Revista Colombiana de Matemáticas
Volumen 39 (2005), páginas 37–55

Polynomial identities for hyper–matrices

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Abstract. We develop an algorithm to construct algebraic invariants for hyper–matrices. We then construct hyper–determinants and exhibit a generalization of the Cayley–Hamilton theorem for hyper–matrices.

Keywords and phrases. Polynomial identities, hyper–matrices.

2000 Mathematics Subject Classification. Primary: $14M12$. Secondary: $15A24$.

Resumen. Se desarrolla un algoritmo para construir invariantes algebraicos para hiper-matrices. A continuación se construyen hiper-determinantes y se muestra una generalización del teorema de Cayley-Hamilton para hiper-matrices.

1. Introduction

Hyper–matrices appear in several contexts in mathematics [22, 23, 24] and in applications such as in the quantum mechanics of entangled states [1, 5, 7, 13], and image processing [2, 3]. Important mathematical problems associated to hyper–matrices are the construction of algebraic invariants and the determination of the minimal number of algebraically independent invariants. In this work we address these problems.

For ordinary matrices the algebraic invariants associated to a matrix $\mathbf{a}$ can be obtained as traces of powers of the given matrix. According to the Cayley–Hamilton theorem only a finite number of powers of a matrix $\mathbf{a}$ are linearly independent and therefore there is a finite number of algebraically independent invariants. A different set of invariants are the discriminants, which are suitable combinations of traces. The advantage of discriminants is that only the first d ones are non trivial while the rest is identically zero.

Then, the Cayley–Hamilton theorem is easily written in terms of discriminants. A different method is to consider alternating products with a second matrix **b**. Generalized discriminants can then be defined and the Cayley–Hamilton theorem takes a simpler form.

In view of further developments we construct algebraic invariants (discriminants) by considering all possible products among a matrix **a** and a second matrix **b**. These products are in a one–to–one correspondence with semi–magic squares of rank 2 (a semi–magic square is a square array of numbers such that the sum of the elements in each row and each column gives the same result). The discriminants can be obtained in terms of semi–magic squares by a counting procedure which we explain in detail. Furthermore, for practical purposes, the discriminants can be constructed using a graphical algorithm in terms of grids which we develop and explain here.

We then proceed to the construction of a Cayley–Hamilton for hyper–matrices. We restrict our considerations to the fourth–rank case. Ordinary matrix multiplication is not defined for hyper–matrices. Therefore, we use the equivalent formalism based on semi–magis squares. We obtain the corresponding discriminants, the determinant and the Cayley–Hamilton theorem.

2. Matrix calculus

2.1. Index notation. For the purposes of dealing with higher–rank matrices (or hyper–matrices from now on) we introduce an adequate notation. We have found that an index notation similar to that of tensor analysis is more convenient. In this case it is necessary to distinguish between covariant and contravariant indices. According to this scheme, a matrix **a** is a second–rank matrix. We can represent the matrix **a** by a second–rank covariant matrix with components a_{ij}; by a second–rank contravariant matrix with components a^{ij}; or, by a matrix of mixed covariance with components $a^i{}_j$.

For two matrices **a** and **b** of mixed covariance the matrix multiplication is defined by the resulting matrix $\mathbf{c} = \mathbf{a} \cdot \mathbf{b}$ with components $c^i{}_j$ given by

$$c^i{}_j = \sum_{k=1}^{d} a^i{}_k \, b^k{}_j \, . \tag{1}$$

From now on the summation convention over repeated indices is assumed. This means that we simply write

$$c^i{}_j = a^i{}_k \, b^k{}_j \, . \tag{2}$$

The unit element **e** for the matrix multiplication has components e^i_j given by

$$e^i_j = \delta^i_j = \begin{cases} 1, & \text{for } i = j; \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

which is known as the Kronecker delta. The inverse matrix $\mathbf{a}^{-1}$ is a matrix with components ${(a^{-1})^i}_j$ satisfying

$$ {(a^{-1})^i}_k \, {a^k}_j = {a^i}_k \, {(a^{-1})^k}_j = \delta^i_j \, . \tag{4} $$

The product of a matrix $\mathbf{a}$ with itself, $\mathbf{a}^2$, is the matrix with components

$$ {(a^2)^i}_j = {a^i}_k \, {a^k}_j \, . \tag{5} $$

Higher powers of $\mathbf{a}$ are defined in a similar way. Then it is direct to construct discriminants and the Cayley–Hamilton theorem. The similarity transformations are now constructed in terms of a matrix $\mathbf{u}$ of mixed covariance with components ${u^i}_j$ and the inverse matrix $\mathbf{u}^{-1}$ with components ${(u^{-1})^i}_j$. Then, the discriminants are invariant under this similarity transformations.

However, for matrices with a different covariance the scheme above does not work anymore. If $\mathbf{a}$ and $\mathbf{b}$ are covariant matrices with components a_{ij} and b_{ij}, then the Cartesian product is defined by the resulting matrix $\mathbf{c}$ with components

$$ c_{ij} = a_{ik} \, I^{kl} \, b_{lj} \, . \tag{6} $$

where $\mathbf{I}$ is a second–rank contravariant matrix with components

$$ I^{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{7} $$

With these conventions we can reproduce all the standard definitions and results. At this point an observation is necessary. We may define the product with an arbitrary matrix $\mathbf{I}$, not given by (7) and we will obtain formally the same results. The only reason to choose $\mathbf{I}$ as in (7) is that generalizes the Cartesian product. In the next section we consider this general case.

