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A modification for Kovarik's method

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Abstract. In this paper, we suggest a modification of Kovarik's approximate orthogonalization method without any necessity to compute an inverse matrix at any iteration. Moreover, the cost of computation in each iteration of our modification is low and constrained to matrix-by-matrix multiplication. It is proved that this modification is linearly convergent with a small asymptotic error constant. Numerical experiments have shown that the number of iterations of our modification is at least as good as that of Kovarik's method. Therefore, despite the quadratic convergence of Kovarik's method, it is expected that the time of implementation of our modification will be lower than that of Kovarik's method. Numerical experiments show the validity of our findings.

Key words and phrases. Approximate Orthogonalization Methods, Quadratic Interpolation, Linear Convergence.

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Resumen. En este artículo sugerimos una modificación del método de ortogonalización aproximada de Kovaric sin necesidad de computar la matriz inversa de ninguna iteración. El costo de computación es relativamente bajo y se limita a multiplicar matrices. Se demuestra que esta modificación es linealmente convergente con una constante de error asintóticamente pequeña. Experimentos numéricos han mostrado que el número de iteraciones de nuestra modificación es por lo menos tan buena como en el método de Kovarik. Por consiguiente, a pesar de la convergencia cuadrática del método de Kovaric, se espera que el tiempo de implementación de nuestra modificación sea tan bajo como la del método de Kovarik. Experimentos numéricos muestran la validez de los hallazgos.

1. Introduction

In [2], Z. KOVARIK proposed his algorithm for approximate orthogonalization of a finite linearly independent set of vectors from a Hilbert space. His algorithm is some kind of iterative version of the classical Gram-Schmidt's and some of its direct applications have been derived for variational finite element formulation of elliptic problems and least squares [3,6]. KOVARIK had shown that his method has quadratic convergence. POPA [5] adapted and extended Kovarik's method to a set of arbitrary vectors in $\mathcal{R}^n$, and proved that if these vectors were put to rows of a matrix, then the transformed matrix columns, in addition to rows, were "quasi-orthogonal" (see [4]).

Suppose that $m \leq n$ and A is a $m \times n$ matrix of rank r. Kovarik's approximate orthogonalization method tries to transform A to a matrix with quasi-orthogonal rows (see (5)). This method is initialized by setting $A_0 = A$ and then generating the sequences of matrices K_k and A_k, $k \geq 0$, as the following:

$$K_k = (I - A_k A_k^T)(I + A_k A_k^T)^{-1}, \quad A_{k+1} = (I + K_k)A_k, \quad k \geq 0. \tag{1}$$

KOVARIK had shown:

Theorem 1. *Suppose that the rows of the matrix A are linearly independent and take*

$$A_\star = \left[(AA^T)^{1/2}\right]^{-1} A.$$

Then,

(a) *the matrix $A_\star$ has mutually orthogonal rows;*
(b) *the sequence $\{A_k\}_{k\geq 0}$ defined by (1) converges to $A_\star$. Moreover,*

$$\|K_0\|_2 < 1$$

and

$$\|A_\star - A_k\|_2 \leq \|K_0\|_2^{2^k}, \quad \forall\ k \geq 1. \tag{2}$$

Relationship (2) tells us that the sequence generated by Kovarik's method has quadratic convergence.

We note that, since the rows of the matrix A are linearly independent, the associated Gram matrix AA^T is symmetric and positive definite. Thus, the matrix $A_\star$ is well defined. On the other hand, if the rows of A are not linearly independent, then the matrix $(AA^T)^{1/2}$ still exists, but is no longer invertible. Thus, instead of $A_\star$, we have to consider its "natural" generalization A_∞ defined by

$$A_\infty = \left[(AA^T)^{1/2}\right]^{+} A,$$

where B^+ is the Moore-Penrose pseudoinverse of B (see [1]). POPA [5] proved that, in this case, sequence $\{A_k\}$ converges to A_∞ and the rows of A_∞ are "quasi-orthogonal".

