Revista Colombiana de Matemáticas
Volumen 38 (2004), páginas 73–85

Rigidity of minimal hypersurfaces of spheres with constant ricci curvature

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Abstract. Let M be a compact oriented minimal hypersurface of the unit n-dimensional sphere S^n. In this paper we will point out that if the Ricci curvature of M is constant, then, we have that either $\mathrm{Ric} \equiv 1$ and M is isometric to an equator or, n is odd, $\mathrm{Ric} \equiv \frac{n-3}{n-2}$ and M is isometric to $S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2}) \times S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2})$. Next, we will prove that there exists a positive number $\epsilon(n)$ such that if the Ricci curvature of a minimal hypersurface immersed by first eigenfunctions M satisfies that $\frac{n-3}{n-2} - \epsilon(n) \leq \mathrm{Ric} \leq \frac{n-3}{n-2} + \epsilon(n)$ and the average of the scalar curvature is $\frac{n-3}{n-2}$, then, the ricci curvature of M must be constant and therefore M must be isometric to $S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2}) \times S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2})$.

Keywords and phrases. Minimal hypersurfaces, spheres, shape operator, Clifford tori.

2000 Mathematics Subject Classification. Primary: 53C42. Secondary: 53A10.

1. Introduction

Let $\phi : M \longrightarrow S^n$ be a minimal immersion of a compact oriented $(n-1)$-dimensional manifold into the unit sphere. We will identify M with the set $\phi(M) \subset \mathbf{R}^{n+1}$, and the space T_mM with the linear subspace $d\phi_m(T_mM)$ of $\mathbf{R}^{n+1}$. The easiest examples of these immersions are the equators, i.e. the totally geodesic S^{n-1}'s in S^n, and the Clifford hypersurfaces, M_{kl}, which are product of spheres, namely: For every pair of positive integers k and l with

$k+l=n-1$,

$$M_{kl}=\{(x,y)\in\mathbf{R}^{k+1}\times\mathbf{R}^{l+1}:|x|^2=\frac{k}{n-1}\ \text{ and }\ |y|^2=\frac{l}{n-1}\}$$

Let ν be a unit normal vector field along M. Notice that $\nu:M\longrightarrow S^n$ satisfies that $\langle\nu(m),m\rangle=0$. For any tangent vector $v\in T_mM$, $m\in M$, the shape operator A is given by $A(v)=-\bar{\nabla}_v\nu$, where $\bar{\nabla}$ denotes the Levi Civita connection in S^n. The shape operator at each $m\in M$ defines a symmetric linear transformation from T_mM to T_mM, the eigenvalues of this linear transformation, $\kappa_1(m),\dots\kappa_{n-1}(m)$, are known as the principal curvatures of M at m. Let us fix some notation: we will denote by Δ the laplacian on M; $\|A\|^2=\sum_{i=1}^{n-1}\kappa_i^2$ will denote the square of the norm of the shape operator, notice that since the second fundamental form II is given by $II(v)=\langle A(v),v\rangle$, then $\|A\|^2=\|II\|^2$; given two linearly independent vectors $v,w\in T_mM$, $k(v,w)$ will represent the sectional curvature of the plane spanned by v and w; for any unit vector $v\in T_mM$, the Ricci curvature is defined by

$$\mathrm{Ric}(v)=\frac{1}{n-2}\sum_{i=1}^{n-2}k(v,v_i)$$

where $\{v,v_1,\dots,v_{n-2}\}$ is an orthonormal basis of T_mM; the scalar curvature is defined by

$$R=\frac{1}{n-1}\sum_{i=1}^{n-1}\mathrm{Ric}(v_i)$$

where $\{v_1,\dots,v_{n-1}\}$ is any orthonormal basis of T_mM. It is known that if $\|A\|^2=n-1$ for all $m\in M$, then M is isometric to a minimal Clifford hypersurface ([3], [7]). We also have that if M is neither an equator nor a Clifford hypersurface, then $\|A\|^2(m)>(n-1)$ for some $m\in M$ [11]. In this paper we will pose the following conjecture:

Conjeture 1.1. *If M is a non-equatorial closed minimal embedded hypersurface in S^n, then $\int_M\|A\|^2\geq\int_M(n-1)$ with equality only if M is a minimal Clifford hypersurface.*

For $n=3$, i.e. for surfaces, we have, by the Gauss-Bonnet theorem and the minimality of M, that

$$\int_M\|A\|^2=\int_M 2+8\pi(g-1)$$

where g is the genus of the surface [10], therefore $\int_M\|A\|^2\leq\int_M 2$ only when M is a sphere or M is a torus; we also have that, if M is a sphere immersed in S^3, then M is an equator [1], therefore, in this case, the conjecture 1.1 is equivalent to the Lawson conjecture: *the only embedded minimal torus in S^3 is the Clifford torus.*

Remark 1.1. *If true, Conjecture 1.1 provides a new proof of Simons' key inequality* $\lambda_1 < -2(n-1)$ *[11] (here* λ_1 *is the first eigenvalue of the stability operator). For any estimate of the form* $\int_M \|A\|^2 \geq \int_M \alpha$ *, (*α *constant) immediately bounds* λ_1 *from above:*

$$\lambda_1 \leq \frac{\int_M J(f)f}{\int_M f^2} = \frac{-\int_M \|A\|^2 - \int_M (n-1)}{\int_M 1} \leq -(n-1+\alpha).$$

Simons' inequality is the crucial step in ruling out any stable minimal hypercones in R^n *other than hyperplanes, when* $n < 8$*. This result in turn has powerful consequences such as the Bernstein theorem in dimensions* $n < 9$ *and the codimension 7 regularity result ultimately proved by Federer [4]. Moreover, suppose one could strengthen Conjecture 1.1 to the effect that if* M *is neither Clifford nor equatorial, then*

$$\int_M \|A\|^2 > \int_M (n-1+1/4).$$

Combining this with Simons' paper and the theorem of Simon & Solomon in [12], one would then obtain a complete classification of area-minimizing hypersurfaces in R^8*, a major advance.*

In this paper we will prove that if the first eigenvalue of the laplacian of M is $n-1$ and the Ricci and scalar curvature satisfy the inequality

$$R \leq \frac{n-3}{n-1} + \frac{2\mathrm{Ric}(v)}{n-1} \quad \text{for any } v \in TM \text{ with } |v| = 1 \qquad (\star)$$

then $\int_M \|A\|^2 \geq \int_M (n-1)$ with equality only if M is Clifford. In particular, for embedded hypersurfaces that satisfy the inequality $(\star)$, we have that Yau's conjecture, *"the first eigenvalue of the laplacian of an embedded hypersurface in* S^n *is* $n-1$*"*, implies the conjecture 1.1. Recently, Huang, X. [5] have published an article on the web with a proof of Yau's conjecture for all dimensions except for surfaces, i.e., in our notation, for $n \geq 4$. Using Huang Theorem and ours we will obtain that if an *embedded* minimal hypersurface M in S^n with $n \geq 4$ satisfies the condition $\star$, and $\int_M \|A\|^2 = \int_M (n-1)$, then M is a Clifford hypersurface. There is a large variety of minimal hypersurfaces in S^n that satisfies the inequality $(\star)$, for example, we will show that if for every $m \in M$, each eigenvalue of the shape operator has multiplicity at least 2, then the condition $(\star)$ is satisfied with the strict inequality, in particular, the Clifford hypersurfaces $M_{k,l}$ with k and l greater than 1 satisfy the condition $(\star)$ with the strict inequality. We also have that for surfaces, the condition $(\star)$ is trivially true because the scalar curvature and the Ricci curvature are the same. Therefore, in this case we obtain the following result that was already proved by Montiel and Ros [9]:

If M is a compact minimal torus immersed in S^3 by first eigenfunctions of the laplacian, then M is isometric to the Clifford torus.