2.2. Alternating products and discriminants. In order to compute the inverse matrix of a matrix $\mathbf{a}$ all what we need is to compute its determinant, $c_d(\mathbf{a}) = \det(\mathbf{a})$. The determinant is the discriminant of order d. There is, however, a second manner for constructing the determinant. Let us consider the Levi–Civita symbol $\epsilon^{i_1 \cdots i_d}$ defined by

$$ \epsilon^{i_1 \cdots i_d} = \begin{cases} 1, & \text{if } i_1 \cdots i_d \text{ is an even permutation of } 1 \cdots d; \\ -1, & \text{if } i_1 \cdots i_d \text{ is an odd permutation of } 1 \cdots d; \\ 0, & \text{otherwise.} \end{cases} \tag{8} $$

Then, the determinant of a matrix $\mathbf{a}$ with components a_{ij} is given by

$$ \det(\mathbf{a}) = \frac{1}{d!} \, \epsilon^{i_1 \cdots i_d} \, \epsilon^{j_1 \cdots j_d} \, a_{i_1 j_1} \, \cdots \, a_{i_d j_d} \, . \tag{9} $$

This is the usual definition of the determinant in matrix calculus. Let us denote the determinant of $\mathbf{a}$ simply by a, that is, $a = \det(\mathbf{a})$. If $a \neq 0$, then we have

$$\mathbf{a}^{-1} = \frac{1}{a} \frac{\partial a}{\partial \mathbf{a}} . \tag{10}$$

In terms of components we have

$$(\mathbf{a}^{-1})^{ij} = \frac{1}{a} \frac{\partial a}{\partial a_{ij}} . \tag{11}$$

Now the notation $(\mathbf{a}^{-1})^{ij}$ is redundant; therefore, we simply write a^{ij} for the components of the inverse matrix. Explicitly we have

$$a^{ij} = \frac{1}{a} \frac{1}{(d-1)!} \epsilon^{ii_1 \cdots i_d} \epsilon^{jj_1 \cdots j_d} a_{i_1 j_1} \cdots a_{i_{(d-1)} j_{(d-1)}} . \tag{12}$$

It is easy to verify that

$$a^{ik} a_{kj} = \delta^i_j . \tag{13}$$

Analogously we can define a covariant Levi–Civita symbol $\epsilon_{i_1 \cdots i_d}$ by

$$\epsilon_{i_1 \cdots i_d} = \begin{cases} 1, & \text{if } i_1 \cdots i_d \text{ is an even permutation of } 1 \cdots d; \\ -1, & \text{if } i_1 \cdots i_d \text{ is an odd permutation of } 1 \cdots d; \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Then, the determinant of a matrix $\mathbf{b}$ with components b^{ij} is given by

$$b = \det(\mathbf{b}) = \frac{1}{d!} \epsilon_{i_1 \cdots i_d} \epsilon_{j_1 \cdots j_d} b^{i_1 j_1} \cdots b^{i_d j_d} . \tag{15}$$

Its inverse matrix $\mathbf{b}^{-1}$ with components b_{ij} is given by

$$b_{ij} = \frac{1}{b} \frac{1}{(d-1)!} \epsilon_{ii_1 \cdots i_d} \epsilon_{jj_1 \cdots j_d} b^{i_1 j_1} \cdots b^{i_{(d-1)} j_{(d-1)}} . \tag{16}$$

For a matrix $\mathbf{b}$ with components b^{ij} we can define the following symbols

$$q_s^{i_1 j_1 \cdots i_s j_s}(\mathbf{b}) = P_j \, b^{i_1 j_1} \cdots b^{i_s j_s} = \frac{1}{s!} b^{|[i_1 j_1} \cdots b^{\, i_s j_s]|} , \tag{17}$$

where P_j denotes the sum over all permutations with respect to the indices j's; $|[\cdots]|$ denotes complete antisymmetry with respect to the indices j's or, equivalently, with respect to the indices i's. Due to the antisymmetry the symbols $\mathbf{q}$ are non trivial only for $s \leq d$. For the first values of s, $\mathbf{q}$ is given by

$$\begin{aligned} q_1^{ij}(\mathbf{b}) &= b^{ij} , \\ q_2^{i_1 j_1 i_2 j_2}(\mathbf{b}) &= \frac{1}{2} (b^{i_1 j_1} b^{i_2 j_2} - b^{i_1 j_2} b^{i_2 j_1}) , \end{aligned}$$

$$\begin{aligned} q_3^{i_1 j_1 i_2 j_2 i_3 j_3}(\mathbf{b}) &= \frac{1}{3!}\,[b^{i_1 j_1}\, b^{i_2 j_2}\, b^{i_3 j_3}(b^{i_1 j_1}\, b^{i_2 j_3}\, b^{i_3 j_2} \\ &\quad + b^{i_1 j_3}\, b^{i_2 j_2}\, b^{i_3 j_1} + b^{i_1 j_2}\, b^{i_2 j_1}\, b^{i_3 j_3}) \\ &\quad + (b^{i_1 j_2}\, b^{i_2 j_3}\, b^{i_3 j_1} + b^{i_1 j_3}\, b^{i_2 j_1}\, b^{i_3 j_2})]\,. \end{aligned} \tag{18}$$

Let us observe that

$$q_d^{i_1 \cdots i_d j_1 \cdots j_d}(\mathbf{b}) = \frac{1}{d!}\, b^{|[i_1 j_1} \,\cdots\, b^{i_d j_d]|} = \det(\mathbf{b})\, \frac{1}{d!}\, \epsilon^{i_1 \cdots i_d}\, \epsilon^{j_1 \cdots j_d}\,. \tag{19}$$