It is known that, if

$$\|A_k A_k^T\|_2 < 1, \tag{3}$$

then the matrix $I + A_k A_k^T$ will be invertible and vice versa [1]. For $k = 0$, (3) is as

$$\|AA^T\|_2 < 1. \tag{4}$$

If (4) is true, then it can be proved [4] that (3) holds true, for all $k \geq 1$. Moreover, assumption (4) is not restrictive. It is sufficient to scale the matrix A, for example, as the following [6]:

$$A^{new} := \frac{1}{\sqrt{\|A\|_1 \|A\|_\infty + 1}} A.$$

So, without loss of generality, it is assumed that the matrix A satisfies (4). In this case, by setting $\delta = 1 - \lambda_{\min}(AA^T)$, where $\lambda_{\min}(AA^T)$ is the smallest nonzero eigenvalue of AA^T, we have [5]:

$$A_\infty := \lim_{k\to\infty} A_k = \left[(AA^T)^{1/2}\right]^+ A$$

and

$$\|A_k - A_\infty\|_2 \leq \delta^{2^k}, \qquad \forall\ k \geq 0.$$

Suppose that the singular value decomposition (SVD) of A is as

$$U^T A V = \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0),$$

where

$$\sigma_1 \geq \cdots \geq \sigma_r > 0, \qquad r = \operatorname{rank}(A).$$

Take

$$\tilde{I} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}.$$

It can be proved [7] that the following "quasi-orthogonality" relation between rows of A_∞ holds true:

$$\langle (A_\infty)_i, (A_\infty)_j \rangle = \langle (U)_i \tilde{I}, (U)_j \rangle. \tag{5}$$

If the rows of A are independent, then $\tilde{I} = I$ and the relation (5) is the same classical orthogonality. Popa [7] showed that a similar relation holds true between the columns of A_∞ and

$$\lim_{k\to\infty} \operatorname{cond}_2(A_k) = \operatorname{cond}_2(A_\infty) = 1.$$

Despite the quadratic convergence of Kovarik's algorithm, there is a difficult computational aspect related to explicitly compute the inverse of $I + A_k A_k^T$, for all k. As it is known, explicit computation of the inverse, because of accumulation and rounding errors, always has some problems [1]. Thus, it is preferred to use some techniques to approximate the inverse. Several modifications have been proposed for Kovarik's method, all of which try to eliminate the necessity

to explicitly compute the inverse. These are upon the use of some approximations for $(I + A_k A_k^T)^{-1}$, which are based on Taylor's series of particular functions and are generally linearly convergent [3,4,6,7].

In this paper, we propose a modification for Kovarik's method that, free from computation of an inverse, is very simple and its computational cost is restricted to matrix-by-matrix multiplication in each iteration. We prove that the modification is linearly convergent with a small asymptotic error constant, so it is expected to have some good convergent properties. Although Kovarik's algorithm is quadratically convergent and our modification is linearly convergent, numerical experiments show that the modification works at least as well as Kovarik's method, both in the number of iterations and in the time of implementation. In other word, we propose a modification for Kovarik's method so that it, with a lower cost, gives results as good as Kovarik's method and free from its numerical problems.

2. **A modification of Kovarik's method**

Suppose that the SVD decomposition of $A_0 = A$ is as

$$U^T A_0 V = \operatorname{diag}(\sigma_1^{(0)}, \ldots, \sigma_r^{(0)}, 0, \ldots, 0),$$

where $r = \operatorname{rank}(A)$. It is shown [5] that the SVD decomposition of A_{k+1} is

$$U^T A_{k+1} V = \operatorname{diag}(\sigma_1^{(k+1)}, \ldots, \sigma_r^{(k+1)}, 0, \ldots, 0),$$

where

$$\sigma_j^{(k+1)} = \left[1 + \frac{1 - (\sigma_j^{(k)})^2}{1 + (\sigma_j^{(k)})^2}\right] \sigma_j^{(k)}, \quad j = 1, \ldots, r, \quad k \geq 0. \tag{6}$$

Thus, the examination of Kovarik's method convergence leads to that of real numbers sequence (6).

Various modifications for Kovarik's method are obtained by the approximation of $1/(1 + (\sigma_j^{(k)})^2)$ (and therefore, $(I + A_k A_k^T)^{-1}$). For example, the approximation

$$\frac{1}{1 + (\sigma_j^{(k)})^2} \approx \sum_{i=0}^{q_k} \left(-(\sigma_j^{(k)})^2\right)^i$$

was used in [7] leading to modification of

$$K_k = (I - A_k A_k^T) \sum_{i=0}^{q_k} \left(-A_k A_k^T\right)^i, \quad A_{k+1} = (I + K_k) A_k, \quad \forall\ k \geq 0$$

for Kovarik's method. Here, the integers q_k are arbitrary odd. For a good convergence property of this modification, the numbers q_k must be chosen large. Moreover, if q_k is even or arbitrary, the method does not always converge.