Before I proceed, I would like to thank my advisor, Professor Bruce Solomon, for his lessons on mathematics and his comments on this paper. I would like also to thank Colciencias for its financial support.

2. **Preliminaries**

Let $\phi : M \longrightarrow S^n$ be a minimal immersion of a compact oriented $(n-1)$-dimensional manifold into the unit sphere. We will identify M with the set $\phi(M) \subset \mathbf{R}^{n+1}$ and the space T_mM with the linear subspace $d\phi_m(T_mM)$ of $\mathbf{R}^{n+1}$. Let $w \in \mathbf{R}^{n+1}$ be fixed. We will define the functions $l_w : M \longrightarrow \mathbf{R}$ and $f_w : M \longrightarrow \mathbf{R}$ by

$$\left.\begin{aligned} l_w(m) &= \langle m, w\rangle \\ f_w(m) &= \langle \nu(m), w\rangle \end{aligned}\right\} \quad \text{for all } m \in M.$$

A direct computation using the minimality of M and the Codazzi equations gives us:

Proposition 2.1. *The gradient and the laplacian of the functions l_w and f_w are given by:*

$$\begin{aligned} \nabla l_w &= w^T & \nabla f_w &= -A(w^T) \\ -\Delta l_w &= (n-1) l_w & -\Delta f_w &= \|A\|^2 f_w \end{aligned}$$

Here w^T denotes the tangential component of w on the tangent space T_mM.

The following lemma is based on the minimax characterization of eigenvalues for elliptic operators.

Lemma 2.1. *Let $M \subset S^n$ be a minimal compact oriented hypersurface. If the first eigenvalue of $-\Delta$ on M is $(n-1)$, then for every smooth function $f : M \longrightarrow \mathbf{R}$ with $\int_M f = 0$ we have that*

$$\int_M |\nabla f|^2 \geq (n-1) \int_M f^2 \quad \textit{with equality only if} \quad -\Delta f = (n-1) f.$$

Our main theorem is based on a technique that uses the group of conformal applications from S^n to S^n; this technique was introduced by Li and Yau in [8]. Let B^{n+1} be the open unit ball in $\mathbf{R}^{n+1}$. For each point $g \in B^{n+1}$ we consider the map

$$F_g(p) = \frac{p + (\mu\langle p, g\rangle + \lambda) g}{\lambda(\langle p, g\rangle + 1)}$$

for all $p \in S^n$, where $\lambda = (1-|g|^2)^{-\frac{1}{2}}$ and $\mu = (\lambda-1)|g|^{-2}$. A direct verification ([9]) shows that F_g is a conformal transformation from S^n to S^n and, for every $v, w \in T_pS^n$, its differential dF_g satisfies

$$\langle dF_g(v), dF_g(w)\rangle = \frac{1-|g|^2}{(\langle p, g\rangle + 1)^2}\langle v, w\rangle.$$

In [8], Li and Yau proved that if $\phi : M \longrightarrow S^n$ is a conformal immersion, then there exists $g \in B^{n+1}$ such that $\int_M F_g \circ \phi = (0, \dots, 0)$. In this paper we will need the same result for immersion which may not be conformal.

Lemma 2.2. *Let M be a compact riemannian manifold. If $\phi : M \longrightarrow S^n$ is a continuous map such that for every $b \in S^n$, the volume of $\phi^{-1}(b) = \{m \in M : \phi(m) = b\}$ vanishes, then there exists $g \in B^{n+1}$ such that $\int_M F_g \circ \phi = (0, \dots, 0)$.*