Then, we define the $\mathbf{b}$–discriminants for a matrix $\mathbf{a}$ by

$$c_s^{\mathbf{b}}(\mathbf{a}) = q_s^{i_1 j_1 \cdots i_s j_s}(\mathbf{b})\, a_{i_1 j_1} \,\cdots\, a_{i_s j_s}\,. \tag{20}$$

For the first values of d we have

$$\begin{aligned} c_2^{\mathbf{b}}(\mathbf{a}) &= \frac{1}{2}\left[\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^2 - \left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle\right], \\ c_3^{\mathbf{b}}(\mathbf{a}) &= \frac{1}{3!}\left[\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^3 - 3\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle \left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle + 2\left\langle (\mathbf{b}\cdot\mathbf{a})^3\right\rangle\right], \\ c_4^{\mathbf{b}}(\mathbf{a}) &= \frac{1}{4!}\Big[\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^4 - 6\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^2 \left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle \\ &\quad + 8\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle \left\langle (\mathbf{b}\cdot\mathbf{a})^3\right\rangle \\ &\quad + 3\left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle^2 - 6\left\langle (\mathbf{b}\cdot\mathbf{a})^4\right\rangle\Big], \\ c_5^{\mathbf{b}}(\mathbf{a}) &= \frac{1}{5!}\Big[\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^5 - 10\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^3 \left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle \\ &\quad + 15\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle \left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle^2 \\ &\quad + 20\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle^2 \left\langle (\mathbf{b}\cdot\mathbf{a})^3\right\rangle - 20\left\langle (\mathbf{b}\cdot\mathbf{a})^2\right\rangle \left\langle (\mathbf{b}\cdot\mathbf{a})^3\right\rangle \\ &\quad - 30\left\langle \mathbf{b}\cdot\mathbf{a}\right\rangle \left\langle (\mathbf{b}\cdot\mathbf{a})^4\right\rangle + 24\left\langle (\mathbf{b}\cdot\mathbf{a})^5\right\rangle\Big], \end{aligned} \tag{21}$$

Using (19), we obtain

$$c_d^{\mathbf{b}}(\mathbf{a}) = \det(\mathbf{b})\, \frac{1}{d!}\, \epsilon^{i_1 \cdots i_d}\, \epsilon^{j_1 \cdots j_d}\, a_{i_1 j_1} \,\cdots\, a_{i_d j_d}\,. \tag{22}$$

Therefore

$$c_d^{\mathbf{b}}(\mathbf{a}) = \det(\mathbf{b})\, \det(\mathbf{a})\,. \tag{23}$$

Let us denote $c_d^{\mathbf{b}}(\mathbf{a})$ simply by a_b; then $a_b = a \cdot b$. If $a \neq 0$ and $b \neq 0$, we obtain

$$\frac{1}{a_b}\,\frac{\partial a_b}{\partial \mathbf{a}} = \frac{1}{a}\,\frac{\partial a}{\partial \mathbf{a}}\,. \tag{24}$$

Therefore

$$\mathbf{a}^{-1} = \frac{1}{a}\,\frac{\partial a}{\partial \mathbf{a}} = \frac{1}{a_b}\,\frac{\partial a_b}{\partial \mathbf{a}}\,. \tag{25}$$

This is the new expression of the Cayley–Hamilton theorem. In fact, for the first values of d we have

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a} - c_1^{\mathbf{b}}(\mathbf{a})\,\mathbf{a} + c_2^{\mathbf{b}}(\mathbf{a})\,\mathbf{b}^{-1} &\equiv 0\,,\\
\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a} - c_1^{\mathbf{b}}(\mathbf{a})\,\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a} + c_2^{\mathbf{b}}(\mathbf{a})\,\mathbf{a} - c_3^{\mathbf{b}}(\mathbf{a})\,\mathbf{b}^{-1} &\equiv 0\,,\\
\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a} - c_1^{\mathbf{b}}(\mathbf{a})\,\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a}&\\
+c_2^{\mathbf{b}}(\mathbf{a})\,\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{a} - c_3^{\mathbf{b}}(\mathbf{a})\,\mathbf{a} + c_4^{\mathbf{b}}(\mathbf{a})\,\mathbf{b}^{-1} &\equiv 0\,.
\end{aligned} \tag{26}$$

There are two particularly interesting instances of these relations. The first case is $\mathbf{b} = \mathbf{I}$. In the second case we would like to have an expression concomitant of the matrix $\mathbf{a}$ alone. To this purpose we choose $\mathbf{b} = \mathbf{a}^{-1}$. In that case all collapses to a useless identity. However, in our generalization to fourth–rank matrices the first case is excluded while the second one is allowed and gives the fourth–rank version of the Cayley–Hamilton theorem.

3. Semi–magic squares and graphical construction of invariants

For hyper–matrices the expressions (21) are nor adequate. Therefore we need a new algorithm for hyper–matrices. We have developed an algorithm based on the use of semi–magic squares which allow to characterize all invariants and furthermore we have developed a graphical algorithm to construct the semi–magic squares. We now introduce both these algorithms and exemplify them with the second–rank case.

