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Representing 3–manifolds by triangulations of S^3 : a constructive approach

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Abstract. A triangulation Δ of S^3 defines uniquely a number $m \leq 4$, a subgraph Γ of Δ and a representation $\omega(\Delta)$ of $\pi_1(S^3\backslash\Gamma)$ into Σ_m. It is shown that every (K, ω), where K is a knot or link in S^3 and ω is transitive representation of $\pi_1(S^3\backslash K)$ in Σ_m, $2 \leq m \leq 3$, equals $\omega(\Delta)$, for some Δ. From this, a representation of closed, orientable 3-manifolds by triangulations of S^3 is obtained. This is a theorem of Izmestiev and Joswig, but, in contrast with their proof, the methods in this paper are constructive. Some generalizations are given. The method involves a new representation of knots and links, which is called a butterfly representation.

Keywords and phrases. Knot, link, 3–manifold, triangulation, representation, branched covering, coloration.

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Resumen. Una triangulación Δ de S^3 define un único número $m \leq 4$, un subgrafo Γ de Δ y una representación $\omega(\Delta)$ de $\pi_1(S^3\backslash\Gamma)$ en Σ_m. Se sabe que cada (K, ω), donde K es un nudo o enlace en S^3 y ω es una representación transitiva de $\pi_1(S^3\backslash K)$ en Σ_m, $2 \leq m \leq 3$, es igual a $\omega(\Delta)$ para algún Δ. De esto se obtiene una representación de 3-variedades cerradas y orientables por triangulaciones de S^3. Este es un teorema de Izmestiev y Joswig pero, en contraste con su prueba, el método en este artículo es constructivo. Este trae consigo una nueva representación de nudos y enlaces llamada representación mariposa. Se dan algunas generalizaciones.

1. Introduction

Recall that if σ is a simplex of a polyhedron, *the star of* σ, $st(\sigma)$, is the set of simplexes of the polyhedron containing σ as a face, and the link of σ, $lk(\sigma)$, is the set of simplexes of $st(\sigma)$ not intersecting the face σ.

Let M, or M^n, be a closed, connected, orientable n-manifold. Let Δ be a triangulation of M. If σ is an $(n-2)$-simplex, the star of σ can be thought of as a *closed chain* of v n-simplices, $\sigma_1, \cdots, \sigma_v$ such that each one shares with the next one precisely one $(n-1)$-face, and σ_v shares precisely one $(n-1)$-face with σ_1. We call v the valence of σ and we denote this number by $v(\sigma)$. We will say that σ is *even* (resp. *odd*) if v is even (resp. odd).

Denote by O_Δ the set of $(n-2)$-simplices of odd valence. Then Δ defines an integer $1 \leq m \leq n+1$, and a, unique up to conjugation, representation $\omega(\Delta) : \pi_1(M \backslash O_\Delta) \to \Sigma_m$ into the symmetric group of m indices, such that no index is left fixed by all the elements of the image of $\omega(\Delta)$ (so the action of the image is *effective*). See Section 2.

Of course, the representation $\omega(\Delta)$ defines uniquely a branched cover of M, which is an n-pseudomanifold $M(\Delta)$. We note that $M(\Delta)$ might not be connected, if $\omega(\Delta)$ is not transitive.

The aim of this paper is to prove the following theorem:

Theorem 6.1 *Let* (K, ω) *be a knot or a link* K *in* S^3 *together with a transitive representation* $\omega : \pi_1\left(S^3 \backslash K\right) \to \Sigma_m$, $2 \leq m \leq 3$, *sending meridians to transpositions (simple representation). Then there is a constructive procedure to obtain a triangulation* Δ *of* S^3 *such that:*

i) $O_\Delta = K$.

ii) $\omega(\Delta) = \omega$.

From this we get, as a corollary, the following Theorem of Izmestiev and Joswig [6].

Corollary 7.1: *Let* M *be a closed, orientable* 3*-manifold. Then there exists a triangulation of* S^3 *such that* M *is homeomorphic to* $M(\Delta)$. *Moreover, there is a simplicial 3-fold simple covering* $\rho : M(\Delta) \to \Delta$, *branched over some knot (dependent on* M*).*

The method of the proof, which is constructive, involves two different ideas. The first idea is to represent each knot or link K in S^3 as a *butterfly*. For this, we mean a polyhedral 3-ball, B_K, with faces identified in pairs by reflections in some particular edges. The result of the identification will be S^3 and the image of the particular edges will be K. The details of such construction are described in Sections 3 and 4.

To obtain the wanted triangulation Δ of S^3 (such that $\omega(\Delta) = \omega$) we associate 4 colors to the vertices of some carefully constructed triangulation of ∂B_K, in such a way that this 4-coloration is compatible with ω (in a way to be specified in Definition 5.2).

Next, we extend this 4-colored triangulation of ∂B_K to B_K by using a constructive theorem of Goodman and Onishi [2] we then claim that this triangulation is the wanted triangulation Δ. Indeed, for this triangulation $O_\Delta = K$ and $\omega(\Delta) = \omega$.

We remark that some of the methods, used in our 4-coloration process, work as well for representations ω more general that the simple ones we are considering here (see Theorem 5.4).

We do not know if our Theorem 6.1 extends to the case $m = 4$. This is an open problem. We will work throughout in the PL category.

2. $\omega(\Delta)$: A representation associated to the triangulation Δ

Let Δ be a triangulation of the n-manifold M^n. If Δ^{n-2} is the $(n-2)$-skeleton of Δ then the group $\pi_1(M^n \backslash \Delta^{n-2})$ is a free group.

If σ is a k-simplex in Δ $(0 \le k \le n)$ we will denote by $b(\sigma)$ its barycenter.

Let $c = \{\sigma_0, \cdots, \sigma_k\}$ be a chain of n-simplices such that $\sigma_i \cap \sigma_{i+1}$, $i = 1, \cdots, k-1$, is a common $(n-1)$-face. This chain defines a unique path starting in $b(\sigma_0)$, and ending in $b(\sigma_k)$, as follows: connect $b(\sigma_i)$ with $b(\sigma_i \cap \sigma_{i+1})$ and with $b(\sigma_{i+1})$ with straight segments for $i = 0, \cdots, k-1$. The union of these segments, oriented by the chain c, defines, in a natural way, a unique linear path $\tilde{c} : [0,1] \to M^n \backslash \Delta^{n-2}$. Now, if for the chain c we have $\sigma_0 = \sigma_k$, then $\tilde{c}$ is a loop in $M^n \backslash \Delta^{n-2}$. In this case we call c a *closed chain.*

Since $I = [0,1]$ is compact, for any loop $\sigma : I \to M \backslash \Delta^{n-2}$, based at $b(\sigma_0)$, there is a unique closed chain of n-simplices $c_\sigma = \{\sigma_0, \cdots, \sigma_k = \sigma_0\}$, associated to σ, such that σ crosses the simplices in the chain consecutively. Of course the same definition applies when Δ triangulates a manifold with boundary.

It is not difficult to see that for any class σ of homotopic loops (based at $b(\sigma_0)$) in $M \backslash \Delta^{n-2}$, we can choose a class representative, such that the closed chain associated to it, has minimal number of simplices. Moreover, this closed chain is unique. In other words we have a one to one correspondence between closed chains (starting at σ_0) and loops based at $b(\sigma_0)$.

A closed chain $c = \{\sigma_0, \cdots, \sigma_k = \sigma_0\}$ defines, in a unique way, a bijection $\omega(\Delta)(\tilde{c})$ of the 0-skeleton σ_0^0 of σ_0, i.e., an element of Σ_{n+1}, as follows:

Take the set $\{1, \cdots, n, n+1\} = C$. We think on C as a set of $(n+1)$-*colors.* We say that an n-simplex is *colored* if and only if every vertex has associated a color and all colors of C are used (one can think of C also as the 0-skeleton σ_0^0 of σ_0).

