿From equation (2.5), and the isomorphism between A, $A \otimes_R R$ and $R \otimes_R A$, we obtain

$$\sum_i \varepsilon(e_{1i})e_{2i} = \sum_i e_{1i}\varepsilon(e_{2i}) = 1_A\,. \qquad (2.11)$$

Then for all $\varphi \in \mathrm{Hom}_R(A,N)$ and for all $a \in A$

$$\begin{aligned}
[\eta_N \rho_N(\varphi)]\,(a) &= \left[\eta_N\left(\sum_i e_{1i}\otimes_R \varphi(e_{2i})\right)\right](a)\\
&= \left[\sum_i \eta_N\left(e_{1i}\otimes_R \varphi(e_{2i})\right)\right](a)\\
&= \sum_i \varepsilon(ae_{1i})\varphi(e_{2i})\\
&= \varphi(\sum_i \varepsilon(ae_{1i})e_{2i})\\
&= \varphi(a)\,.
\end{aligned}$$

This last equality follows from the equation (2.11). We have already proved that $\eta_N\rho_N = \mathrm{Id}_{\mathrm{Hom}_R(A,N)}$. Moreover, for all $a\otimes_R n \in A\otimes_R N$ we have that

$$\begin{aligned}
\rho_N(\eta_N(a\otimes_R n)) = \sum_i e_{1i}\otimes_R \eta_N(a\otimes_R n)(e_{2i}) = \sum_i e_{1i}\otimes_R \varepsilon(ae_{2i})n\\
= a\otimes_R \sum_i e_{1i}\varepsilon(e_{2i})n\,.
\end{aligned}$$

Therefore $\rho_N(\eta_N(a\otimes_R n)) = a\otimes_R n$, which proves that $\rho_N\eta_N = \mathrm{Id}_{A\otimes_R N}$. ☑

Definition 2.6. Given a Frobenius extension $i : R \to A$, a Frobenius system is an ordered quadruple $\mathcal{F} = (R, A, \varepsilon, \Delta)$ where ε and Δ are chosen in such a way that they satisfy proposition (2.5)

A Frobenius system is more than two sets with two operations. In fact, it is important to consider the $R-$module homomorphism i and the multiplication in the ring A. This last operation defines clearly another $A-$bimodule homomorphism $m : A\otimes_R A \to A$.

2.1. **Examples of Frobenius systems.** It is easy to prove that thethe following three examples are Frobenius systems. The first two were studied by KHOVANOV [10]. The last one was studied by LEE [14]

- $\mathcal{F}_1 = (R_1, A_1, \varepsilon_1, \Delta_1)$ with $R_1 = \mathbb{Z}$, and $A_1 = \mathbb{Z}[X]/\langle X^2\rangle$

$$\varepsilon_1(1) = 0; \quad \varepsilon_1(X) = 1; \quad \Delta_1(1) = 1\otimes X + X\otimes 1; \quad \Delta_1(X) = X\otimes X$$

- $\mathcal{F}_2 = (R_2, A_2, \varepsilon_2, \Delta_2)$ with $R_2 = \mathbb{Z}[c]$, $A_1 = \mathbb{Z}[X,c]/\langle X^2\rangle$

$$\begin{aligned}
&\varepsilon_2(1) = -c \qquad && \Delta_2(1) = 1\otimes X + X\otimes 1 + cX\otimes X\\
&\varepsilon_2(X) = 1 && \Delta_2(X) = X\otimes X
\end{aligned}$$

- $\mathcal{F}_3 = (R_3, A_3, \varepsilon_3, \Delta_3)$ with $R_3 = \mathbb{Q}$, $A_3 = \mathbb{Q}[X]/\langle X^2 - 1\rangle$

$$\epsilon_3(1) = 0 \qquad \Delta_3(1) = 1 \otimes X + X \otimes 1$$
$$\varepsilon_3(X) = 1 \qquad \Delta_3(X) = X \otimes X + 1 \otimes 1$$

3. The category of two dimensional cobordisms and TQFT functors

Topological quantum field theories (TQFTs) were introduced by ATIYAH in [2] and their relations to Frobenius systems were described by ABRAMS in [1]. A TQFT is a functor between the category of 2-dimensional cobordism and the category of $R-$modules, so we will initially present the category of 2-dimensional cobordisms, which is denoted here by **2-Cob**.

The objects in **2-Cob** are disjoint unions of orientable, compact 1-manifolds embedded in $\mathbb{R}^2 \times \{0\}$ or $\mathbb{R}^2 \times \{1\}$. Given two objects $\mathcal{O}_1$ and $\mathcal{O}_2$ in **2-Cob** a morphism $\mathcal{C} : \mathcal{O}_1 \to \mathcal{O}_2$ is an oriented topological surface embedded in $\mathbb{R}^2 \times [0, 1]$, equipped with an orientation preserving homeomorphism from the boundary $\partial\mathcal{C}$ to the disjoint union $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$. The following figures can be used to exemplify what a cobordism is. The boundaries typify the objects of the category and the surfaces represent morphisms between objects.

