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The Igusa local zeta function for $x^q - a$

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Abstract. In this short note we compute for the polynomial $x^q - a$, $a \in K((\pi))$, its Igusa local zeta function and the corresponding Poincaré series.

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Resumen. En esta breve nota calculamos la función zeta local de Igusa y su correspondiente serie de Poincaré para el binomio $x^q - a$, $a \in K((\pi))$

Our purpose in this short note is to compute the Igusa local zeta function for $x^q - a \in K((\pi))$, where K is a non–archimedean local field of positive characteristic. Let us begin by recalling what is meant by the Igusa local zeta function of a non constant polynomial. If K be a non-archimedean local field of arbitrary characteristic, let $\mathcal{O}_K$ be its ring of integers and $\mathfrak{P}_K$ its maximal ideal. Let π be a fixed uniformizing parameter of K, i.e. $\mathfrak{P}_K = \pi\mathcal{O}_K$, and let the residue field of K, i.e. $\mathcal{O}_K/\mathfrak{P}_K = \mathbb{F}_q$, the finite field with $q = p^r$ elements. Let υ denote the valuation of K satisfying $\upsilon(\pi) = 1$. For $x \in K^\times$, let $|x|_K = q^{-\upsilon(x)}$ and $|0|_K = 0$. Let $f(x) \in \mathcal{O}_K[x]$, $x = (x_1, \ldots, x_n)$, be a nonconstant polynomial. To these data one associates Igusa's local zeta function,

$$Z(f, s) := \int_{\mathcal{O}_K^n} |f(x)|_K^s \, |dx|, \quad s \in \mathbb{C}, \quad \mathrm{Re}\,(s) > 0, \tag{1}$$

where $|dx|$ denotes the Haar measure on K^n, normalized such that $\mathcal{O}_K^n$ has measure 1.

In order to compute (1), Igusa has suggested to use his stationary phase formula (SPF), which we pass to explain ([1], [4, p. 168]):

Let $\overline{E}$ a subset of $\mathbb{F}_q^n$ and let $\overline{S}$ its subset consisting of all $\overline{a}$ in $\overline{E}$ such that $\overline{f}(\overline{a}) = \nabla\overline{f}(\overline{a}) = 0$. Let E and S denote the preimages of $\overline{E}$, $\overline{S}$ under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ and let N the number of zeros of $\overline{f}(x)$ in $\overline{E}$. Then we have

$$\begin{aligned}\int_E |f(x)|_K^s\,|dx| =& q^{-n}\big(\mathrm{Card}\,(\overline{E}) - N\big) + q^{-n}\big(N - \mathrm{Card}\,(\overline{S})\big)\frac{(1-q^{-1})q^{-s}}{1-q^{-1-s}} \\ &+ \int_S |f(x)|_K^s\,|dx|\,.\end{aligned} \tag{2}$$

On the other hand, the Igusa local zeta function of a non constant polynomial is related to the number of solutions of congruences modulo $\pi^m\mathcal{O}_K$ and to exponential sums modulo $\pi^m\mathcal{O}_K$ ([1], [2]). More precisely, if

$$N_m = \mathrm{Card}\,\left\{x \in \big(\mathcal{O}_K/\pi^m\mathcal{O}_K\big)^n \mid f(x) \equiv 0 \mod \pi^m\mathcal{O}_K\right\}, \tag{3}$$

and $P(f,t)$ is the Poincaré series $P(f,t) = \sum_{m=0}^{\infty} N_m\big(q^{-n}t\big)^m$, where $N_0 = 1$, then

$$P(f,t) = \frac{1 - tZ(f,s)}{1-t}, \tag{4}$$

where $t = q^{-s}$. Thus the computation of the Igusa local zeta function of a polynomial is an equivalent process to the one of computing its Poincaré series.

If K is a non-archimedean local field of characteristic $p > 0$, it is known that $K = \mathbb{F}_q((\pi))$, the field of formal power series with coefficients in $\mathbb{F}_q$, $q = p^r$, in which case π is a uniformizing parameter.