If two n-simplices share a $(n-1)$-face and one of them is colored, the other can be colored in just one way.

Start now with a closed chain $c = \{\sigma_0, \sigma_1, \cdots, \sigma_k = \sigma_0\}$. Color σ_0. Then we can color successively $\sigma_1, \cdots, \sigma_{k-1}$ in just one way. Then the color in σ_{k-1} induces a coloration of σ_0, which in general do not coincide with the original one but it is a permutation of the set $\{1, \cdots, n, n+1\} = C$ of colors.

This permutation is by definition $\omega(\Delta)(c)$. This process of coloring a chain of simplices starting from a fixed one σ_0 will be referred to as *propagation* for shortness. The sequence of $k+1$ vertices of c (one for each simplex of the chain c) with the same color i, is the i-*orbit*. Then the permutation $\omega(\Delta)(c)$ is the map sending the first vertex of each orbit to its last. In Fig. 1 we see that if the color of v_1 is i, the i-orbit is the sequence $v_1, v_3, v_3, v_5, v_5, v_1$.

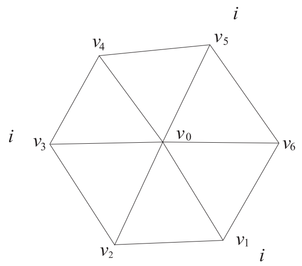

Figure 1: The i-orbit.

Proposition 2.1. *If M^n is a closed, connected, orientable n-manifold with a triangulation Δ, $\sigma_0 \in \Delta$, there is a canonical representation $\omega(\Delta) : \pi_1(M\backslash\Delta^{n-2}, b(\sigma_0)) \to \Sigma_{n+1}$ associated to Δ, unique up to conjugation in Σ_{n+1}.*

Proof. Define $\omega(\Delta)(\sigma) = \omega(\Delta)(c_\sigma)$, where σ is an element of $\pi_1(M\backslash\Delta^{n-2}, b(\sigma_0))$ and c_σ is the chain uniquely associated to it ☑

Let α be an $(n-2)$-face of Δ. A meridian $[\mu_\alpha]$ of α is an element of $\pi_1(M\backslash\Delta^{n-2}, b(\sigma_0))$ having a representative μ_α of the following form. Note that $st(\alpha)$ is a closed chain c, which has associated a closed path $\tilde{c}$. The representative of the meridian $[\mu_\alpha]$ is the composition of 3 paths $t \circ c \circ t^{-1}$, where $\tilde{c}$ is the loop associated to $st(\alpha)$ and the path t (called the *tail* of μ_α) starts at $b(\sigma_0)$ and ends at the barycenter of a simplex in $st(\alpha)$. The property given by the following proposition is then trivial (look at Figure 3).

Proposition 2.2. *If $\mu_\alpha \in \pi_1(M\backslash\Delta^{n-2}, b(\sigma_0))$ is a meridian of an $(n-2)$-face α, then $\omega(\Delta)(\mu_\alpha) = id$ if and only if α is even.*

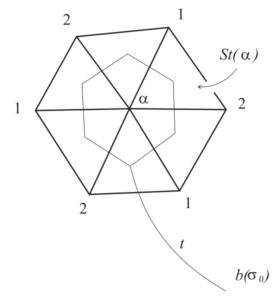

Figure 2: μ_α for α even.

Let O_Δ be the set of the $(n-2)$-simplices of odd valence. Because of the last proposition and the Van Kampen Theorem, then $\omega(\Delta)$ defines a unique, up to conjugation, homomorphism, which we denote also by

$$\omega(\Delta) : \pi_1(M \backslash O_\Delta, b(\sigma_0)) \to \Sigma_{n+1} = \Sigma_{\sigma_0^0}.$$

Let S be the complement in σ_0^0 of the set of fixed-points of σ_0^0 under the action of the image of $\omega(\Delta)$. Then $\omega(\Delta)$ acts effectively in $S \subseteq \sigma_0^0$, and the image of $\omega(\Delta)$ lies in Σ_S, where $\Sigma_S \subseteq \Sigma_{\sigma_0^0}$ is induced by the inclusion $S \subseteq \sigma_0^0$.

In this way a triangulation Δ of a manifold M^n defines a number (the cardinal of S) m, $1 \leq m \leq n+1$ and a representation, unique up to conjugation $\omega(\Delta) : \pi_1(M \backslash O_\Delta, b(\sigma_0)) \to \Sigma_m$, such that no index is left fixed by the image of $\omega(\Delta)$.

We call the representation $\omega(\Delta) : \pi_1(M \backslash O_\Delta, b(\sigma_0)) \to \Sigma_m$ *the canonical representation of* Δ.

Definition 2.3. *We say that a triangulation of an n-manifold M^n is $(n+1)$-colored if all simplices can be colored simultaneously with $(n+1)$-colors.*

The following theorem follows from the definitions.

Theorem 2.4. *If Δ is $(n+1)$-colored then all $(n-2)$-simplices of Δ are even and $\omega(\Delta)$ is trivial.*

Remark 2.1. *In general, even if $\omega(\Delta)$ is effective it does not need to be transitive. But it will be transitive if the cardinal m of the set S is ≤ 3.*

3. Knots and butterflies

In this section, we study a special class of balls with faces which are identified by topological reflections. These balls are called butterflies since the identifications of each pair of faces recall how a butterfly closes its wings. Here are formal definitions.

Let F be a connected, closed, orientable surface. A *polygon* (or n-gon, $n \geq 1$) in F is a tame embedding of the 2-disk in F together with a set of $n \geq 1$ points in its boundary which are called the *vertices of the polygon.* The closures of the connected components of the complement of the vertices in the boundary of a polygon are called the *edges of the polygon. (*An edge is an arc if $n \geq 2$ or a circle if $n = 1$.) A *polygonization* of F is a decomposition of F in a union of a finite number of polygons such that (i) the interiors of the distinct polygons of the decomposition are disjoint; and (ii) if two arbitrary polygons intersect, their intersection is at the same time a union of vertices and edges, and a connected 0-, or 1-dimensional manifold. (Therefore they can intersect in just one vertex, or in an arc formed by various edges, or in a circle; and, in this case, F must be S^2.) The union R of the boundaries of the polygons of the decomposition is then a connected graph embedded in F, since it is a union of vertices and edges. We say that *the graph R polygonizes F.*

Example 3.1. *Figures 3, 6a and 8 show different polygonalizations of S^2.*

Let A and B be two polygons of a polygonization of S intersecting in exactly one edge α of R and assume that A and B have the same number of edges. (This number might be one for the trivial polygonization of S.) Select a topological reflection $\alpha : A \to B$ which is orientation reversing in S, fixes each point of α, and sends vertices (resp. edges) of $R \cap A$ into vertices (resp. edges) of $R \cap B$. The reflection along α will be denoted by α also, and we say that $\alpha : A \to B$ is an α-reflection.

Definition 3.2. *Given $n \in \mathbb{N}$, $n \geq 1$, an n-butterfly (B, R, T) is a 3-ball B with a polygonization of its boundary S by a graph R into $2n$ polygons, together with a subset T of n mutually disjoint edges of R, such that the polygons are identified by α-reflections in pairs, $\alpha \in T$. (As we said before, to be identified, two faces must share exactly an edge $\alpha \in T$. The identification of a pair of faces is then achieved by an α-reflection along this common edge.) The $2n$ polygons of an n-butterfly are called wings, the union of the edges along which we made the reflections is called the trunk T and n is called the butterfly number.*

The result of the identification of pairs of wings of an n-butterfly (B, R, T) is a 3-manifold $M(B, R, T)$ homeomorphic to the 3-sphere S^3 (see [4]). Denote by $p : B \to M(B, R, T)$ the natural projection.