3.1. Semi–magic squares. In order to construct algebraic invariants we consider products of $\mathbf{a}$'s, with components a_{ij} and $\mathbf{b}$'s, with components b^{ij}. In order for the result to be an invariant all indices must be contracted. This means that we must consider an equal number n of $\mathbf{a}$'s and $\mathbf{b}$'s. Since both $\mathbf{a}$ and $\mathbf{b}$ have two indices each, $\mathbf{a}$ can be contracted at most with 2 indices belonging to $\mathbf{b}$'s and the same is true for $\mathbf{b}$'s. The way in which the n $\mathbf{a}$'s are contracted with the n $\mathbf{b}$'s can be represented by an $n \times n$ square array of numbers $\mathbf{s}$, where the components s_{IJ} denote the number of contractions between the Ith $\mathbf{a}$ and the Jth $\mathbf{b}$. Monomial algebraic invariants are characterized by the way in which the indices of $\mathbf{a}$ are contracted with the indices of $\mathbf{b}$. This contraction scheme can be represented by a semi–magic square. Graphically

$$\mathbf{s} = \begin{array}{c} \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{array} \\ \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} \end{array} . \tag{27}$$

If the Ith $\mathbf{a}$ is contracted once with the Jth $\mathbf{b}$, then $s_{IJ} = 1$; if the Ith $\mathbf{a}$ is contracted twice with the Jth $\mathbf{b}$, then $s_{IJ} = 2$; if there is no contraction between the Ith $\mathbf{a}$ and the Jth $\mathbf{b}$, then $s_{IJ} = 0$. Since all indices must be contracted the sum of the elements of each row and each column must be equal to 2, that is,

$$\begin{aligned} \sum_{I=1}^{n} s_{IJ} &= 2\,, \\ \sum_{J=1}^{n} s_{IJ} &= 2\,. \end{aligned} \tag{28}$$

Arrays with this property are known as semi–magic squares [17]. Therefore, the number of possible algebraic invariants is determined by the number $H_n(2)$ of semi–magic squares $\mathbf{s}$. Semi–magic squares of rank r are defined by the relations

$$\begin{aligned} \sum_{I=1}^{n} s_{IJ} &= r\,, \\ \sum_{J=1}^{n} s_{IJ} &= r\,. \end{aligned} \tag{29}$$

The number of different possible semi–magic squares $H_n(r)$ is given by [14, 16, 17]

$$\begin{aligned} H_1(r) &= 1\,, \\ H_2(r) &= r+1\,, \\ H_3(r) &= 6 + 15\binom{r-1}{1} + 19\binom{r-1}{2} + 12\binom{r-1}{3} + 3\binom{r-1}{4} \\ &= \frac{1}{8}\left(r^4 + 6\,r^3 + 15\,r^2 + 18\,r + 8\right) \\ &= \frac{1}{2}\,R\,(R+1)\,, \\ H_4(r) &= 24 + 258\binom{r-1}{1} + 1468\binom{r-1}{2} + 4945\binom{r-1}{3} \\ &\quad +10532\binom{r-1}{4} + 14620\binom{r-1}{5} + 13232\binom{r-1}{6} \\ &\quad +7544\binom{r-1}{7} + 2464\binom{r-1}{8} + 352\binom{r-1}{9}\,. \end{aligned} \tag{30}$$

where $R = (r+1)(r+2)/2$.

For $r = 2$ the result is: $H_n(2) = \{1, 3, 21, 282, \cdots\}$.

For $n = 1$ and $n = 2$ the corresponding semi–magic squares are given by

$$s_{2,1} = \{(2)\}\,,$$

$$s_{2,2} = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}. \qquad (31)$$

Since each column and each row represents the same matrix, semi–magic squares which are related by the permutation of rows and/or columns represent the same algebraic invariant. For example, for $n = 2$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ are the same invariant. Therefore, semi–magic squares can be classifed into equivalence classes related by permutations of rows and/or columns. Then, we need to take care only of the representatives $s_{r,n} = \{s_i, i = 1, \cdots, p_r(n)\}$ for each equivalence class, where $p_r(n)$ is the number of equivalence classes for rank r and order n. For $r = 2$ the number of equivalence classes $p_2(n)$ is given by the number of integer partitions of n, that is, $p(n)$. Therefore, $p_2(n) = \{1, 2, 3, 5, \cdots\}$. For the first values of n the representatives of each equivalence class are

$$\begin{aligned}
s_{2,1} &= \{(2)\}, \\
s_{2,2} &= \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \\
s_{2,3} &= \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}, \\
s_{2,4} &= \left\{ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right\}. \qquad (32)
\end{aligned}$$

The algebraic invariants to which each semi–magic square corresponds are given by

$$\begin{aligned}
(2) &= \langle \mathbf{b}\,\mathbf{a} \rangle, \\
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \langle \mathbf{b}\,\mathbf{a} \rangle^2, \\
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} &= \langle (\mathbf{b}\,\mathbf{a})^2 \rangle,
\end{aligned}$$

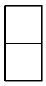
$$
\begin{aligned}
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} &= \big\langle \mathbf{b}\,\mathbf{a} \big\rangle^3 , \\
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} &= \big\langle \mathbf{b}\,\mathbf{a} \big\rangle \big\langle (\mathbf{b}\,\mathbf{a})^2 \big\rangle , \\
\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} &= \big\langle (\mathbf{b}\,\mathbf{a})^3 \big\rangle , \\
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} &= \big\langle \mathbf{b}\,\mathbf{a} \big\rangle^4 , \\
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} &= \big\langle \mathbf{b}\,\mathbf{a} \big\rangle^2 \big\langle (\mathbf{b}\,\mathbf{a})^2 \big\rangle , \\
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} &= \big\langle \mathbf{b}\,\mathbf{a} \big\rangle \big\langle (\mathbf{b}\,\mathbf{a})^3 \big\rangle , \\
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} &= \big\langle (\mathbf{b}\,\mathbf{a})^2 \big\rangle^2 , \\
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} &= \big\langle (\mathbf{b}\,\mathbf{a})^4 \big\rangle . \qquad (33)
\end{aligned}
$$