(a) (b)

FIGURE 1. Examples of $2-$cobordisms

If $\mathcal{C}_1 : \mathcal{O}_1 \to \mathcal{O}_2$ and $\mathcal{C}_2 : \mathcal{O}_2 \to \mathcal{O}_3$ are morphisms, the composition $\mathcal{C}_1 \circ \mathcal{C}_1 : \mathcal{O}_1 \to \mathcal{O}_3$ is produced by placing $\mathcal{C}_1$ atop $\mathcal{C}_2$. **2-Cob** is in fact a tensor category whose monoidal structure is given by the disjoint union, which is denoted here by the symbol $\bigsqcup$. The cobordism identity for the 1-manifold $\mathcal{O}$ is the cylinder $\mathcal{O} \times I$ symbolized by $\mathrm{Id}_{\mathcal{O}}$. The empty 1-manifold is denoted by $\emptyset$. Furthermore, it can be easily proved ([1] and [13]) that **2-Cob** is generated by only five cobordisms. That is to say, every connected 2-cobordism can be formed by the composition and the disjoint union of the following five cobordisms

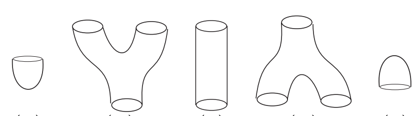

(a) (b) (c) (d) (e)

FIGURE 2. The generators of **2-Cob**

Given a commutative ring R, a $(1+1)-$dimensional topological quantum field theory ($(1+1)-$TQFT), is a monoidal functor (it preserves the monoidal structure) between the category of 2-**Cob** of the 2-dimensional cobordisms and the category $\mathcal{M}_R$. That means that the following statements are verified:

- $\mathcal{T}$**1:** $\mathcal{T}(\mathcal{O} \times I) = \mathrm{Id}_{\mathcal{T}\mathcal{O}}$
- $\mathcal{T}$**2:** $\mathcal{T}(\mathcal{C}' \circ \mathcal{C}'') = (\mathcal{T}\mathcal{C}') \circ (\mathcal{T}\mathcal{C}'')$
- $\mathcal{T}$**3:** $\mathcal{T}(\mathcal{O}' \sqcup \mathcal{O}'') = \mathcal{T}\mathcal{O}' \otimes_R \mathcal{T}\mathcal{O}''$, $\qquad \mathcal{T}(\mathcal{C}' \sqcup \mathcal{C}'') = \mathcal{T}\mathcal{C}' \otimes_R \mathcal{T}\mathcal{C}''$
- $\mathcal{T}$**4:** $\mathcal{T}\emptyset = R$

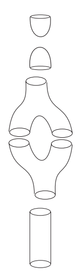
The first two statements express that $\mathcal{T}$ is a functor, the last two state that $\mathcal{T}$ is monoidal

Proposition 3.1. *There exists a one-to-one correspondence between the $(1+1)-$TQFTs and Frobenius systems.*

Proof. Let $\mathcal{T}$ be a $(1+1)-$TQFT. By definition we have a $R-$module A assigned to the circle S^1. Applying $\mathcal{T}3$, $\mathcal{T}4$, and the fact that every compact non-empty 1-manifold is the disjoint union of copies of S^1, it is possible to complete the assignations for the objects in the category, in the following way:

$$\emptyset \longrightarrow R$$
$$\bigcirc \longrightarrow A$$
$$\bigcirc\bigcirc \longrightarrow A \otimes_R A, \text{ etc.}$$

Naturally, there also exists an assignation for the generating cobordisms. Because of $\mathcal{T}1$, the assignation for $S^1 \times I$ is Id_A. The assigned homomorphisms for the other generators can be named by ε, i, m and Δ. From $\mathcal{T}2$, $\mathcal{T}3$ and the relation displayed in the Figure (3), it follows that the relation (2.5) holds. Figures (5) and (6) imply that m is associative and commutative, and that Δ is coassociative and cocommutative. Hence, $(R, A, \Delta, \varepsilon)$ defines a Frobenius system.



$\rightarrow \quad \varepsilon : A \to R$

$\rightarrow \quad i : R \to A$

$\rightarrow \quad m : A \otimes_R A \to A$

$\rightarrow \quad \Delta : A \to A \otimes_R A$

$\rightarrow \quad \mathrm{Id}_A : A \to A$

FIGURE 3. Correspondence between the generating 2−cobordisms and morphisms in the Frobenius Algebra

Conversely, if we have a Frobenius system $(R, A, \Delta, \varepsilon)$, we can assign R to $\emptyset$, and A to S^1, and, by Figure (3), complete the assignations for all the

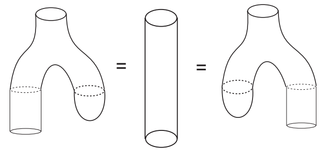


FIGURE 4. Cobordism correspondence for $(\mathrm{Id} \otimes_R \varepsilon)\Delta = \mathrm{Id} = (\varepsilon \otimes_R \mathrm{Id})\Delta$