Definition 3.3. *In the identification $p : B \to M(B, R, T) = S^3$ the image $p(T)$ of the trunk T is a (linked) graph K embedded in S^3, and we say that the graph (knot, or link) K in S^3 can be represented by an n-butterfly, or that K admits an n-butterfly representation, or that (B, R, T) is an n-butterfly representation of K.*

Let (B, R, T) be a butterfly and $p : B \to M(B, R, T) = S^3$ the natural projection. We now make a classification of the set of vertices of R. A vertex V of R will be called an A-vertex iff $V \in T$. (It will be generically represented by A). The vertex V of R will be called an E-vertex iff it is not an A-vertex but $V \in p^{-1}p(A)$. (It will be generically represented by E). Finally the vertex V of R will be called a B-vertex iff $V \notin p^{-1}p(A)$ for any A-vertex A. (It will be generically represented by B). In the proof of Theorem 3.4 there is an example of a butterfly with vertices of types A and E, and in Example 3.5 there are some vertices of type A and B.

In the sequel we will consider definitions and concepts about knots, links and their projections as explained in [1], [8] and [12].

It is obvious that the trivial knot is the only one admitting a 1-butterfly representation (just identify the northern and southern hemispheres of S by reflection in the equator $R = T$).

Let us recall that for each rational number p/q in lowest terms ($p > q > 0$) there is a knot (if p is odd) or a link (if p is even) denoted by the same number p/q.

The following theorem is a translation of [10, page 164] to our language of butterflies, see [11, page 78].

Theorem 3.4. *Every 2-bridge knot or link p/q can be represented as a 2-butterfly. Except, for the trivial knot or the link $2/1$ (Hopf link) the butterfly will have E-vertices.*

Proof. For shortness we give the proof only for the rational knot 5/3. Without difficulty, it could be generalized to every other 2-bridge knot or link. We start from the following 2-butterfly and we show that performing the reflections illustrated by arrows in Figure 3, we obtain the knot $\frac{5}{3}$.

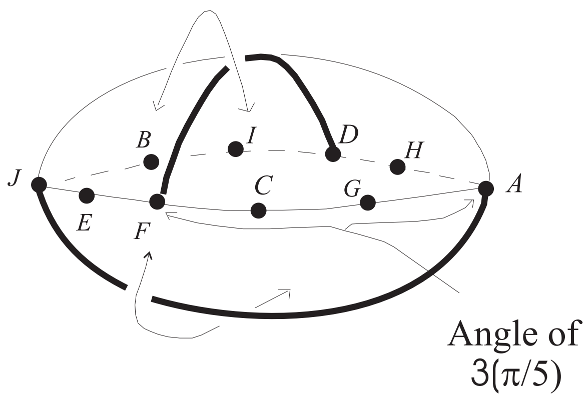

Figure 3: The rational knot 5/3.

This butterfly has four wings; $FDHAGC$; $FEJBID$; $JEFCGA$ and $AHDI$ BJ, which are identified two by two by the reflections made along the edges FD and AJ, that are indicated by arrows. The trunk is $\{FD, AJ\}$, so that F, D, A and J are A-vertices. The vertices C, G, H, I, B and E are E-vertices. Now, we make a sequence of deformations in order to be able to visualize the knot $\frac{5}{3}$. First, we stretch out the points placed on the butterfly "equator", obtaining a cylinder. Next we rotate the "upper lid" of the cylinder an angle $3(\frac{\pi}{5})$.

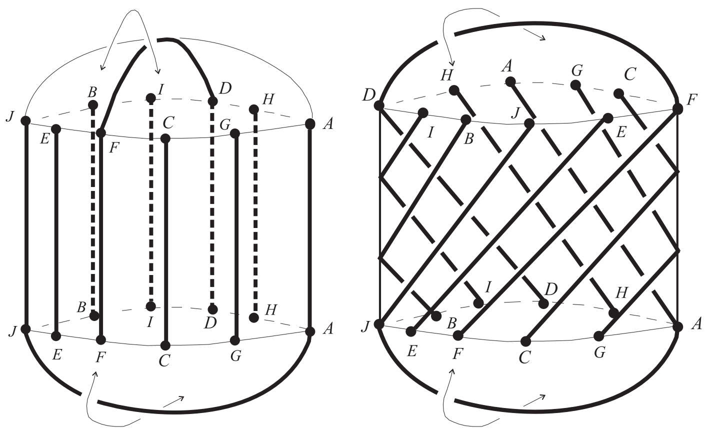

Figure 4a: Stretching. Figure 4b: Rotating.

Finally, making the identifications indicated by the arrows on the "upper and lower lids" of the cylinder, the knot $\frac{5}{3}$ becomes visible.

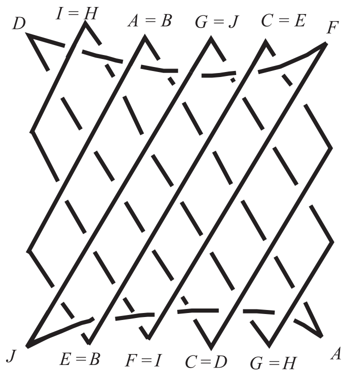

Figure 5: Identifying.

☑

Example 3.5. *Similarly Thurston showed in* [15] *that the Borromean-rings admits a* 6*-butterfly representation. In this butterfly there are 12 A-vertices and 8 B-vertices. See Figures 6a and 6b.*

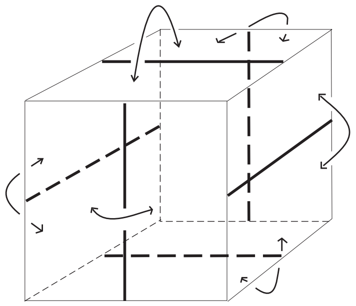

Figure 6a: A butterfly for the Borromean rings.

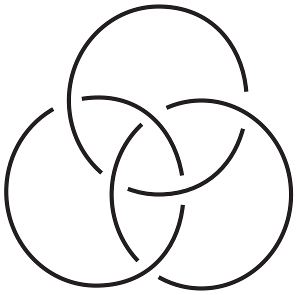

Figure 6b: The Borromean rings.

4. **Representing knots as butterflies**

Actually, every knot or link can be represented by an n-butterfly, for some $n \in \mathbb{N}$. This representation will depend on the diagram of the knot or link, so for each knot or link we have an infinite number of butterfly representations.

Theorem 4.1. *Every knot or link admits an n-butterfly representation, for some $n \in \mathbb{N}$.*

Proof. There is a more detailed proof in [4]. Let K be a knot (or link) which is embedded in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, we assume that it is as flat as possible as in the following picture.

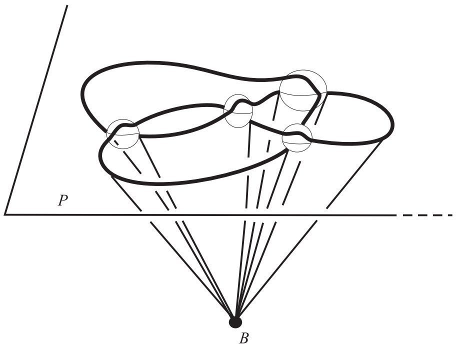

Figure 7: The cone over a knot.

To get a butterfly that represents the knot we cut along a cone of the knot.