Let us observe that block semi–magic squares can be decomposed in terms of lower order semi–magic squares as

$$
\begin{aligned}
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= (2)^2 , \\
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} &= (2)^3 ,
\end{aligned}
$$

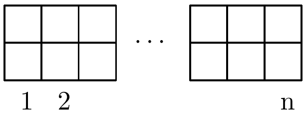
$$
\begin{aligned}
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} &= (2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \\
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} &= (2)^4 , \\
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} &= (2)^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \\
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} &= (2) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} , \\
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 .
\end{aligned}
\tag{34}
$$

3.2. Graphical construction of invariants. Each semi–magic square determines an algebraic invariant. On the other hand, semi–magic squares are obtained by considering all possible permutations of indices. However, for large values of n this algorithm becomes unpractical. In order to avoid this difficulty we now develop a graphical algorithm for the construction and characterization of algebraic invariants which allows to simplify this task. Let us represent the matrix $\mathbf{a}$ by a vertical grid with two boxes, namely



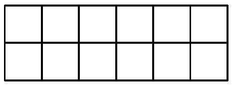
(35)

The product of n matrices is represented by



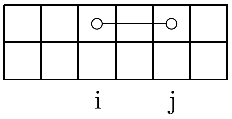
(36)

Each algebraic invariant is characterized by the way in which indices are contracted. We can always choose to keep fix the indices of the first row and look at how the indices in the second row are contracted with the indices in the first row. A void grid indicates that no permutation has been performed



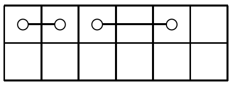
(37)

A permutation of the indices ith and jth is indicated by



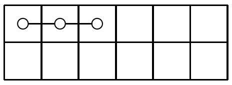
(38)

A double permutation is indicated by



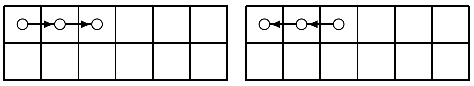
(39)

A cyclic permutation is indicated by



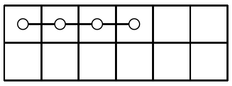
(40)

In this case, however, it is necessary to take into account the sense in which the permutation is performed. There are 2 possibilities



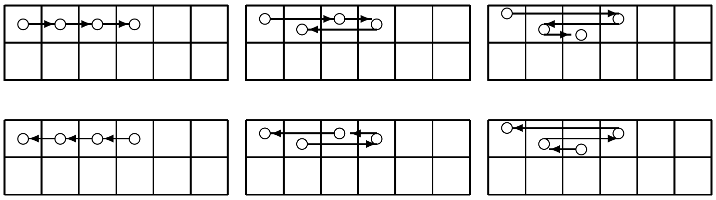
(41)

When the sense of the permutation is irrelevant we use the right–oriented grid. The next possibility is



(42)

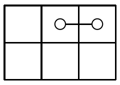
In this case there are 6 possibles senses for the permutation, namely,



(43)

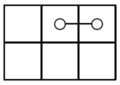
As in the previous case, when the sense of the permutation is irrelevant we use only the right–oriented grid.

The semi–magic square corresponding to a given grid is obtained as follows. The number of empty boxes in each column corresponds to the diagonal entries in the semi–magic square. The lines correspond to the off–diagonal terms. For example



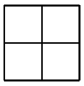
$$\sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} . \qquad (44)$$

The multiplicity is the number of possible ways in which the given permutation can be performed over the n boxes (indices). The parity is given by the number of lines for the permutation. For example

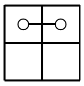


$$= -3 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} . \tag{45}$$

Let us perform the explicit construction of the semi–magic squares and discriminants for the first values of n. For $n = 2$ the result is

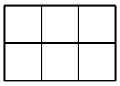


$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} ,$$

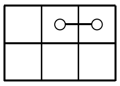


$$= - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{46}$$

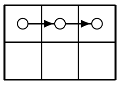
For $n = 3$ the result is



$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} ,$$

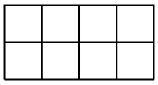


$$= -3 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} , \tag{47}$$

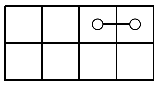


$$= 2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} .$$

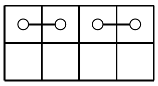
For $n = 4$ the result is



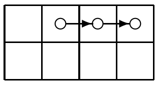
$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} ,$$



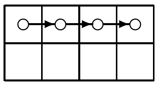
$$= -6 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} ,$$



$$= 3 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} ,$$



$$= 8 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} ,$$



$$ = \quad -6 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} . \tag{48}$$

The number of semi–magic squares depends on the number of possible permutations. For large values of n and r this counting becomes quite involved. Therefore, it is advisable to have an easy recipe to obtain the correct counting.