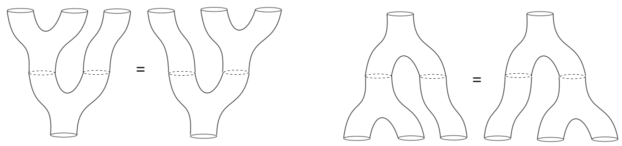


FIGURE 5. The equalities for these cobordisms show that $m(m \otimes_R \mathrm{Id}) = m(\mathrm{Id} \otimes_R m)$ and $(\Delta \otimes_R \mathrm{Id})\Delta = (\mathrm{Id} \otimes_R \Delta)\Delta$

other generating cobordisms. We assign the composition and disjoint sum of cobordisms to the composition and tensor product of morphisms in $\mathcal{M}_R$ to obtain in this way a TQFT. It only remains to prove that the assignations are well defined. That is, we need to prove that given two equivalent cobordisms we will obtain the same morphism in $\mathcal{M}_R$. We can do that by proving that every relation in the 2-**Cob** category leads to a relation in $\mathcal{M}_R$. Some of these relations and their equivalencies in the Frobenius system are displayed in figures 3, 5, 6 and 10. But there are more relations in 2-**Cob**. For a completed list of them, see [13]. We only have to translate every equality in 2-**Cob** to the language of $\mathcal{M}_R$ and prove that the resulting equality holds

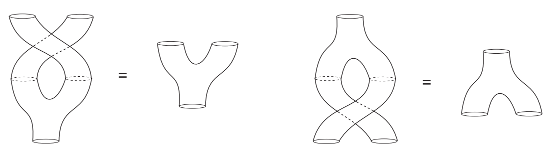


FIGURE 6. These equalities show that m is commutative and Δ is cocommutative. ☑

4. **The Khovanov invariant**

As we did in [5], we are going to work with *tangles* considered in the sense of [12]. Given an even integer n, an n-tangle T is a proper embedding of the union of $n/2$ arcs and a finite number of circles into a 3-ball $\mathbf{B}$. The n ends of the arcs are the only points of T that are in $\partial\mathbf{B}$. A (0)-tangle with k embedded circles is a k component *link*, and a link with one component is a *knot*. Thus, in defining an invariant for tangles, we are also defining an invariant for knots and links.

To explain the Khovanov invariant, consider a link with n crossings, denote respectively with n_+ and n_- the number of positive crossings and negative crossings. In Figure (7), we observe the Khovanov bracket of the figure eight knot. This knot is drawn in the upper left corner of the figure. We could use simpler examples like the trefoil knot or the Hopf link, but these examples have been used before in several papers, see for example [10], [3] or [4]. Below the knot we can find the ordered pair $(n_+, n_-) = (2, 2)$ revealing that there are 2 positive and 2 negative crossings. The graph is called the associated n *dimensional cube* (4 dimensional, in our case). The sixteen nodes of this graph, *vertices*, are marked by the strings of four characters, each of them is either equal to 0 or to 1 and corresponds to a crossing of the knot. Every *edge* of the cube is marked with the same characters, 0's or 1's and exactly one *, which indicates the coordinate change from zero to one in the given edge. To each edge we assign a sign: $(-)$ if the number of 1's before * is odd, $(+)$ otherwise.

As we go from the left corner to the cube to the right, the labels on the vertices change from 0000 to 1111. For every vertex of the cube we define its *height*, which is the sum of its coordinates. In our example the height of every vertex is a number between 0 and 4. The cube is drawn in such a way that the vertices of height r are projected down to the point marked by $r - n_-$ over a line below the cube. Finally, every vertex of the cube of a link L supports a $1-$manifold that we will call the *resolution* of L. A resolution in a vertex labeled by a combination of bits A is obtained replacing every crossing in the diagram of L, by a *0-resolution* or by a *1-resolution* depending if there is a 0 or a 1 in A. These type of resolutions are defined in the next figure. For example, if the vertex is marked by 0110, then the corresponding resolution of the knot assigned to this vertex will be one that has a *0-resolution* for the first and last crossings, and a *1-resolution* for the second and the third crossings. A diagram of a tangle L with n crossing has 2^n resolutions.

4.1. **The category Kob and the chain complex.** Now we follow [4] and interpret the mentioned cube, as a chain complex. To do that, let us define the abelian category we are going to work in, and name this category **Kob**. The objects in this category are column vectors whose elements are formal $\mathbb{Z}$-linear combinations of resolutions with the same height. These resolutions are 1-manifolds possibly with boundary. Given two objects in this category

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \vdots \\ \mathcal{O}_n \end{pmatrix} \qquad \mathcal{O}^1 = \begin{pmatrix} \mathcal{O}_1^1 \\ \mathcal{O}_2^1 \\ \vdots \\ \mathcal{O}_m^1 \end{pmatrix}$$

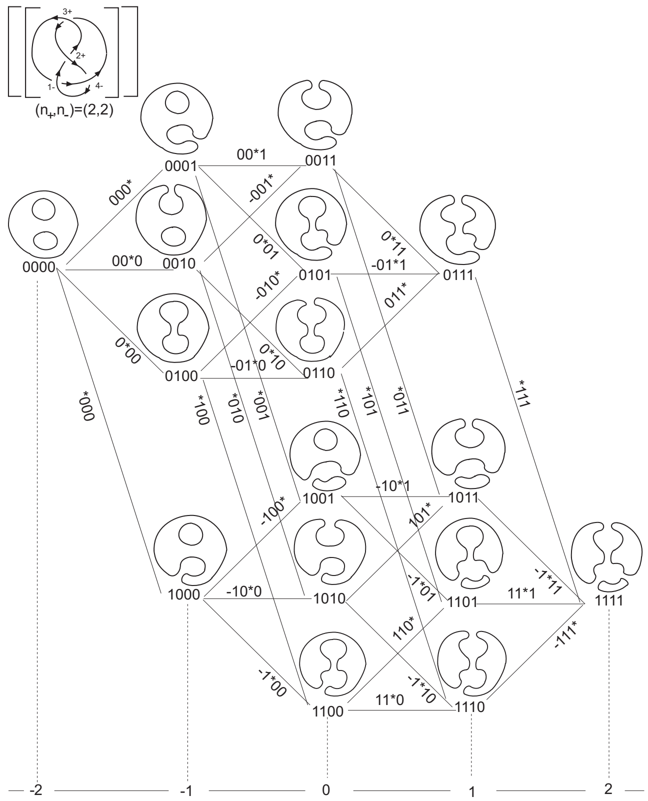


FIGURE 7. The Khovanov bracket for the figure 8 knot.