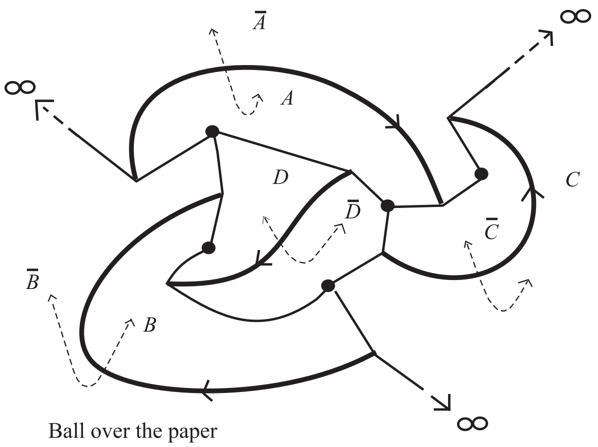

Figure 8: The butterfly.

The preview picture illustrates the boundary of the 4-butterfly representation of the Figure-eight knot (that is, the rational 5/3 knot). We envision that the interior of the butterfly is placed over the paper. We can see 20 edges. Three of them go to the point at infinity. The trunk has 4 edges that form the knot before cutting the cone off and we have 8 wings identified in pairs. ☑

A regular diagram of a knot (or of a link), like the one in Fig. 7 or 9a, can be thought of as a disjoint union of arcs in a plane. These arcs will be called the *arcs* of the diagram.

We remark that the last theorem is constructive. Now, we make explicit the algorithm for constructing a butterfly from a given regular diagram.

Algorithm for constructing a butterfly associated to a diagram of a knot: Let D_K be a regular diagram of a knot (or link) K, which we assume oriented only in order to fix notation. Without lost of generality we can assume that K is not the trivial knot (for this case we have a 1-butterfly representation). We also assume S^3 oriented. The positive orientation of S^3 in our figures will be given by a right handed screw.

Step 1 The diagram D_K of K is a finite collection T of disjoint, oriented arcs (no circles!) in a plane P (the plane of the paper; see Fig. 9a). We assume, as we can, that the projection of D_K onto P is connected and has no kinks.

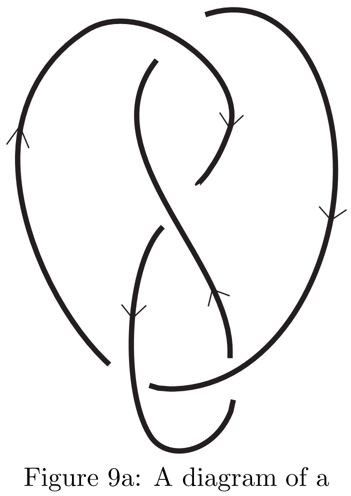

Figure 9a: A diagram of a knot.

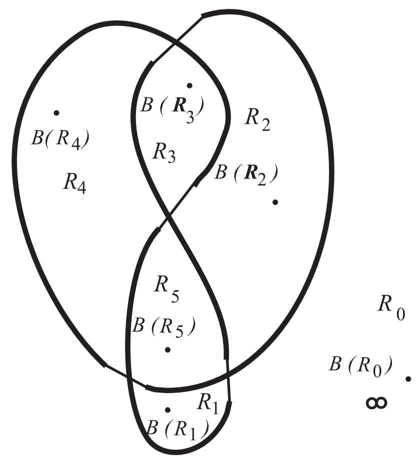

Figure 9b: The regions.

Step 2 We consider the regions R_i determined in the plane P by the projection of K (see Fig. 9b). In the interior of each bounded region we choose a point and label it by $B(R_i)$. For the unbounded region R_0, we label $B(R_0)$ the point at the infinity. These points are B-vertices.

Step 3 (Recall that the end points of the arcs of the diagram D_K are the A-vertices.) Each A-vertex, end of an arc α of the collection T, is joined, using an arc in P, with each of the B-vertices that belong to the adjacent regions to the arc α. So the paper becomes polygonalized by the graph R which is the union of the trunk T and the added arcs. (See Fig. 8).

Step 4 Over the plane of the paper P, we assume that there is a 3-ball B_K, with the induced orientation, whose boundary is the polygonalized plane P, oriented as the boundary of the oriented 3-ball B_K.

Step 5 The adjacent faces to each arc α of S are identified by an α-reflection that is indicated by double arrows. We denote by $\bar{A}$ the wing (or face) identified with the wing A. The face A will be placed at the right side of the oriented arc α. (See Fig. 8).

Then (B_K, R, T) is the wanted butterfly. It will be called the *butterfly associated to the diagram D_K of the knot K.* For shortness, we denote it simply by B_K.

For each arc α of the trunk T, there are two adjacent wings in B_K that are identified by a reflection along α and they have the following shape:

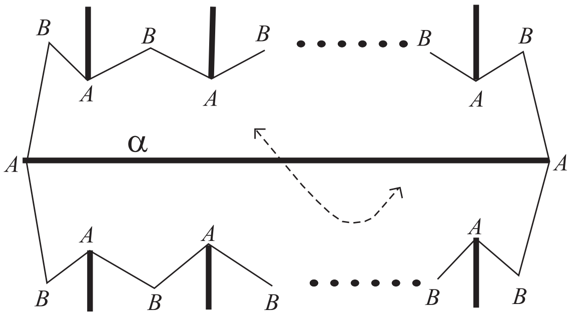

Figure 10: Two wings.

On each wing we distinguish two types of vertices:

A-*vertices* are the ends of the different arcs of T.

B-*vertices* are those that come from the vertex of the cone once we cut it off. They correspond to the points $B(R_i)$ given in the algorithm.

Remark 4.1. *M. Toro (*[16]*) explains a programm in Mathematica for constructing the butterfly associated to a diagram of a knot.*

Example 4.2. *Applying this algorithm to the regular diagram of the Borromean rings shown in Fig. 6b, we recover the butterfly representation discovered by Thurston (Fig. 6a).*

5. A canonical triangulation Δ_K of (S^3, ω)

If (B, R, T) is a butterfly representation of a knot K, each (oriented) edge α of the trunk T defines a meridian generator μ_α of the knot group of K as follows. Take an interior (base) point O inside B and run an oriented arc from O to an interior point in A and another oriented arc from the corresponding point in $\bar{A}$ back to O. The (oriented) union of these two arcs represents a meridian generator of the knot group that will be denoted by μ_α. If ω is a representation of the knot group into the symmetric group Σ_n of the numbers $1, 2, \cdots, n$ we can endow the (oriented) edge $\alpha \in T$ with the permutation $\omega(\mu_\alpha)$. (If the representation ω sends meridians to elements of Σ_n of order two, then the permutations $\omega(\mu_\alpha)$ and $(\omega(\mu_\alpha))^{-1}$ coincide and the orientation of the trunk becomes irrelevant).

In the sequel we will assume that for $n < m$ there is a natural inclusion of Σ_n as a subgroup of Σ_m induced by the inclusion $\{1, 2, \cdots, n\} \subset \{1, 2, \cdots, m\}$. In this way an element of Σ_n acts in the set $\{1, 2, \cdots, m\}$ fixing the numbers $\{n + 1, \cdots, m\}$.

We will understand that the boundary of a butterfly is *triangulated* if we have a triangulation for the butterfly boundary, such that all the triangles become identified by couples when we identify the butterfly wings using the reflections along the edges of the trunk.

We need to extend slightly the definition of a triangulation of an M^n manifold to be $(n+1)$-colored (see Definition 2.3) to the case in which the number of colors l be $\geq (n+1)$.