Let $f(x) = x^q - a$, where $a = a_0 + a_1\pi + a_2\pi^2 + \cdots$, $a_i \in \mathbb{F}_q$. Our goal is to compute the integral

$$Z = \int_{\mathcal{O}_K} |x^q - a|_K^s\,|dx|, \tag{5}$$

We can suppose that $a \in \mathcal{O}_K^\times$, i.e. $a_0 \neq 0$, since otherwise an application of SPF produces

$$Z = 1 - q^{-1} + q^{-1}\int_{\mathcal{O}_K} |\pi^q x^q - a|_K^s\,|dx|, \tag{6}$$

and we can factorize a suitable power of π, or repeat the application of SPF.

If $a_o \neq 0$, the first application of SPF yields

$$Z = 1 - q^{-1} + q^{-1} \int_{\mathcal{O}_K} |a_0^q + \pi^q x^q - (a_0 + a_1\pi + a_2\pi^2 + \cdots)|_K^s \, |dx|, \tag{7}$$

$$1 - q^{-1} + q^{-1} \int_{\mathcal{O}_K} |\pi^q x^q - (a_1\pi + a_2\pi^2 + \cdots)|_K^s \, |dx| \, . \tag{8}$$

At this moment, we have several possibilities:

First possibility: There exists $a_i \neq 0$, with $1 \leq i \leq q-1$. In this case if j is the minimun index with this property, we have

$$Z = 1 - q^{-1} + q^{-1} t^j \int_{\mathcal{O}_K} |\pi^{q-j} x^{q-j} - a_j - \pi(\cdots)|_K^s \, |dx|, \tag{9}$$

and an application of SPF to the last integral yields

$$Z = 1 - q^{-1} + q^{-1} t^j \, . \tag{10}$$

Second possibility: $a_i = 0$ for $i = 1, \ldots, q-1$ and $a_q \neq 0$. In this case

$$Z = 1 - q^{-1} + q^{-1} t^q \int_{\mathcal{O}_K} |x^q - (a_q + a_{q+1}\pi + a_{q+2}\pi^2 + \cdots)|_K^s \, |dx|, \tag{11}$$

and we repeat the process with the last integral.

Therefore, if there exists $a_i \neq 0$, with $q \nmid i$, we deduce that Z is a rational function of $t = q^{-s}$, and in fact is a polynomial in t.

In the other case, i.e. for all i such that $q \nmid i$, $a_i = 0$, we have $a = a_{n_0} + a_{n_1 q}\pi^{n_1 q} + a_{n_2 q}\pi^{n_2 q} + \cdots$, where $0 = n_0 < n_1 < n_2 < \cdots$. Then the first application of SPF produces

$$Z = 1 - q^{-1} + q^{-1} t^q \int_{\mathcal{O}_K} |x^q - (a_{n_1 q}\pi^{(n_1 - 1)q} + a_{n_2 q}\pi^{(n_2 - 1)q} + \cdots)|_K^s \, |dx|, \tag{12}$$

and by a systematic application of SPF we obtain

$$Z = 1 - q^{-1}\big(1 + q^{-1}t^q + \cdots + q^{-(n_1-1)} t^{q(n_1-1)}\big) \tag{13}$$

$$+ q^{-n_1} t^{n_1 q} \int_{\mathcal{O}_K} |x^q - (a_{n_1 q} + a_{n_2 q}\pi^{(n_2 - n_1)q} + \cdots)|_K^s \, |dx|, \tag{14}$$

so that

$$Z = 1 - q^{-1} \sum_{k=0}^{\infty} \big(q^{-1} t^q\big)^{n_k} \sum_{i=0}^{n_{k+1} - n_k - 1} \big(q^{-1} t^q\big)^i, \tag{15}$$

and therefore

$$Z = \frac{1 - q^{-1}}{1 - q^{-1}\, t^{q}} \sum_{k=0}^{\infty} \left(q^{-1}\, t^{q}\right)^{n_k} - \left(q^{-1}\, t^{q}\right)^{n_{k+1}} . \tag{16}$$

Finally, if $|t| < \sqrt[q]{q}$, we have

$$Z = \frac{1 - q^{-1}}{1 - q^{-1}\, t^{q}}, \tag{17}$$

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