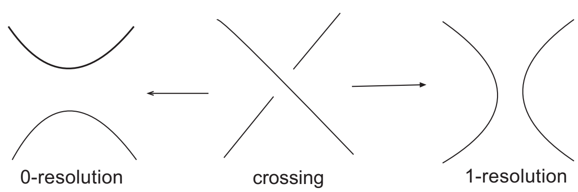


FIGURE 8. Resolutions for a crossing

The morphisms between these objects will be matrices whose entries are formal sums of cobordisms between them, that is, 2-manifold embeddings in $\mathbb{R}^2 \times [0,1]$ and boundaries entirely in $\mathbb{R}^2 \times \{0,1\} \bigcup E \times [0,1]$, where E is the boundary of the objects in $\mathbb{R}^2 \times \{0\}$ or $\mathbb{R}^2 \times \{1\}$. The upper boundary is the resolution $\mathcal{O}_i$ and the lower boundary is the resolution $\mathcal{O}_j^1$. The morphisms in this abelian category are added using the usual matrix addition and the morphism composition is modeled by matrix multiplication. The following figure is useful to understand this situation:

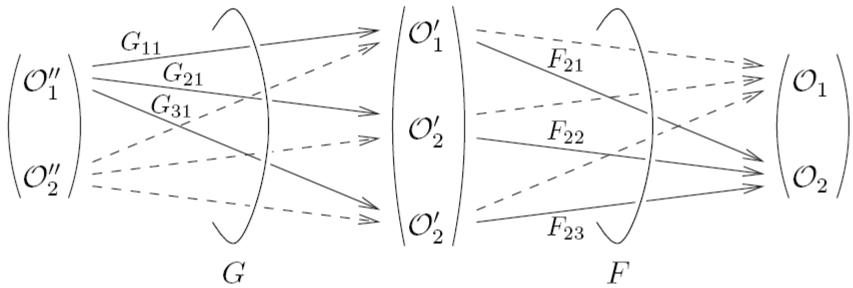
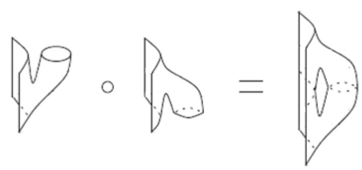


FIGURE 9. Morphisms in **Kob** and their composition

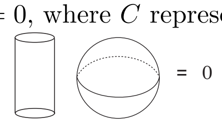
Given two morphisms $F = (F_{ik})$ and $G = (G_{kj})$ between objects of this category, then $F \circ G$ is well defined, if the number of columns of F is the same as the number of rows in G, and

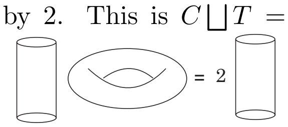
$$FG = \sum_k F_{ik} G_{kj} \,,$$

where $F_{ik}G_{kj}$ is realized as in the previous section, putting the second cobordism G_{jk} on top of F_{ij}, just as the following figure displays.

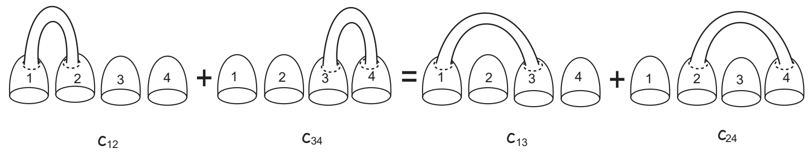


Furthermore the morphisms in **Kob** satisfy the following three relations.

(1) Relation S. If a cobordism has a sphere, this cobordism can be replaced by zero. This is: $C \bigsqcup S = 0$, where C represents any cobordism and S is a sphere. For example 

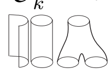
(2) Relation T. If a torus appears in a cobordism, it is possible to remove the torus and write down the rest of the cobordism multiplied by 2. This is $C \bigsqcup T = 2C$. Here T denote a torus. For example, 

(3) Relation $4Tu$.



This relation may be stated as follows. Given a Cobordism C, mark four disks on it and label them by 1,2,3,4. Denote by C_{ij} the cobordism formed by removing the disk marked with i and j and joining the boundaries of the resulting holes with a tube, as shown in the figure. Then the $4Tu$ relation says that $C_{12} + C_{34} = C_{13} + C_{24}$

It is an easy exercise to prove that **Kob** is in fact a category.