Definition 5.1. *Let Δ be a triangulation of an n-manifold. Let $l \geq (n+1)$. Let $C = \{1, \cdots, l\}$. An n-simplex σ is l-colored iff to each vertex is assigned a color from C, and the colors of its vertices are pairwise different. A triangulation Δ is l-colored iff all n-simplexes can be l-colored simultaneously. In a l-coloration of Δ with $C = \{1, \cdots, l\}$, $C(v)$ will denote the color of the vertex v.*

Definition 5.2. *If ω is a representation of the knot group of K into the symmetric group Σ_n, $n \geq 2$, (B, R, T) is a butterfly representation of K and the boundary of this butterfly is triangulated by Δ and l-colored by $C = \{1, 2, \cdots, l\}$, $l > n$, we say that C is compatible with ω if and only if under the α-reflection (α is the arc shared by A and $\bar{A}$) the color k matches with color $\omega(\mu_\alpha)(k)$, that is, for every vertex $v \in A$ we have $C(\alpha(v)) = \omega(\mu_\alpha)(C(v))$.*

Example 5.3. ***The knot 3/1.*** *The Fig. 11 illustrates a triangulation Δ of a butterfly (B, R, T) that represents the rational knot $3/1$ (trefoil knot). It is 4-colored by $C = \{1, 2, 3, 4\}$ as shown. The representation $\omega : \pi_1 \left(S^3 \backslash 3/1\right) \to \Sigma_3$ is the map of the group of $3/1$ into Σ_3 sending the two generating meridians μ_α and μ_β (associated to the trunk T) to the permutations $(12)(3)$ and $(13)(2)$, respectively. It is easy to see that this 4-coloration is compatible with ω.*

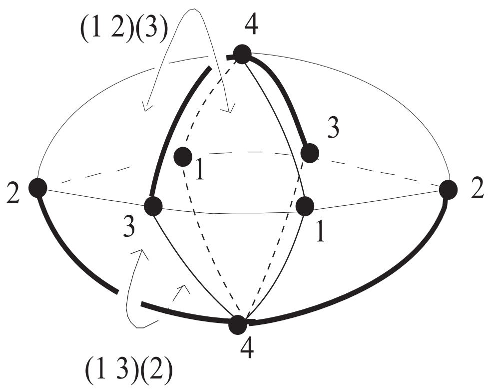

Figure 11: A 4-coloration compatible with ω.

The next is the principal result of this section.

Theorem 5.4. *Let (K, ω) be a knot (or a link) with a given regular diagram D_K, together with a transitive representation $\omega : \pi_1 \left(S^3 \backslash K\right) \to \Sigma_m$, $1 \leq m \leq 3$, sending meridians to transpositions. Then there is a triangulation Δ_∂ of* *the boundary* $\partial(B_K)$ *of the butterfly* B_K *and a* 4*-coloration of* Δ_∂ *which is compatible with* ω*.*

Proof. Our purpose is to find a triangulation of $\partial(B_K)$ and a 4-coloration compatible with ω. In other words, we need a triangulation of $\partial(B_K)$ such that if two triangles are identified by an α-reflection whose associated transposition is $(i\ j)$, $i, j \in \{1,2,3\}$, the α-reflection sends colors (i, j) to colors (j, i), and leaves the remaining two colors fixed. (The number 4 remains always fixed.) In particular, in an α-reflection with permutation $(i\ j)$ the vertices upon the arc α are colored with one of the remaining numbers.

First of all, we construct an initial triangulation Δ, and we assign four colors to the vertices in such a way they are compatible with ω. After that we will refine Δ in order that colors along adjacent vertices be different.

Let us describe the set of vertices of Δ. This set contains all the A-, and B-vertices given in the Algorithm before. Moreover, it also contains a point, which is taken in the interior of each arc of the diagram. We denote generically these points by D and we say that they are D*-vertices.*

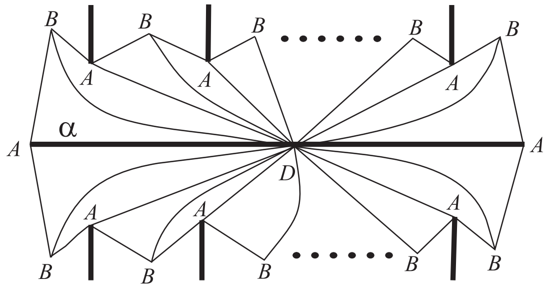

Figure 12: Putting a D-vertex.

Now, we describe the edges of Δ. On one hand, the curves AB and AD that are contained in the boundary of the wings are edges of Δ. On the other hand, we notice that each wing has only one D-vertex in its boundary. From that point we trace disjoint curves (except in D) toward every B-, and A-vertex (that is not an end of the arc that contains the point D). These additional curves are contained in the same wing that contains the point D and are also edges of Δ by definition.

We already have the triangulation Δ. We remark that each triangle has one vertex of each type.

Now, we color the vertices of Δ. In order to get a 4-coloration compatible with ω, first of all, we color the A-, and D-vertices which are the only ones lying in the edges α of the trunk. If the permutation associated to the edge α is $(i\ j)$, the vertices of α are to be given a color in $\{1,2,3,4\} - \{i, j\}$.

Assign the number 4 to D-vertices and the color $\{1,2,3\} - \{i, j\}$ to A-vertices. The color 4 given to D-vertices is compatible with ω because ω fixes

4. We observe that the colors given to the A-vertices are also compatible with ω. In fact, this is trivial for the A-vertices that are ends of the arc α; they remain fixed under the α-reflection. Hence, let us consider an A-vertex that belongs to the adjacent wings to α but is not an end point of α. Its corresponding point under the α-reflection is another A-vertex, and these two A-vertices arise from the same crossing point of the diagram. So, their colors also match under the permutation $(i\ j)$ associated to α because ω associates to each arc a transposition in Σ_3 and the only possibilities for the A-vertices that arise from the same crossing point are the following: A-vertices belong to arcs with the same associated transposition (Fig. 13a) or belong to arcs with different associated transpositions (Fig. 13b). (Actually, the fact that the colors of the A-vertices match is a consequence of the Wirtinger relations that are verified at each crossing point of the diagram).

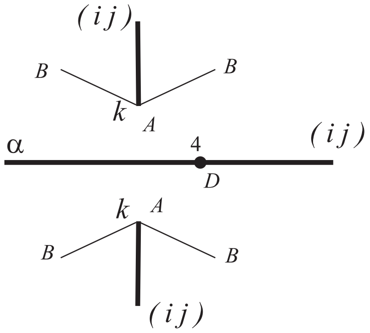

Figure 13a: Colors compatible with $(i\ j)$.

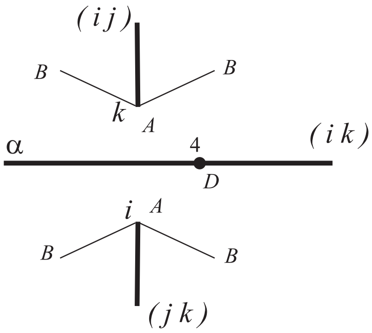

Figure 13b: Colors compatible with $(i\ k)$.

Now, we proceed to color B-vertices. Recall that for each region R_i, determined by the projection of the knot, there is exactly one B-vertex denoted by $B(R_i)$.

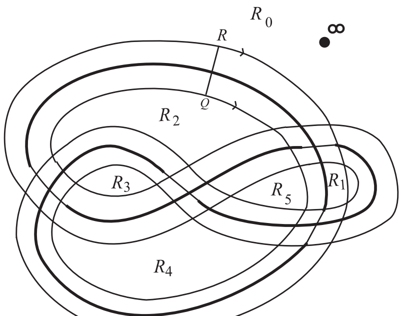

Figure 14: Touring the regions.