Proof. For every measurable set $T \subset M$ we will denote its volume by $|T|$. Let us define the map $H : B^{n+1} \longrightarrow B^{n+1}$ in the following way

$$H(g) = \frac{1}{|M|}\int_M F_g \circ \phi = \frac{1}{|M|}\big(\int_M \langle F_g \circ \phi, e_1\rangle, \dots, \int_M \langle F_g \circ \phi, e_1\rangle\big)$$

where $e_1 = (1, 0, \dots, 0), \dots, e_{n+1} = (0, \dots, 0, 1)$. Notice that $H(g) \in B^{n+1}$ since

$$\begin{aligned}\big|H(g)\big|^2 &= \frac{1}{|M|^2}\sum_{i=1}^{n+1}\big(\int_M \langle F_g \circ \phi, e_1\rangle\big)^2\\ &\le \frac{1}{|M|^2}\sum_{i=1}^{n+1}\big(\int_M 1^2\big)\big(\int_M \langle F_g \circ \phi, e_1\rangle^2\big)\\ &= \frac{1}{|M|}\sum_{i=1}^{n+1}\big(\int_M \langle F_g \circ \phi, e_1\rangle^2\big)\\ &= \frac{1}{|M|}\int_M 1 = 1.\end{aligned}$$

We need to show that $H(g) = (0, \dots, 0)$ for some $g \in B^{n+1}$. We will achieve this by showing that H can be extended continuously to $\partial B^{n+1} = S^n$ with $H(b) = b$ for all $b \in S^n$ since every continuous map from $\overline{B^{n+1}}$ to $\overline{B^{n+1}}$ which fixes $\partial B^{n+1} = S^n$ must be onto. Using the hypothesis of the lemma we have

$$\forall b_0 \in S^n \quad \lim_{k\to\infty} |\{m : 1 + \langle \phi(m), b_0\rangle < \frac{1}{k}\}| = |\phi^{-1}(-b_0)| = 0 \tag{1}$$

For every $b \in S^n$ and $\delta > 0$ let us define $M_\delta(b) = \{m \in M : \langle \phi(m), b\rangle + 1 < \delta\}$. Let b_0 be a fixed vector in S^n and ϵ be a positive number. By (1) we can find a positive integer k such that $k > \frac{1}{\epsilon}$ and $|M_{\frac{1}{k}}(b_0)| < \frac{\epsilon}{8}|M|$. A direct verification shows that if $b \in S^n$ and $|b - b_0| < \frac{1}{2k}$ then $M_{\frac{1}{2k}}(b) \subset M_{\frac{1}{k}}(b_0)$.

We will prove the lemma by showing that there exists a positive number δ such that for any $g \in B^{n+1}$ with $|\frac{g}{|g|} - b_0| < \frac{1}{2k}$ and $|g| > 1-\delta$ we have:

$$\langle H(g), \frac{g}{|g|}\rangle > 1 - \frac{\epsilon}{2}. \tag{2}$$

Once we have (2), the lemma will follow by noticing that

$$\begin{aligned}\langle H(g), b_0\rangle &= \langle H(g), \frac{g}{|g|}\rangle + \langle H(g), b_0\rangle - \langle H(g), \frac{g}{|g|}\rangle \\ &> 1 - \frac{\epsilon}{2} - |\langle H(g), \frac{g}{|g|} - b_0\rangle| \\ &> 1 - \frac{\epsilon}{2} - \frac{1}{2k} > 1 - \epsilon.\end{aligned}$$

Let us start the proof of (2). Notice that for every $m \notin M_{\frac{1}{2k}}(\frac{g}{|g|})$ we have

$$\frac{1}{2k} \leq 1 + \langle \phi(m), \frac{g}{|g|}\rangle = \frac{|g| - 1 + 1 + \langle \phi(m), g\rangle}{|g|}.$$

Then,

$$\frac{1}{2k}|g| + (1 - |g|) \leq 1 + \langle \phi(m), g\rangle$$

and therefore,

$$\frac{1-|g|^2}{|g|(1+\langle\phi(m)\rangle)} \leq \frac{1-|g|^2}{(\frac{1}{2k}|g| + (1-|g|))|g|}$$

For a fixed $g \in B^{n+1}$ with $|\frac{g}{|g|} - b_0| < \frac{1}{2k}$, we will use the inequality above to estimate the function $\langle F_g(\phi(m)), \frac{g}{|g|}\rangle$ defined on the complement of $M_{\frac{1}{2k}}(\frac{g}{|g|})$.