It is also not difficult to show that if a resolution in a cube with height r has k circles, then all the resolutions with height $r+1$ will have either $k+1$ or $k-1$ circles. We can now regard the cube as a chain complex in the following way: The r-th object of the complex is going to be the vector of all resolutions with height r. Each edge in the cube corresponds to a cobordism between the resolutions at its extreme points. This cobordism is always the identity composed with a single saddle cobordism. For example if $\mathcal{O}'_i =$  and $\mathcal{O}'_k =$  then the cobordism $\mathcal{C} : \mathcal{O}'_i \to \mathcal{O}'_k$ between them will be  , the differential d of the cube will be determined by matrices of cobordisms. A composition of two consecutive edges, $F_{ik}G_{kj}$ and $F_{il}G_{lj}$, corresponds to the addition of two one-handles. Since handles of he same index can be added in any order, this means that two-dimensional faces of the cube commute. Figure (10) shows a possible case, when we have consecutive edges in the resolution of a link. Now we assign to this cobordism the sign of

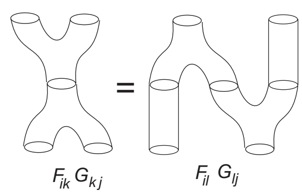


Figure 10. Composition of cobordisms when the difference of the number of their circles, that are central objects, is 2

the corresponding edge. Clearly, the number of cobordisms with sign $(-)$ in each face of the cube is going to be odd, then the equality obtained is

$$F_{ik}G_{kj} = -F_{il}G_{lj} \, ,$$

and the differential defined in the cube satisfies

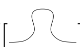
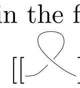
$$d^2 = \sum_k F_{ik}G_{kj} = 0 \, .$$

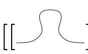
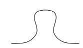
Therefore, we have a chain complex. For a tangle T we denote its associated chain complex by $[[T]]$. It is called the Khovanov bracket of T.

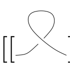
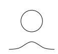
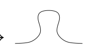
4.2. **Invariance by the Reidemeister moves.** Our goal now is to show that given a tangle T, the chain complex

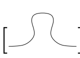
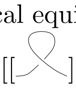
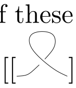
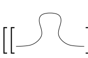
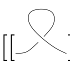
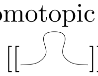
$$[[T]] = (\cdots 0 \longrightarrow [[T]]^{-n_-}) \longrightarrow \cdots \longrightarrow [[T]]^{n_+}) \longrightarrow 0 \cdots)$$

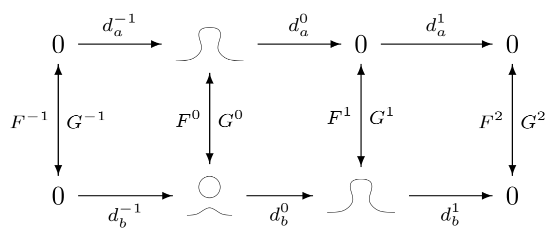
is an invariant of tangles, if it is regarded as a complex in the category **Kob**. That is, given two equivalent tangles their associated chain complexes are homotopically equivalent. Now we will explain the details which were not given in [4].

4.3. **First move.** We need to show that the associated chain complex for the diagrams in the first Reidemeister move are homotopically equivalent. This is [[]]∼ [[]]. We have :

[[]]=(0 ⟶  ⟶ 0)

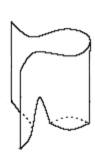
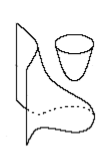
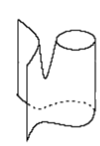
[[]]=(0 ⟶  ⟶  ⟶ 0)

To prove the homotopical equivalence of these complexes we have to find morphisms F :[[]]⟶[[]] and G :[[]]⟶[[]]. such that FG is homotopic to the identity map of [[]] and GF is homotopic to the identity of [[]]. We have the following diagram.

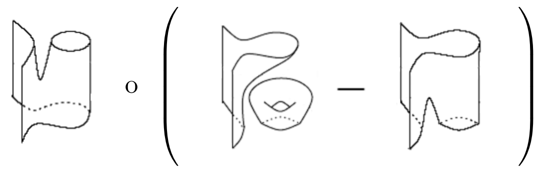


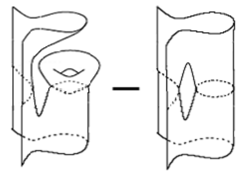
Clearly F^{-1}, G^{-1}, F^1, G^1, F^2, G^2, d_a^{-1}, d_b^{-1}, d_a^1 and d_b^1 are the zero morphism and hence the first and the third square are commutative.

Define:

$F^0 =$  $-$  ; $G^0 =$  and $d_b^0 =$ 

We have that $F^1 d_a^0 = 0$ and

$d_b^0 \circ F^0 =$ 

$=$  $= 0\,.$

This proves that the second square of the diagram is also commutative, so that F is a morphism of complexes. Since in every case

$$G^{i+1} d_b^i = d_a^i G^i = 0$$

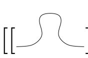
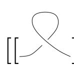
G is clearly a morphism of complexes.