We note the obvious fact that starting in a region R_i, it is possible to tour all the regions going transversely across the arcs of the link. For instance, here is a curve visiting all the regions, in the case of a knot: Our initial point is a point Q in a given region. Then we move following a parallel curve to the knot until we again find Q. Once there, we traverse the knot and we choose another point in the adjacent region to the initial one, that we label R. We continue touring the regions moving along a parallel curve to the knot until we find R and we stop there. In this way we have passed through every region because every region is bounded by the knot (see Fig. 14). If we have a link, an obvious modification of the above method gives the touring curve.

Now, let us give the coloring rule for the B-vertices. We start by assigning any of the numbers $1, 2$ or 3 to one of them. Once it is colored the others become colored in the following way: if R_{i_1} and R_{i_2} are two regions that share an arc to which is associated the transposition $(i\ j)$ and if we have assigned the color $k \in \{1, 2, 3\}$ to $B(R_{i_1})$, then the vertex $B(R_{i_2})$ gets the color $(i\ j)k$, i.e., the image of k under the transposition $(i\ j)$.

This coloring rule and the fact that starting in any region we can travel to any other region guarantees that it is enough to choose the color of only one of the $B(R_i)$ to get all the B vertices colored. Moreover, since we have three options for coloring the first one, each of the three gives a different coloration of the $B(R_i)$ vertices. (Incidentally, we will see later, in Remark 8.1, that these different colorations might give rise to different triangulations).

We need to prove that given the color of $B(R_i)$, then the color of $B(R_j)$ is independent of the chosen path between $B(R_i)$ and $B(R_j)$. In fact, it is enough to observe what happens with the colors of B-vertices of the four regions sharing a crossing point.

Let $R_{i_1}, R_{i_2}, R_{i_3}$ and R_{i_4} be the four adjacent regions to a fixed crossing point. If we give the color, for example, to $B(R_{i_1})$ then the colors of $B(R_{i_2}), B(R_{i_3})$ and $B(R_{i_4})$ are determined. In fact, the two possible cases are:

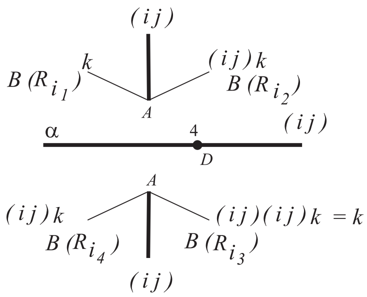

Figure 15a: The three arcs have the same associated transposition.

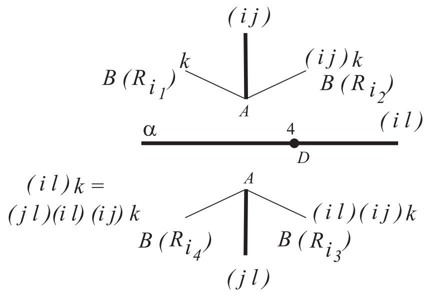

Figure 15b: The three arcs have different associated transpositions.

So, the color of one of the $B(R_i)$ determines the color of the rest. Furthermore, because of the coloration rule, we have that these colors are compatible with ω. (Again, all this is a consequence of the Wirtinger relations that are verified at each crossing point of the diagram.) In this way, we have colored all the vertices of the triangulation Δ.

However, this coloration is not in general a 4-coloration, because we could get two adjacent vertices with the same color. Since the only vertices that are given the number 4 are D-vertices and they are not adjacent, then the only adjacent vertices that could have the same color have to be A or B-vertices. Let us assume that we have a fixed A-vertex that has color k and is an end of an arc α with associated transposition $(i\ j)$. Suppose that it connects to a B-vertex with the same color k. In Fig. 16a we illustrate what happens at the crossing point, at which the fixed A belongs.

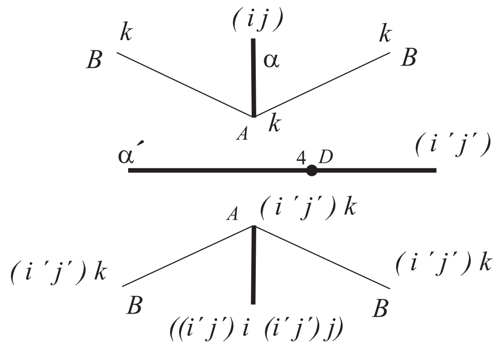

Figure 16a: To be adjusted.

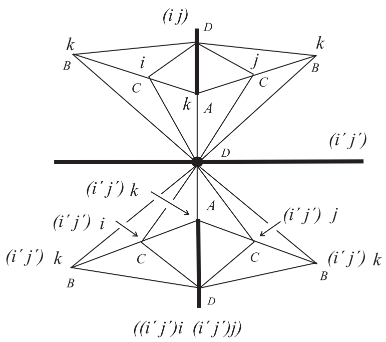

Figure 16b: Adjusted.

Actually, there are two B-vertices that connect to the same fixed A-vertex, and both have to have the same color k because the number k is fixed under the action of $(i\ j)$. Now, let α' be the arc that pass over at this crossing point. Suppose that the associated transposition of α' is $(i'\ j')$, then we see that the A and B-vertices placed on the other side of the arc α' have the same color $(i'\ j')k$. Therefore there are four vertices to be adjusted by subdivision. We take a point in the interior of each of these four edges in such a way that all four become identified by the reflections performed along α and α'. We call these points C-*vertices.* We add them to the initial set of vertices of Δ. Now, we trace curves from each of these C-vertices to connect them to the two D-vertices placed in the two adjacent wings to C. We add these new edges to the triangulation Δ. See Fig. 16b. To color these four C-vertices, we take any of them and color it with $i \in \{1, 2, 3\} - \{k\}$, if it belongs to the arc whose ends are colored with k. Immediately, because of the Wirtinger relations, the colors of the other three are determined and they are compatible with ω. We see this situation in Fig. 16b. Moreover, the coloration of the four C points around a crossing point is local, that is, it is independent of the coloration of any other C-vertex placed at a different crossing point (if it exists). Since, it is possible to color the four C-vertices wherever they appear, we get a 4-colored triangulation Δ_∂ of ∂B_K compatible with ω. This completes the proof of the theorem. ☑

Remark 5.1. *Notice that any triangle in the triangulation Δ_∂ has always a D-vertex, i.e., a vertex labeled with the color 4. Moreover, the colors of the vertices of any edge that is not on the trunk T belong to the set $\{1, 2, 3\}$.*

Now, we extend this triangulation Δ_∂ of ∂B_K to the butterfly B_k by using a constructive theorem of Goodman and Onishi that says that given a 4-colored triangulation of S^2 it is possible to extend it to a 4-colored triangulation of the 3-ball B^3 (see [2]).

Once we have this triangulation of B_K, identifying the butterfly wings by $p : B_k \to S^3$ we not always obtain a triangulation of S^3. We have to be careful! The obtained "triangulation" might not be a good triangulation, since we could get two tetrahedra sharing more than one face or one tetrahedron with two of its faces identified between them. If this occurs, we need to make a special type of subdivision, that we call *antipyramidal subdivision*, to each of the problematical tetrahedra. In the following we explain how to perform this subdivision.

Let $v_1v_2v_3$ be a triangle in a triangulation Δ of a 2-manifold M^2. We suppose that the colors of the vertices are the same subindexes. Let B be its barycenter. Let B_1, B_2 and B_3 be the barycenter of the triangles Bv_2v_3, Bv_1v_3 and Bv_1v_2, respectively, that we color with their subindexes. The *antipyramidal subdivision* has the following 7 triangles $B_1B_2B_3$, $v_1v_2B_3$, $v_1v_3B_2$, $v_2v_3B_1$, $v_1B_2B_3$, $v_2B_1B_3$ y $v_3B_1B_2$. We extend the coloration of the initial triangle to this new triangulation.