In what follows, we suggest a modification for Kovarik's method, based on a special quadratic interpolation. To clarify the issue, we simplify (6) in the following way:

$$\sigma_j^{(k+1)} = \frac{2\sigma_j^{(k)}}{1 + (\sigma_j^{(k)})^2}. \tag{7}$$

Consider the function

$$f(t) = \frac{1}{1+t}, \quad 0 \le t \le 1.$$

We know that $f(0) = 1$ and $f(1) = 0.5$. We are going to approximate this function with a quadratic polynomial passing the points (0,1) and (1,0.5) which is as good as possible. Suppose that this polynomial is

$$p(t) = a_0 + a_1 t + a_2 t^2.$$

Since $p(0) = 1$ and $p(1) = 0.5$, we will have

$$a_0 = 1, \quad a_2 = -0.5 - a_1.$$

Therefore, $p(t)$ is as

$$p(t) = 1 + a_1 t - (a_1 + 0.5)t^2,$$

where a_1 is a parameter. Due to the existence of the parameter a_1 in $p(t)$, a class of approximations for $f(t)$ is obtained. Different choices for a_1 lead to different modifications for Kovarik's method. For example, if we choose a_1 so that

$$\int_0^1 f(t)\,dt = \int_0^1 p(t)\,dt,$$

then we will obtain $a_1 \approx -0.841$, leading to iterative scheme

$$\begin{aligned}
\sigma_j^{(k+1)} &= 2p((\sigma_j^{(k)})^2)\sigma_j^{(k)} \\
&= \left[2 - 1.682(\sigma_j^{(k)})^2 + 0.682(\sigma_j^{(k)})^4\right]\sigma_j^{(k)} \\
&= \left[1 + \left(1 - (\sigma_j^{(k)})^2\right)\left(1 - 0.682(\sigma_j^{(k)})^2\right)\right]\sigma_j^{(k)}
\end{aligned}$$

and, hence, the modification of

$$K_k = (I - A_k A_k^T)(I - 0.682 A_k A_k^T), \quad A_{k+1} = (I + K_k)A_k, \quad k \ge 0 \tag{8}$$

for Kovarik's method.

Also, we can choose a_1 such that least square error

$$\int_0^1 \left(f(t) - p(t)\right)^2 dt$$

is minimum. With some calculations, we obtain $a_1 \approx -0.848$ for which iterative scheme

$$\begin{aligned}\sigma_j^{(k+1)} &= 2p((\sigma_j^{(k)})^2)\sigma_j^{(k)} \\ &= \left[2 - 1.696(\sigma_j^{(k)})^2 + 0.696(\sigma_j^{(k)})^4\right]\sigma_j^{(k)} \\ &= \left[1 + \left(1 - (\sigma_j^{(k)})^2\right)\left(1 - 0.696(\sigma_j^{(k)})^2\right)\right]\sigma_j^{(k)}\end{aligned}$$

and hence the modification

$$K_k = (I - A_k A_k^T)(I - 0.696 A_k A_k^T), \quad A_{k+1} = (I + K_k)A_k, \quad k \geq 0 \tag{9}$$

are resulted. The convergence of these modifications will be examined later.

To make $p(t)$ a good approximation for $f(t)$, we choose the parameter a_1 such that $p(t)$ is near $l(t)$, where

$$l(t) = 1 - 0.5t$$

is the chord connecting the points $(0, 1)$ and $(1, 0.5)$. Since $f(t) \leq l(t)$, for all $t \in [0, 1]$, and since $p(t)$ must be a good approximation for $f(t)$, then we have $p(t) \leq l(t)$. Therefore,

$$|l(t) - p(t)| = l(t) - p(t) = (a_1 + 0.5)(t^2 - t), \quad \forall\ t,\ 0 \leq t \leq 1.$$

On the other hand, since $t^2 \leq t$, for all $t \in [0, 1]$, then we must have $a_1 + 0.5 < 0$ or $a_1 < -0.5$.