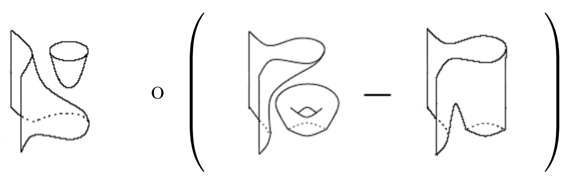
To prove the equivalence between the two complexes, it only remains to show that:

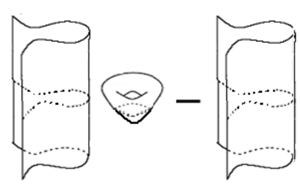
$$GF \sim I_a \tag{4.1}$$

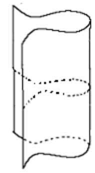
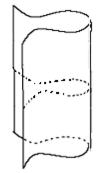
and

$$FG \sim I_b \tag{4.2}$$

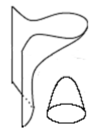
where I_a and I_b denote the identities of [[]] and [[]] respectively. To prove 4.1 we have to show that there exists h_a^r such that $G^r F^r - I^r = h_a^{r+1} \circ d_a^r + d_a^{r-1} h_a^r$. We see that $d_a^r = 0$ for all r; then we just have to show that $G^r F^r = I^r$. Which is trivial for $r \neq 0$. For $r = 0$ we have

$d_b^0 F^0 =$ 

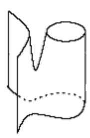
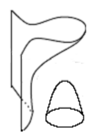
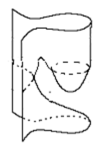
$=$ 

$$= 2$$  $$-$$  $$= I_a^0$$

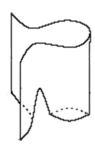
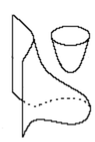
In order to prove 4.2, let $h_b^r = 0$ for every $r \neq 1$, and let

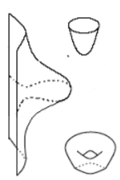
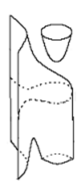
$$h_b^1 = -$$ 

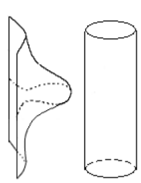
Clearly, $F^r G^r - I_b^r = h_b^{r+1} d_b^r + d_b^{r-1} h_b^r = 0$, when $r \neq 0, 1$. For $r = 1$, we have that $F^1 G^1 = 0 = h_b^2 d_b^1$ and

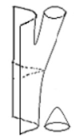
$$d_b^0 h_b^1 =$$  $$\circ \left(- \right.$$  $$\left. \right) = -$$  $$= -I_b^1$$

and hence we obtain $F^1 G^1 - I_b^1 = h_b^2 d_b^1 + d_b^0 h_b^1$. We have $d_b^{-1} h_b^0 = 0$. Thus, for $r = 0$, we just have to prove that $F^0 G^0 - I_b^0 = h_b^1 d_b^0$. But

$$F^0 G^0 = \left(\right.$$  $$-$$  $$\left. \right) \circ$$ 

$$=$$  $$-$$ 

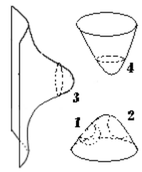
$$I_b^0 =$$ 

$h_b^1 d_b^0 = -$ 

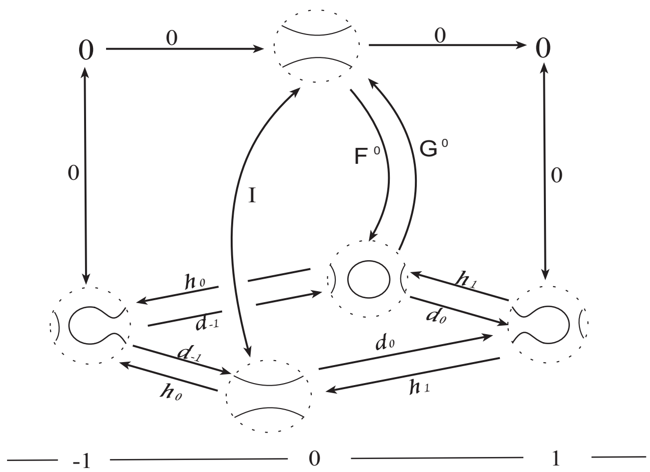
Therefore

$$F^0 G^0 - I_b^0 - h_b^1 d_b^0 = C_{12} - C_{13} - C_{24} + C_{34} = 0$$

This last equality is obtained applying the relation $4Tu$ to the cobordism



4.4. **Second move.** We have to prove that the diagrams involved in the second Reidemeister movement give rise to complexes that are equivalent under homotopy



The two complexes displayed above are:

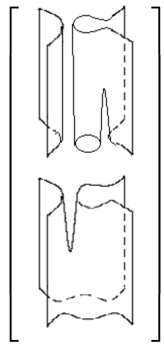
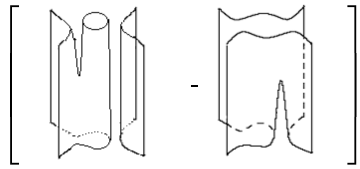
$$[[$$  $$]] = (0 \longrightarrow$$  $$\longrightarrow 0)$$

$$[[$$  $$]] = (0 \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0)$$

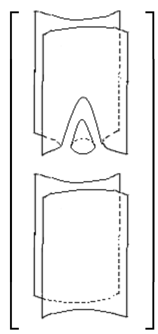
where

$T^{-1} = \Big[$  $\Big]$; $T^0 = \Big[$  $\Big]$ and $T^1 = \Big[$  $\Big]$

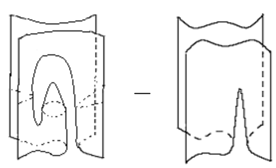
Then, $d_a^r = 0$ for all r, $d_b^r = 0$ if $r \neq -1, 0$ and