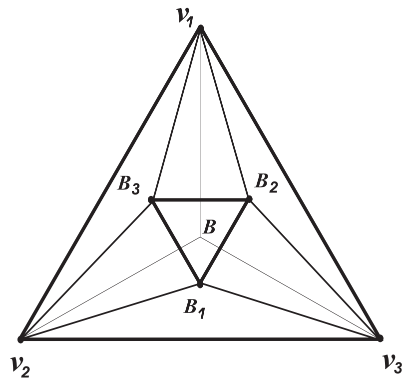

Figure 17: The antipyramidal subdivision.

Similarly, for dimension 3, if $v_1v_2v_3v_4$ is a tetrahedron with barycenter B, its antipyramidal subdivision contains the following 13 tetrahedra $B_1B_2B_3B_4$,

$v_1v_2v_3B_4$, $v_1v_2v_4B_3$, $v_1v_3v_4B_2$, $v_2v_3v_4B_1$, $v_1v_2B_3B_4$, $v_1v_3B_2B_4$, $v_2v_3B_1B_4$, $v_3v_4B_1B_2$, $v_1B_2B_3B_4$, $v_2B_1B_3B_4$, $v_3B_1B_2B_4$ and $v_4B_1B_2B_3$, where B_i is the barycenter of $Bv_jv_kv_l$, with j,k,l different numbers in $\{1,2,3,4\}$. As in the 2 dimension case, this subdivision has the advantage that the coloration of the original tetrahedron extends to it, i.e., it does not spoil it.

Definition 5.5. *The triangulation of B_K just defined, which is 4-colored, will be denoted by Δ_{B_K} and called the canonical triangulation of B_K. It induces a triangulation of S^3, denoted by Δ_K, and called the canonical triangulation of S^3.*

Notice that this canonical triangulation depends on the given diagram D_K. See an example in Section 8.

The following lemma classifies the edges in the canonical triangulation Δ_K of S^3 and will be essential in the proof of the Main Theorem.

Lemma 5.1. *The edges α in Δ_K such that $p^{-1}(\alpha) \subset \partial B_K$ are of three types:*

a) *If $p^{-1}(\alpha) \subset T$, the trunk of B_K, then $p^{-1}(\alpha)$ consists of precisely one edge.*
b) *If $p^{-1}(\alpha) \not\subset T$ and $Int(p^{-1}(\alpha)) \subset Int(A \cup \bar{A})$, for some wing A, then $p^{-1}(\alpha)$ consists of precisely two edges.*
c) *If $p^{-1}(\alpha) \not\subset T$ and $p^{-1}(\alpha)$ is contained in the boundary of some wings, then $p^{-1}(\alpha)$ consists of precisely four edges.*

Proof. Types a) and b) come directly from the identification p, since the reflections are made along the trunk T, so the points in the trunk are identified with no other point. Also the points in the interior of the wings always are identified in pairs.

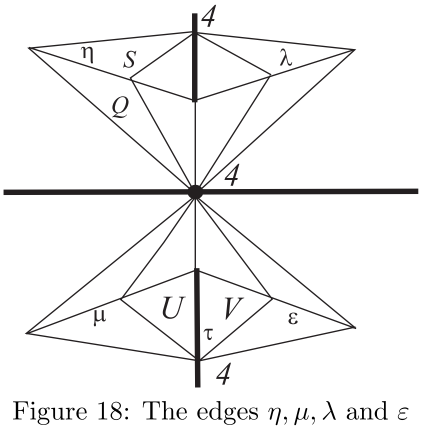

Figure 18: The edges η, μ, λ and ε are identified.

Now the Type c) is true because if $p^{-1}(\alpha) \not\subset T$ and $p^{-1}(\alpha)$ is contained in the boundary of some wings, so if η is an edge in $p^{-1}(\alpha)$ then it is an edge or a part of an edge that joins an A-vertex with a B-vertex. Therefore, we see that under p, η is identified with 3 more edges μ, γ and ε, each one placed in a different regions adjacent to the same crossing point of K. See Figures 12 and 18. ☑

We finish this section proving that the set of odd edges in Δ_K coincides with the knot K.

Proposition 5.6. *If Δ_K is the canonical triangulation of S^3 then $O_{\Delta_K} = K$.*

Proof. We first show that each edge of Δ_K not in K is even. The edges of K form the trunk T of B_K. The other edges α of Δ_K belong to one of three cases:

Case 1 $Int(\alpha) \subset Int(B_K)$. Since Δ_{B_K}, the canonical triangulation of B_K, is 4-colored then α is even. See comment after Theorem 2.4.

Case 2 $p^{-1}(\alpha) \subset \partial B_K$ and $Int(p^{-1}(\alpha)) \subset Int(A \cup \bar{A})$ for some wing A. Then $p^{-1}(\alpha)$ is the union of two edges β and γ, where $\beta \subset A$ and $\gamma \subset \bar{A}$ and they are identified by p. See b) of Lemma 5.1.

The $st(\alpha)$ is decomposed by ∂B_K in $st(\beta)$ and in $st(\gamma)$ in B_K. We want to show that the valences of β and γ have the same parity, i.e., $v(\beta) \equiv v(\gamma) (\mathrm{mod}\, 2)$. Notice that $v(\beta)$ *is even if and only if the two vertices of $(st(\beta) \cap \partial B_K) \backslash \beta$ have the same color.* But then the corresponding vertices $(st(\gamma) \cap \partial B_K) \backslash \gamma$ have the same color, because they are obtained by applying the permutation $\omega(\mu_\delta)$, where $\delta = A \cap \bar{A}$. This shows that in fact $v(\beta) \equiv v(\gamma) (\mathrm{mod}\, 2)$, then $v(\alpha) \equiv 0 (\mathrm{mod}\, 2)$ as we wanted to prove.

Case 3 $p^{-1}(\alpha) \subset \partial B_K$ and $Int(p^{-1}(\alpha)) \not\subset Int(A \cup \bar{A})$. This is the Type c) of Lemma 5.1 therefore $p^{-1}(\alpha)$ is the union of four edges $\eta, \mu, \lambda, \varepsilon$ that are not in T but are in the boundary of wings. Thus $st(\alpha)$ is fragmented into four stars in B_K, i.e., $st(\alpha) = st(\eta) \cup st(\mu) \cup st(\lambda) \cup st(\varepsilon)$. We want to show that $v(\alpha) \equiv 0 (\mathrm{mod}\, 2)$.

This will follow if we prove that $v(\eta) \equiv v(\mu) \equiv v(\lambda) \equiv v(\varepsilon) \equiv 0 (\mathrm{mod}\, 2)$. Consider, for instance, $st(\eta)$. The set $st(\eta) \cap \partial B_K$ consists of two triangles, S and Q (see Fig. 18). Since the vertices of $S - \eta$ and $Q - \eta$ are colored by 4, see Remark 5.1, then $v(\eta)$ is even (compare the italics in Case 2). Thus $O_{\Delta_K} \subseteq K$.

To see that $O_{\Delta_K} \supseteq K$ simply notice that if τ is an edge of T the valence $v(\tau)$ is odd because the two vertexes forming the link $lk(\tau)$ in ∂B_K have different colors. This completes the proof. ☑

6. Proof of the main theorem

Now we are in place of proving The Main Theorem.

Theorem 6.1. ***(Main Theorem)*** *Let (K, ω) be a knot or a link K in S^3 together with a transitive representation $\omega : \pi_1 \left(S^3 \backslash K\right) \to \Sigma_m$, $2 \leq m \leq 3$,* *sending meridians to transpositions (simple representation). Then there exists a triangulation* Δ_K *of* S^3 *such that:*

$$\text{(i)}\ \ O_{\Delta_K} = K \qquad \text{and} \qquad \text{(ii)}\ \ \omega\,(\Delta_K) = \omega\ .$$

Proof. We take the canonical triangulation Δ_K of S^3 associated to the butterfly B_K and to the representation ω, constructed above. By Proposition 5.6 we already know that $O_{\Delta_K} = K$.