The special choice of $a_1 = -0.5$ implies that $p(t)$ equals $l(t)$ so that iterative scheme

$$\sigma_j^{(k+1)} = 2\left[1 - (\sigma_j^{(k)})^2\right]\sigma_j^{(k)}$$

and modification

$$A_{k+1} = 2(I - A_k A_k^T)A_k, \quad k \geq 0$$

are obtained for Kovarik's method. The above method is the same as that obtained by using two first terms of Newman's series [1]

$$(I + A_k A_k^T)^{-1} = I - A_k A_k^T + (A_k A_k^T)^2 - (A_k A_k^T)^3 + \cdots .$$

Having known the choice of the parameter a_1 in $p(t)$ (namely, a_1 satisfying $a_1 + 0.5 < 0$), we can pose the class of methods

$$\begin{aligned}\sigma_j^{(k+1)} &= 2p((\sigma_j^{(k)})^2)\sigma_j^{(k)} \\ &= \left[2 + a_1(\sigma_j^{(k)})^2 - 2(a_1 + 0.5)(\sigma_j^{(k)})^4\right]\sigma_j^{(k)} \\ &= \left[1 + \left(1 - (\sigma_j^{(k)})^2\right)\left(1 - \alpha(\sigma_j^{(k)})^2\right)\right]\sigma_j^{(k)}\end{aligned} \tag{10}$$

leading to the modification class of

$$K_k = (I - A_k A_k^T)(I - \alpha A_k A_k^T), \quad A_{k+1} = (I + K_k)A_{k+1}, \quad k \geq 0 \tag{11}$$

for Kovarik's method. Here, $\alpha = -2a_1 - 1$ and since $a_1 + 0.5 < 0$, then

$$\alpha = -2a_1 - 1 > 0. \tag{12}$$

We note that the modifications (8) and (9) for Kovarik's method are special cases of (11) corresponding to the choices of $\alpha = 0.682$ and $\alpha = 0.696$, respectively. In the next section, we prove that if the parameter α is chosen in a special interval, then (11) is always convergent and, in general, the order of its convergence is linear.

3. **Convergence proof**

It is clear that the parameter α must be chosen such that the class (10) (or (11)) will be convergent. We can write the sequence (10) as

$$x_{k+1} = h(x_k), \quad k \geq 0 \tag{13}$$

in which

$$h(x) = \left(1 + (1 - x^2)(1 - \alpha x^2)\right) x.$$

The above sequence starts from an initial approximation $x_0 \in (0, 1]$. If $x^* = \lim_{k\to\infty} x_k$, then x^* is a fixed point of function $h(x)$ and we have

$$x^* = \left(1 + (1 - (x^*)^2)(1 - \alpha(x^*)^2)\right) x^*.$$

Therefore,

$$x^*(1 - (x^*)^2)(1 - \alpha(x^*)^2) = 0$$

which results in

$$x^* = 0, \pm 1, \pm 1/\sqrt{\alpha}.$$

On the other hand, from (3) and that $x_k = \sigma_j^{(k)}$ is an eigenvalue of A_k, we have $0 < x_k < 1$, and hence $0 < x^* \leq 1$.

Theorem 2. *If $\alpha \in (0, 1)$, and if the numbers a and b are chosen such that $0 < a < 1 < b < 1/\sqrt{\alpha}$, then the sequence (13) converges to $x^* = 1$, for any initial approximation of $x_0 \in (0, 1]$.*

Proof. If $[a, b]$ consists of only fixed point of $x^* = 1$ and $h : [a, b] \longrightarrow [a, b]$, it is clear that, from the continuity of $h(x)$, sequence (13) converges to $x^* = 1$. Since $[a, b]$ does not include fixed point of 0, therefore $a > 0$. (that is $[a, b] = (0, b]$.) From $h(a) > a$, we conclude that $1 - (1 + \alpha)a^2 + \alpha a^4 > 0$, which is a quadratic polynomial in terms of a^2 with the roots of

$$a^2 = \frac{(1 + \alpha) \pm |1 - \alpha|}{2\alpha}.$$

Therefore, a must be such that

$$\begin{cases} a < 1 \ \ or \ \ a > 1/\sqrt{\alpha} & 1 - \alpha > 0 \\ a < 1/\sqrt{\alpha} \ \ or \ \ a > 1 & 1 - \alpha < 0. \end{cases}$$