$d_b^{-1} =$  $d_b^0 =$ 

Define F in the following way: $F^r = 0$ if $r \neq 0$ and

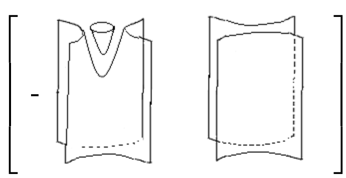
$F^0 =$ 

It can be seen from the graphic that, if $r \neq 0$, then $d_b^r \circ F^r = F^{r+1} \circ d_a^r = 0$. Moreover $F^1 \circ d_a^0 = 0$ and

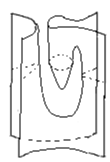
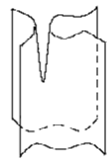
$d_b^0 \circ F^0 =$  $= 0$

Then, it has been proven that each of the squares from this complex is commutative, and hence F is a morphism of complexes.

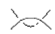
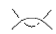
Let us define G : [[]] $\rightarrow$ [[]] in the following way: $G^r = 0$ if $r \neq 0$ and

$G^0 =$ 

Clearly, $G^r d_b^{r-1} = d_r^{r-1} G^{r-1}$ for all $r \neq 0$. Furthermore $d_a^{-1} G^{-1} = 0$ and

$G^0 d_b^{-1} = -$  $+$  $= 0$

Therefore, every square determined by G in the complex is also commutative and hence G is a complex morphism.

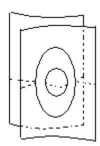
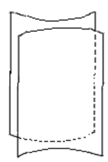
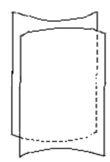
Now we must show that the two complexes are actually equivalent by homotopy. That is, that there exist complex morphisms h_a : [[]]^r $\rightarrow$ [[]]$^{r-1}$ and h_b : [[]]^r $\rightarrow$ [[]]$^{r-1}$ such that

$$G^r F^r - I_a^r = h_a^{r+1} d_a^r + d_a^{r-1} h_a^r \tag{4.3}$$

and

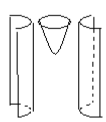
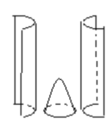
$$F^r G^r - I_b^r = h_b^{r+1} d_b^r + d_b^{r-1} h_b^r \,. \tag{4.4}$$

It should be noticed that, if $r \neq 0$, then $G^r F^r = 0 = I_a^r$ and $h_a^{r+1} d_a^r + d_a^{r-1} h_a^r$. Therefore the equation 4.3 is trivially obtained. Furthermore

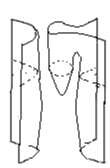
$G^0 F^0 = -$  $+$  $=$  $= I_a^0$

which implies that 4.3 is true for every r.

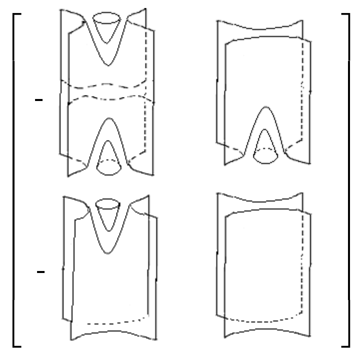
To prove 4.4 define h_b^r in the following way: $h_b^r = 0$, if $r \neq 0$, 1 and

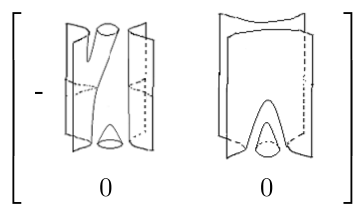
$h_b^0 = \Big[$ -  $\quad 0 \Big]$ $\qquad$ $h_b^1 = \Big[$  $\quad 0 \Big]$

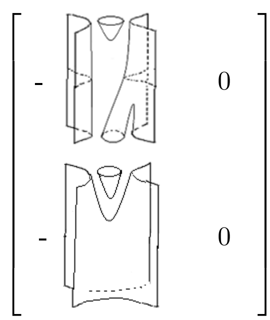
Then $F^{-1} G^{-1} = 0$, $d_b^{-2} h_b^{-1} = 0$ and

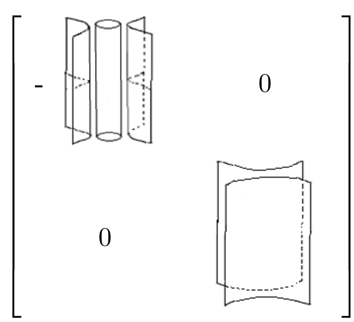
$h_b^0 d_b^{-1} = -$  $= -I_b^{-1}$

Therefore, we have that $F^{-1}G^{-1} - I_b^{-1} = h_b^0 d_b^{-1} + d_b^{-2} h_b^{-1}$. In addition we have that

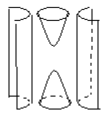
$F^0 \circ G^0 =$ 

$h_b^1 \circ d_b^0 =$ 

$d_b^{-1} \circ h_b^0 =$ 

$I_b^0 =$ 

and clearly, $F^0 \circ G^0 - I_b^0 = h_b^1 \circ d_b^0 + d_b^{-1} \circ h_b^0$. The equality for the element in the upper left corner is obtained applying the $4Tu$ relation to the cobordism



. The equalities for the other elements are trivial.