We will show that $\omega\,(\Delta_K) = \omega$.

Take then a meridian μ_α of an edge α of K. Then $p^{-1}(\alpha)$ is an edge of the trunk T in ∂B_K. Denote $p^{-1}(\alpha)$ by τ (see Fig. 18), so

$$p^{-1}(st(\alpha)) = st(\tau) \text{ in } B_K.$$

The set $st(\tau) \cap \partial B_K$ consists of two triangles U and V sharing the edge τ. Let u (resp. v) be the vertex in $U - \tau$ (resp. $V - \tau$). Let w be the vertex $lk(U)$ in B_K.

Assume $\omega(\mu_\alpha)$ is the transposition (i, j). Then the vertexes of τ are colored k and 4. Thus, u is colored i or j (say j). Then w is colored i.

Recall that the canonical triangulation Δ_{B_K} is 4-colored. In this coloration σ_0 is colored. We take this as the starting coloration of σ_0 needed to obtain the permutation $\omega(\Delta_K)(\mu_\alpha)$. Call s_i the vertex of σ_0 having the color i.

Since $\omega(\mu_\alpha)(i) = j$ we want to prove that $\omega(\Delta_K)(\mu_\alpha)s_i = s_j$, where s_j is the vertex of σ_0 with color j.

Recall that μ_α consists of a tail t starting in the base tetrahedron σ_0 (inside B_K), going from $b(\sigma_0)$ to the tetrahedron of $st(\tau)$ with base U, going around α through $st(\alpha)$ and coming back to $b(\sigma_0)$ by the reverse of the tail t. Note that the orbit of s_i (resp. s_j) through t under propagation ends in the vertex w of U (resp. u) because Δ_{B_K} is 4-colored. Going around $st(\tau)$, the vertex w of U propagates to the vertex v of V because $v(\tau)$ is odd. When we match U and V by p, the vertex $v \in V$ matches with the vertex $u \in U$ which under propagation through t^{-1} ends in s_j. Therefore $\omega(\Delta_K)(\mu_\alpha)$ and $\omega(\mu_\alpha)$ coincide when applied to the 3 colors i, k, 4 of U. Therefore they coincide in the remaining one. Thus $\omega(\Delta_K)(\mu_\alpha) = \omega(\mu_\alpha)$ for all elements α. This ends the proof. ☑

7. The theorem of Izmestiev and Joswig

In a very interesting paper by I. Izmestiev and M. Joswig ([7] and [6]) they show that a triangulation of a manifold N gives rise, in a natural way, to a branched covering over N. For any triangulation Δ they associated a group $\Pi(\Delta)$, called the group of projectivities of Δ. This group has some similarities with the fundamental group, even though it is not a topological invariant. In fact, the action of it on the set of vertices of a simplex of Δ, permits the construction of branched coverings over N. In this way, they show that any closed orientable 3-manifold M arises as a branched covering over S^3 from some triangulation of S^3.

Their proof uses the theorem that asserts that any closed orientable 3-manifold M is a simple 3-branched covering over S^3 with a knot K as branched set (see [3] and [9]). They start from a tubular neighborhood R of the knot K and give a triangulation for it. Now, using handlebody decomposition, they attach triangulated k-handles to finally find the triangulation Δ of S^3 from which M arises as a branched covering of S^3. But their proof is not constructive. Nevertheless, this result is important because it shows that branched coverings arise automatically once we have a triangulation of the manifold.

The aim of the precedent sections was to obtain this same theorem in a different way, which it turns out to be constructive.

Corollary 7.1. ***(Izmestiev-Joswig)***. *Let M be a closed, orientable 3-manifold. Then there exists a triangulation Δ of S^3, such that M is homeomorphic to $M(\Delta)$; and there exists a natural simplicial 3-fold simple covering $\rho : M(\Delta) \to \Delta$ branched over some knot (dependent on M).*

Proof. In fact, it was shown independently in [3] and in [9] that for any 3-manifold M there is a simple 3-fold covering branched over a knot. In other words, for each 3-manifold there is a knot (K, ω) in S^3 together with a transitive representation $\omega : \pi_1(S^3 \backslash K) \to \Sigma_3$ sending meridians to transpositions. Apply Theorem 6.1 to obtain a triangulation Δ of S^3, such that $\omega(\Delta) = \omega$. Then $M(\Delta)$ is connected (Remark 2.1) and being the 3-fold branched covering of S^3 over K associated to $\omega(\Delta) = \omega$ coincides with M, up to homeomorphism. We finally note that the construction of the branched covering $\rho : M(\Delta) \to S^3$ is simplicial. ☑

8. An example

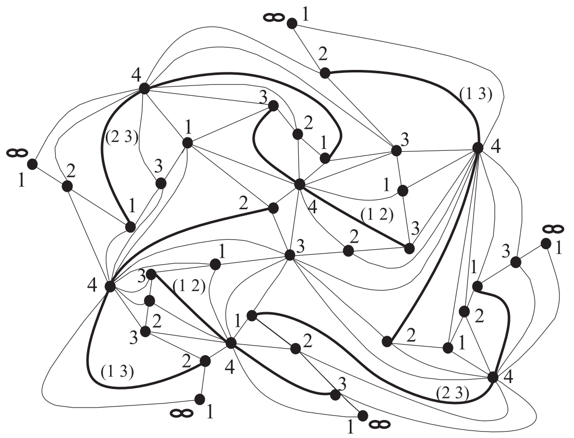

Figure 19: A 4-colored triangulation.

This section is written to substantiate our claim that the theorems in this paper are constructive. We will make a construction for a concrete knot.

Fig. 19 shows a regular diagram for the rational knot $K = 9/2$ (the knot 6_1 in [1]) with a 4-colored triangulation for the boundary ∂B_K of its associated butterfly B_K. For simplicity, we have placed the point at the infinity in several places in the picture, but we note that ∞ represents only one point.

To obtain the triangulation Δ_{B_K} (and with it the triangulation Δ_K of S^3, which involves some antipyramidal subdivisions) one only needs to apply Goodman and Onishi algorithm that can be seen in [2].

Remark 8.1. *By no means is the triangulation Δ_K unique. To start with, the Goodman-Onishi algorithm gives many possible triangulations. Moreover, if we refer to Section 5, one sees that the potential C-vertices to be added to the triangulation will depend on the initial color attributed to $B(R_1)$. And different triangulations might arise according to the choice of color of $B(R_1)$.*

Figures 16a and 16b show the crossing points where we have to subdivide to adjust coloration discrepancies (i.e., where we adjoin C-vertices) when we give different colors to ∞. Each figure corresponds to a different canonical triangulation. For example, Fig. 19 illustrates one of the 4-colorations that we could obtain for the Fig. 20a.

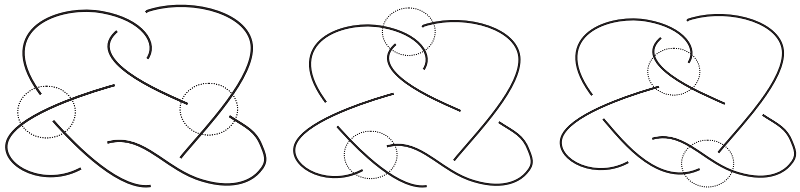

Figure 20a: ∞ *is colored with 1.* *Figure 20b:* ∞ *is colored with 2.* *Figure 20c:* ∞ *is colored with 3.*

Of course, it remains to find the relationship between triangulations of S^3 giving rise to the same 3-manifold.

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