Similarly, from $h(b) < b$ we conclude that $1 - (1+\alpha)b^2 + \alpha b^4 < 0$, which is a quadratic polynomial in terms of b^2 with the roots of

$$b^2 = \frac{(1+\alpha) \pm |1-\alpha|}{2\alpha}.$$

Therefore, b must be such that

$$\begin{cases} 1 < b < 1/\sqrt{\alpha} & 1-\alpha > 0 \\ 1/\sqrt{\alpha} < b < 1 & 1-\alpha < 0. \end{cases}$$

Hence, a and b must be such that

$$\begin{cases} 0 < a < 1 < b < 1/\sqrt{\alpha} & 1-\alpha > 0 \\ 0 < a < 1/\sqrt{\alpha} < b < 1 & 1-\alpha < 0. \end{cases}$$

In any case, $h : [a,b] \longrightarrow [a,b]$. Since we expect that $[a,b]$ includes fixed point of $x^* = 1$, we must choose $0 < a < 1 < b < 1/\sqrt{\alpha}$, corresponding to $1-\alpha > 0$. Moreover, by (12), we get $\alpha = -2a_1 - 1 > 0$, which results in $0 < \alpha < 1$. ☑

In the next theorem, we show that any modification of the class (11) is linearly convergent.

Theorem 3. *If $\alpha \in (0,1)$, then the sequence (13) is linearly convergent to $x^* = 1$, for any initial approximation of $x_0 \in (0,1]$. Moreover, the asymptotic error constant is $|2\alpha - 1|$.*

Proof. Let $e_k = x_k - 1$ denote the error in the kth iteration of the sequence (13). From

$$\begin{aligned} x_{k+1} &= \left(1 + (1-x_k^2)(1-\alpha x_k^2)\right) x_k \\ &= 2x_k - (\alpha+1)x_k^3 + \alpha x_k^5, \end{aligned}$$

we obtain

$$\begin{aligned} e_{k+1} &= 2e_k - (\alpha+1)(e_k+1)^3 + \alpha(e_k+1)^5 \\ &\approx (-1+2\alpha)e_k. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = |2\alpha - 1|. \tag{14}$$

Relationship (14) shows that the sequence (13) is linearly convergent with asymptotic error constant $|2\alpha - 1|$. ☑

By combining the above two theorems, we obtain the following result.

Theorem 4. *If $\alpha \in (0,1)$, then the class of modifications (11) is linearly convergent with the asymptotic error constant $|2\alpha - 1|$.* ☑

For a rapid convergence, we must choose the parameter α such that the asymptotic error constant $|2\alpha - 1|$ will be as small as possible. Suppose that $x_0 \in (0, 1]$ is arbitrary. If the sequence (13) is generated for different values of α, $0 < \alpha < 1$, then we will observe that:

(a) For $0.506 < \alpha < 1$, the convergence is monotonic, while for $0 < \alpha \leq 0.506$, it is not.
(b) For α approaching zero or one, the convergence is slow.
(c) For $0.506 < \alpha \leq 0.6$, the convergence is rapid and monotonic.
(d) For $0.4 \leq \alpha < 0.506$, the convergence is rapid but not monotonic.
(e) For $\alpha = 0.507$ the most rapid convergence is obtained.

So, we take $\alpha = 0.507$ and obtain the modification

$$K_{k+1} = (I - A_k A_k^T)(I - 0.507 A_k A_k^T), \quad A_{k+1} = (I + K_k)A_k, \quad k \geq 0 \tag{15}$$

for Kovarik's method that is linearly convergent with an asymptotic error constant equal to 0.014.

4. **Numerical experiments**

In this section, Kovarik's method, KO, is numerically compared with (15), MK. To this end, we use the stop criterion $\|A_{k+1} - A_k\|_2 < 10^{-4}$. The computing platform is a PC with a PIV processor at 2.8 MHz and 512 Mb RAM. Test square matrices are as the following:

Matrix H: Hilbert matrix $n \times n$, with elements $a_{ij} = 1/(i + j - 1)$;

Matrix P: Pascal matrix $n \times n$, with elements $a_{1j} = a_{i1} = 1$, $a_{ij} = a_{i-1,j} + a_{i,j-1}$;

Matrix A: a $n \times n$ matrix with elements $a_{ij} = |i - j|$;