For each order we have a different number of possible permutations P_n. We can represent them as

$$\begin{aligned}
P_2 &= \mathbf{0} - \mathbf{1}\,, \\
P_3 &= \mathbf{0} - 3\cdot\mathbf{1} + 2\cdot\mathbf{2}\,, \\
P_4 &= \mathbf{0} - 6\cdot\mathbf{1} + 3\cdot\mathbf{1}^2 + 8\cdot\mathbf{2} - 6\cdot\mathbf{3}\,, \\
P_5 &= \mathbf{0} - 10\cdot\mathbf{1} + 15\cdot\mathbf{1}^2 + 20\cdot\mathbf{2} - 20\cdot(\mathbf{2}\,\mathbf{1}) - 30\cdot\mathbf{3} + 24\cdot\mathbf{4}\,, \\
P_6 &= \mathbf{0} - 15\cdot\mathbf{1} + 45\cdot\mathbf{1}^2 - 15\cdot\mathbf{1}^3 + 40\cdot\mathbf{2} - 120\cdot(\mathbf{2}\,\mathbf{1}) \\
&\quad +40\cdot\mathbf{2}^2 - 90\cdot\mathbf{3} + 90\cdot(\mathbf{3}\,\mathbf{1}) + 144\cdot\mathbf{4} - 120\cdot\mathbf{5}\,.
\end{aligned} \tag{49}$$

The number of terms involving some given permutations is given by the coefficients in (49). This can be easily verified in the graphical construction above.

4. Hyper–matrices. The fourth–rank case

There is not a natural multiplication operation for hyper–matrices in the sense that the product of two hyper–matrices be again a hyper–matrix of the same rank. Therefore, the construction of algebraic invariants must be performed using the semi–magic square technique developed above. This algorithm can be easily extended to hyper–matrices of any arbitrary even–rank r.

Our construction is based on alternating products. To this purpose let us consider a fourth–rank matrix $\mathbf{A}$ with components A_{ijkl} and a fourth–rank matrix $\mathbf{B}$ with components B^{ijkl}. The discriminants are represented by semi–magic squares of rank 4. They are constructed using the graphical algorithm of the section above.

4.1. Alternating products and discriminants. In analogy with (9), for a fourth–rank matrix $\mathbf{A}$ with componentes A_{ijkl} we can define

$$\det(\mathbf{A}) = \frac{1}{d!}\,\epsilon^{i_1\cdots i_d}\,\cdots\,\epsilon^{l_1\cdots l_d}\,A_{i_1 j_1 k_1 l_1}\,\cdots\,A_{i_d j_d k_d l_d}\,. \tag{50}$$

Let us denote $A = \det(\mathbf{A})$. In analogy with (10) we define

$$\mathbf{A}^{-1} = \frac{1}{A}\,\frac{\partial A}{\partial \mathbf{A}}\,. \tag{51}$$

In term of components

$$A^{ijkl} = \frac{1}{A} \frac{\partial A}{\partial A_{ijkl}} \,. \tag{52}$$

Then

$$\begin{aligned} A^{ijkl} \quad &= \quad \frac{1}{(d-1)!} \frac{1}{A} \epsilon^{ii_1 \cdots i_{(d-1)}} \cdots \epsilon^{ll_1 \cdots l_{(d-1)}} \\ &\quad \times A_{i_1 j_1 k_1 l_1} \cdots A_{i_{(d-1)} j_{(d-1)} k_{(d-1)} l_{(d-1)}} \,. \end{aligned} \tag{53}$$

This hyper–matrix satisfies

$$A^{ik_1 k_2 k_3} A_{jk_1 k_2 k_3} = \delta^i_j \,. \tag{54}$$

The definitions (50) and (52–54) were used in previous works [18, 19, 20, 21] concerning the applications of fourth–rank geometry to the formulation of an alternative theory for the gravitational field.

As an example of the relation above let us consider the simple case $d = 2$. The determinant (50) is then given by

$$A = A_{1111} A_{2222} - 4 A_{1112} A_{1222} + 3 A^2_{1122} \,. \tag{55}$$

The components of the hyper–matrix A^{ijkl} are given by

$$\begin{aligned} A^{1111} \quad &= \quad \frac{1}{A} A_{2222} \,, \\ A^{1112} \quad &= \quad -\frac{1}{A} A_{1222} \,, \\ A^{1122} \quad &= \quad \frac{1}{A} A_{1122} \,. \end{aligned} \tag{56}$$

and similar expressions for the other components. In order to check the validity of eq. (56) let us consider the cases 11 and 12. We can then verify that

$$\begin{aligned} A^{1ijk} A_{1ijk} \quad &= \quad 1 \,, \\ A^{1ijk} A_{2ijk} \quad &= \quad 0 \,, \end{aligned} \tag{57}$$

and similar relations for the other indices.

The determinant for a fourth–rank matrix $\mathbf{B}$ with components B^{ijkl} is given by

$$\det(\mathbf{B}) = \frac{1}{d!} \epsilon_{i_1 \cdots i_d} \cdots \epsilon_{l_1 \cdots l_d} B^{i_1 j_1 k_1 l_1} \cdots B^{i_d j_d k_d l_d} \,. \tag{58}$$

Let us now consider a fourth–rank matrix $\mathbf{B}$ with components B^{ijkl}. In a way similar to (17) we define

$$Q_s^{i_1 j_1 k_1 l_1 \cdots i_s j_s k_s l_s}(\mathbf{B}) = \frac{1}{s!} B^{|[i_1 j_1 k_1 l_1} \cdots B^{i_s j_s k_s l_s]|} \,. \tag{59}$$