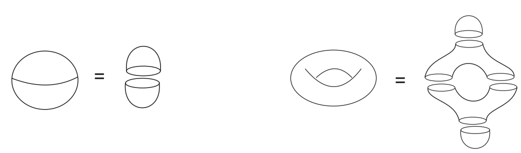
4.5. **Third move.** We have to prove that [[]] $\sim$ [[]]. As BAR–NATAN says in [4], "this is the easiest and the hardest move". We only have to draw the complexes, use the obvious cobordism, and apply only isotopies. It does not require the use of the S, T and $4Tu$ relations. But, because it involves more crossings, the work is long and tedious, although easy to carry out. For an alternative shorter proof see [4].

5. The homology theory

In the previous section we presented a homotopy type invariant for links. The resolutions of a link being always closed 1-manifolds, we can then apply a $(1+1)-$TQFT functor to the subcategory of **Kob** generated by the closed 1-manifolds, to get a chain complex in the category $\mathcal{M}_R$. We expect that the obtained chain complex of $R-$modules will be a homotopy type invariant for links. Actually, since a $(1+1)-$TQFT preserves the composition and disjoint union our hope to get that invariant is very high . Unfortunately, in the previous section, we used not only the properties of a functor, but also the S, T and $4Tu$ relations. Then, to prove that we obtain an invariant of links we need to prove that the functor also preserves these relations. This is possible, as is expressed in the following theorem

Theorem 5.1. *Let A be a commutative ring with two generators 1 and X, and let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a Frobenius system satisfying $\varepsilon(1) = 0$ and $\Delta(1) = 1 \otimes X + X \otimes 1$. Then $\mathcal{F}$ defines an invariant of links.*

Proof. It must be shown that when a $(1+1)-$TQFT is applied to this $\mathcal{F}$ the relations S, T, and $4Tu$ are respected. We are going to use 1_R for the unit of R and simply 1 for the unit of A. Let $C \bigsqcup S$ a cobordism formed by the



a b

FIGURE 11. The sphere and the torus are regarded as a composition of fundamental (generating) cobordisms

disjoint union between an arbitrary cobordism C and a sphere S. As suggested by figure (11a), we apply the $(1+1)-$TQFT $\mathcal{T}$ to obtain

$$\mathcal{T}(C\bigsqcup S) = \mathcal{T}(C) \otimes \varepsilon i\,.$$

But $\varepsilon i(1_R) = \varepsilon(1) = 0$, and so

$$\mathcal{T}(C\bigsqcup S) = 0\,. \tag{5.1}$$

Let $C\bigsqcup T$ a cobordism formed by the disjoint union between an arbitrary cobordism C and a torus T. Figure (11b) shows that in applying the $(1+1)-$TQFT $\mathcal{T}$ we obtain

$$\mathcal{T}(C\bigsqcup T) = \mathcal{T}(C) \otimes \varepsilon m\Delta i\,.$$

But $\varepsilon m\Delta(1) = \varepsilon(m(1\otimes_R X + X\otimes_R 1)) = 2$, so

$$\mathcal{T}(C\bigsqcup S) = 2\mathcal{T}(C)\,. \tag{5.2}$$

Finally, to prove that the relation $4tu$ is preserved. Indeed, in **Kob** we have

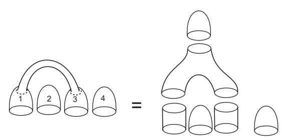


FIGURE 12. One term in the relation 4Tu is regarded as a composition of fundamental (generating) cobordisms

the equality $C_{12}+C_{34} = C_{13}+C24$, as in section (4.1). Applying $\mathcal{T}$ to the left side of the equality we obtain

$$\begin{aligned}&[\Delta i\otimes_R i\otimes_R i + i\otimes_R i\otimes_R \Delta i](1_R)\\ &\qquad = \Delta(1)\otimes_R 1\otimes_R 1 + 1\otimes_R 1\otimes_R \Delta(1)\\ &\qquad = 1\otimes_R X\otimes_R 1\otimes_R 1 + X\otimes_R 1\otimes_R 1\otimes_R 1\\ &\qquad\quad + 1\otimes_R 1\otimes_R 1\otimes_R X + 1\otimes_R 1\otimes_R X\otimes_R 1\end{aligned}$$

Applying $\mathcal{T}$ to the right member of the equality we obtain

$$\begin{aligned}&[(Id\otimes_R i\otimes_R Id)\Delta i\otimes_R i + i\otimes_R (Id\otimes_R i\otimes_R Id)\Delta i](1_R)\\ &\qquad = (Id\otimes_R i\otimes_R Id)(1\otimes_R X + X\otimes_R 1)\otimes_R 1\\ &\qquad\quad + 1\otimes_R (Id\otimes_R i\otimes_R Id)(1\otimes_R X + X\otimes_R 1)\,.\end{aligned}$$

So we obtain the same result. This, together with equations (5.1) and (5.2), proves that $\mathcal{T}$ preserves the three relations and hence the chain complex obtained in the category $\mathcal{M}_R$ is an invariant of knots and links. ☑

Obviously, the homology of this chain complex is an invariant of links. From this, we can conclude that this is the case, when the Frobenius system is either example number (1) or example (3) from section 2.1.

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