Matrix M: a $n \times n$ matrix with elements $a_{ij} = \max\{i, j\}$;

Matrix R: a $n \times n$ random matrix with elements belonging to $[0, 1]$ (normal distribution with average 0 and variance 1);

In the next step, some test matrices were obtained from the following integral equation of the first kind [3]: for a given function $y \in L^2([0, 1])$, find $x \in L^2([0, 1])$ such that

$$\int_0^1 k(s, t)x(t)dt = y(s), \qquad s \in [0, 1]. \tag{16}$$

We discretized (16) by a collocation algorithm with the collocation points

$$s_i = \frac{i - 1}{n - 1}, \qquad i = 1, 2, \ldots, n,$$

obtaining a symmetric and positive semidefinite system

$$Ax = b, \tag{17}$$

where A and b given by

$$A_{ij} = \int_0^1 k(s_i, t)k(s_j, t)dt, \qquad b_i = y(s_i). \tag{18}$$

We considered two particular cases. In the first, denoted by **C1**,

$$k(s,t) = \frac{1}{1 + |s - 0.5| + t}, \qquad y(s) = \begin{cases} \ln\left(\dfrac{2.5 - s}{1.5 - s}\right) & s \in [0,\, 0.5) \\ \ln\left(\dfrac{1.5 + s}{0.5 + s}\right) & s \in [0.5,\, 1]. \end{cases}$$

The right hand side y was computed such that the equation (16) had the solution $x(t) = 1,\ \forall\, t \in [0, 1]$. Then

$$A_{ij} = \int_0^1 k(s_i, t)k(s_j, t)dt = \begin{cases} \dfrac{1}{\alpha_i(1 + \alpha_j)} & \alpha_i = \alpha_j \\ \dfrac{1}{\alpha_i - \alpha_j} \ln \dfrac{(1 + \alpha_j)\alpha_i}{(1 + \alpha_i)\alpha_j} & \alpha_i \neq \alpha_j \end{cases} \tag{19}$$

where

$$\alpha_i = 1 + |s_i - 0.5|, \qquad i = 1, 2, \ldots, n.$$

For $n \geq 3$, the rank of matrix A is given by

$$\operatorname{rank}(A) = \begin{cases} (n+1)/2 & n = odd \\ n/2 & n = even. \end{cases}$$

The second case, denoted by **C2**, is derived from the determination of the charge distribution generated from a given electric field. In this case,

$$k(s,t) = \frac{1}{\sqrt{(1 + (s - t)^2)^3}}, \qquad y(s) = s \tag{20}$$

and the exact values A_{ij} from (18) were approximated by rectangular ('midpoint') quadrature formula, with 16 equally spaced points in $[0, 1]$.

In table 1, iter and time indicate the number of iterations and the time of computations, respectively. According to table 1, iter and time of MK method, in general, are better than those of KO. Although MK method has linear convergence, in contrast with quadratic convergence of KO, table 1 shows that MK has better performance than KO. Note that the number of computational arithmetic of MK is much less than that of KO. So, MK method, besides simplicity, has a good performance.

		MK		KO	
MAT	n	iter	time(s)	iter	time(s)
H	100	68	0.344	70	0.547
	200	71	2.343	72	3.469
	500	70	34.828	74	51.234
P	100	232	1.000	230	1.719
	200	431	13.781	428	20.109
	500	739	22.461	739	31.011
A	100	17	0.125	18	0.188
	200	19	0.640	20	0.969
	500	21	10.438	23	15.907
M	100	19	0.141	20	0.203
	200	21	0.734	28	1.063
	500	23	11.407	25	17.250
R	100	14	0.125	15	0.187
	200	18	0.625	18	0.922
	500	18	8.750	20	13.594
C1	100	17	0.666	17	1.753
	200	15	9.108	16	20.172
	500	20	12.082	26	33.458
C2	100	66	4.252	70	8.735
	200	66	37.701	64	82.280
	500	72	50.017	81	102.118

Table 1.

5. **Conclusion**

We represented a modification for Kovarik's method based on a special quadratic interpolation. Our modification does not need inversion in every iteration. In addition, the cost of computations is much less than that of Kovarik's method. On the other hand, numerical experiments show that the number of iterations is less than that of Kovarik's method. Hence, we expect that, in general case, the time of computations will be less than that of Kovarik's method which is confirmed by numerical experiments.

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