For the first values of s we obtain

$$\begin{aligned} Q_1^{ijkl}(\mathbf{B}) &= B^{ijkl}\,, \\ Q_2^{i_1j_1k_1l_1i_2j_2k_2l_2}(\mathbf{B}) &= \frac{1}{2}\,[B^{i_1j_1k_1l_1}\,B^{i_2j_2k_2l_2} - (B^{i_1j_1k_1l_2}\,B^{i_2j_2k_2l_1} \\ &\quad +B^{i_1j_1k_2l_1}\,B^{i_2j_2k_1l_2} + B^{i_1j_2k_1l_1}\,B^{i_2j_1k_2l_2} \\ &\quad +B^{i_2j_1k_1l_1}\,B^{i_1j_2k_2l_2}) + (B^{i_1j_1k_2l_2}\,B^{i_2j_2k_1l_1} \\ &\quad +B^{i_1j_2k_1l_2}\,B^{i_2j_1k_2l_1} + B^{i_2j_1k_1l_2}\,B^{i_1j_2k_2l_1})]\,, \end{aligned} \tag{60}$$

etc. Let us observe that

$$\begin{aligned} Q_d^{i_1j_1k_1l_1\cdots i_dj_dk_dl_d}(\mathbf{B}) &= \frac{1}{d!}\,B^{|[i_1j_1k_1l_1}\,\cdots\,B^{i_dj_dk_dl_d]|} \\ &= \det(\mathbf{B})\,\frac{1}{d!}\,\epsilon^{i_1\cdots i_d}\,\cdots\,\epsilon^{l_1\cdots l_d}\,. \end{aligned} \tag{61}$$

Then, the **B**–discriminants of **A** are defined by

$$C_s^{\mathbf{B}}(\mathbf{A}) = Q_s^{i_1j_1k_1l_1\cdots i_sj_sk_sl_s}(\mathbf{B})\,A_{i_1j_1k_1l_1}\,\cdots\,A_{i_sj_sk_sl_s}\,. \tag{62}$$

For $s = 2$ we obtain

$$\begin{aligned} C_2^{\mathbf{B}}(\mathbf{A}) &= \frac{1}{2}\Big[(B^{ijkl}\,A_{ijkl})^2 - 4\,B^{ijkl}\,A_{jklm}\,B^{mnpq}\,A_{npqi} \\ &\quad +3\,B^{ijkl}\,A_{klmn}\,B^{mnpq}\,A_{pqij}\Big]\,. \end{aligned} \tag{63}$$

For $s > 3$ the corresponding expressions are too long to be exhibited here. Instead we will determine the **B**–discriminants with semi–magic squares.

In analogy with (23) we obtain

$$C_d^{\mathbf{B}}(\mathbf{A} = \det(\mathbf{B})\,\det(\mathbf{A})\,. \tag{64}$$

Let us denote $C_d^{\mathbf{B}}(\mathbf{A})$ simply as A_B; then $A_B = A \cdot B$. If $A \neq 0$ and $B \neq 0$, we obtain

$$\frac{1}{A_B}\,\frac{\partial A_B}{\partial \mathbf{A}} = \frac{1}{A}\,\frac{\partial A}{\partial \mathbf{A}}\,. \tag{65}$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{A}\,\frac{\partial A}{\partial \mathbf{A}} = \frac{1}{A_B}\,\frac{\partial A_B}{\partial \mathbf{A}}\,. \tag{66}$$

This is the statement of the Cayley–Hamilton theorem for hyper–matrices. For $s = 2$ we have

$$\begin{aligned} &(B^{ijkl}\,A_{ijkl})\,A_{abcd} - 4\,A_{(a|ijk}\,B^{ijkl}\,A_{l|bcd)} \\ &+3\,A_{(ab|ij}\,B^{ijkl}\,A_{kl|cd)} - C_2^{\mathbf{B}}(\mathbf{A})\,(B^{-1})_{abcd} \equiv 0\,. \end{aligned} \tag{67}$$

If we now choose $\mathbf{B} = \mathbf{A}^{-1}$, that is $\mathbf{B}^{-1} = \mathbf{A}$, the above expression reduces to

$$A_{(ab|ij}\, A^{ijkl}\, A_{kl|cd)} - \frac{1}{2}\, (A_{(mn|ij}\, A^{ijkl}\, A_{kl|pq)}\, A^{mnpq})\, A_{abcd} \equiv 0\,. \tag{68}$$

4.2. Semi–magic squares. The algebraic invariants which can be constructed in this case are given by the semi–magic squares of rank 4. Their number is $H_n(4) = \{1, 5, 120, 7558, \cdots\}$. As for the second–rank case we must take care only of the representatives for each equivalence class. The number of equivalence classes $p_4(n)$ is given by the generating function

$$\sum_{n=0}^{\infty} p_4(n)\, x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{n!}}\,, \tag{69}$$

For the first values of n $p_4(n)$ is given by

$$p_4(n) = \{1,\, 1,\, 3,\, 9,\, 36,\, \cdots\}\,. \tag{70}$$

4.3. Construction of discriminants. Each semi–magic square determines an algebraic invariant. However, in this case, the corresponding invariant can no longer be represented by mean of traces, as was done for ordinary matrices. The semi–magic squares of order 1 and 2 correspond to the following invariants

$$\begin{aligned}
(4) &= B^{ijkl}\, A_{ijkl}\,, \\
\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} &= \left(B^{ijkl}\, A_{ijkl}\right)^2\,, \\
\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} &= B^{i_1j_1k_1l_1}\, A_{l_1i_2j_2k_2}\, B^{i_2j_2k_2l_2}\, A_{l_2i_1j_1k_1}\,, \\
\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} &= B^{i_1j_1k_1l_1}\, A_{k_1l_1i_2j_2}\, B^{i_2j_2k_2l_2}\, A_{k_2l_2i_1j_1}\,.
\end{aligned} \tag{71}$$

It is obvious from the expressions above that semi–magic squares are more practical for representing algebraic invariants.

The corresponding discriminants are linear combinations of the monomial algebraic invariants (semi–magic squares) of the same order. In order to determine the coefficients of this linear combination we proceed in a way similar to that for ordinary matrices. The hyper–matrices can be contracted according to the allowed number of possible permutations. The possible permutations are the same as described in Section 3.2. However, this time there are three additions involved. It is obvious that the number of terms which must be computed growth very fast, as $(n!)^{r-1}$. Therefore, even when this algorithm provides a direct answer, a more practical way to evaluate the coefficients is necessary. Then we must consider the graphical algorithm developed in Section 3.2.

4.4. The Cayley–Hamilton theorem. Let us write the Cayley–Hamilton theorem in terms of almost–magic rectangles. For the first values of d we obtain

$$\begin{pmatrix} 4 & \mathbf{0} \\ 0 & \mathbf{4} \end{pmatrix} - 4 \begin{pmatrix} 3 & \mathbf{1} \\ 1 & \mathbf{3} \end{pmatrix} + 3 \begin{pmatrix} 2 & \mathbf{2} \\ 2 & \mathbf{2} \end{pmatrix} - C_2^{\mathbf{B}}(\mathbf{A})\,\mathbf{B}^{-1} \quad \equiv \quad 0\,,$$

$$\begin{aligned} &\frac{1}{2}\left[\begin{pmatrix} 4 & 0 & \mathbf{0} \\ 0 & 4 & \mathbf{0} \\ 0 & 0 & \mathbf{4} \end{pmatrix} - 4 \begin{pmatrix} 3 & 1 & \mathbf{0} \\ 1 & 3 & \mathbf{0} \\ 0 & 0 & \mathbf{4} \end{pmatrix} + 3 \begin{pmatrix} 2 & 2 & \mathbf{0} \\ 2 & 2 & \mathbf{0} \\ 0 & 0 & \mathbf{4} \end{pmatrix}\right] \\ &- 4 \begin{pmatrix} 4 & 0 & \mathbf{0} \\ 0 & 3 & \mathbf{1} \\ 0 & 1 & \mathbf{3} \end{pmatrix} + 3 \begin{pmatrix} 4 & 0 & \mathbf{0} \\ 0 & 2 & \mathbf{2} \\ 0 & 2 & \mathbf{2} \end{pmatrix} + 4 \begin{pmatrix} 3 & 1 & \mathbf{0} \\ 0 & 3 & \mathbf{1} \\ 1 & 0 & \mathbf{3} \end{pmatrix} \\ &+ 6 \begin{pmatrix} 3 & 0 & \mathbf{1} \\ 0 & 3 & \mathbf{1} \\ 1 & 1 & \mathbf{2} \end{pmatrix} + 12 \begin{pmatrix} 3 & 1 & \mathbf{0} \\ 1 & 2 & \mathbf{1} \\ 0 & 1 & \mathbf{3} \end{pmatrix} - 12 \begin{pmatrix} 3 & 1 & \mathbf{0} \\ 1 & 1 & \mathbf{2} \\ 0 & 2 & \mathbf{2} \end{pmatrix} \\ &-12 \begin{pmatrix} 3 & 0 & \mathbf{1} \\ 1 & 2 & \mathbf{1} \\ 0 & 2 & \mathbf{2} \end{pmatrix} + 3 \begin{pmatrix} 2 & 2 & \mathbf{0} \\ 2 & 0 & \mathbf{2} \\ 0 & 2 & \mathbf{2} \end{pmatrix} - 12 \begin{pmatrix} 2 & 2 & \mathbf{0} \\ 2 & 1 & \mathbf{1} \\ 0 & 1 & \mathbf{3} \end{pmatrix} \\ &+ 6 \begin{pmatrix} 2 & 2 & \mathbf{0} \\ 1 & 1 & \mathbf{2} \\ 1 & 1 & \mathbf{2} \end{pmatrix} + 12 \begin{pmatrix} 2 & 1 & \mathbf{1} \\ 2 & 1 & \mathbf{1} \\ 0 & 2 & \mathbf{2} \end{pmatrix} - 6 \begin{pmatrix} 2 & 1 & \mathbf{1} \\ 1 & 2 & \mathbf{1} \\ 1 & 1 & \mathbf{2} \end{pmatrix} \\ &\qquad\qquad - C_3^{\mathbf{B}}(\mathbf{A})\,\mathbf{B}^{-1} \quad \equiv \quad 0\,. \end{aligned} \tag{72}$$

The above is the Cayley–Hamilton theorem for fourth–rank matrices.

5. Concluding remarks

We have developed an algorithm to construct algebraic invariants for hyper–matrices. We constructed hyper–determinants and exhibit an extension of the Cayley–Hamilton theorem to hyper–matrices.

These algebraic invariants were considered by Cayley [6]; see [8, 9] for an updated account.

Higher–rank tensors look similar to hyper–matrices and the results presented here are a first step for the construction of algebraic invariants for higher–rank tensors. Higher–rank tensors appear in several contexts such as in Finsler geometry [4, 15], fourth–rank gravity [18, 19, 20, 21], dual models for higher spin gauge fields [10, 11, 12].

Acknowledgements. This work was partially done at the *Abdus Salam* International Centre for Theoretical Physics, Trieste. This work was partially supported by Dirección de Investigación – Bogotá, Universidad Nacional de Colombia.

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(Recibido en mayo de 2005. Aceptado en agosto de 2005)

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