$$\begin{aligned}\langle F_g(\phi(m)), \frac{g}{|g|}\rangle &= \frac{\langle \phi(m), \frac{g}{|g|}\rangle + (\mu\langle\phi(m), g\rangle + \lambda)|g|}{\lambda(\langle\phi(m), g\rangle + 1)} \\ &= \frac{\langle\phi(m), g\rangle + (\lambda - 1)\langle\phi(m), g\rangle + \lambda|g|^2}{|g|\lambda(\langle\phi(m), g\rangle + 1)} \\ &= \frac{1}{|g|} - \frac{1-|g|^2}{|g|(\langle\phi(m), g\rangle + 1)} \\ &\geq \frac{1}{|g|} - \frac{1-|g|^2}{|g|(\frac{1}{2k}|g| + (1-|g|))}.\end{aligned}$$

Since the last expression is independent of m and converge to 1 when $|g|$ goes to 1, we can find $\delta > 0$ such that for all $m \notin M_{\frac{1}{2k}}(\frac{g}{|g|})$, if $1 - \delta < |g|$ then

$$\langle F_g(\phi(m)), \frac{g}{|g|}\rangle > 1 - \frac{\epsilon}{4}.$$

Hence for any $g \in B^{n+1}$ with $|\frac{g}{|g|} - b_0| < \frac{1}{2k}$ and $|g| > 1 - \delta$ we have:

$$\begin{aligned}\langle H(g), \frac{g}{|g|}\rangle &= \frac{1}{|M|}\Big(\int_{M\setminus M_{\frac{1}{2k}}(\frac{g}{|g|})} \langle F_g\circ\phi, \frac{g}{|g|}\rangle + \int_{M_{\frac{1}{2k}}(\frac{g}{|g|})} \langle F_g\circ\phi, \frac{g}{|g|}\rangle\Big)\\ &\geq \frac{1}{|M|}(1-\frac{\epsilon}{4})(|M| - |M_{\frac{1}{2k}}(\frac{g}{|g|})|) - \frac{1}{|M|}\big|M_{\frac{1}{2k}}(\frac{g}{|g|})\big|\\ &> 1-\frac{\epsilon}{4} - 2\frac{1}{|M|}\big|M_{\frac{1}{2k}}(\frac{g}{|g|})\big|\\ &> 1-\frac{\epsilon}{4} - 2\frac{1}{|M|}\big|M_{\frac{1}{k}}(b_0)\big|\\ &> 1-\frac{\epsilon}{2}.\end{aligned}$$

This completes the proof of (2) and henceforth the proof of the lemma. ☑

3. The average of the norm of the shape operator of a minimal hypersurface of the unit sphere

In this section we will make some estimates on the average of the norm of the shape operator $av = \frac{\int_M \|A\|^2}{\int_M 1}$ for minimal hypersurface on spheres that satisfies the condition $(\star)$. We will prove that $av \geq \frac{n-1}{2}$ for these immersions, moreover, if the immersion is given by first eigenfunctions, then, $av \geq n-1$ with equality only if M is a Clifford hypersurface. We start this section with the following lemmas.

Lemma 3.1. *Let M be an oriented closed minimal hypersurface in S^n. If $w \in \mathbf{R}^{n+1}$ is a fixed vector such that the function $1 + f_w(m)$ is always positive on M, then*

$$\int_M \|A\|^2 = \int_M \frac{1-|w|^2}{(1+f_w)^2}\|A\|^2 + \int_M \frac{|w^T|^2\|A\|^2 + l_w^2\|A\|^2 - 2|A(w^T)|^2}{(1+f_w)^2}.$$

Proof. Let us define $f : M \longrightarrow \mathbf{R}$ by $f = \ln(1+f_w)$. A direct verification shows that $\nabla f = \frac{\nabla f_w}{(1+f_w)}$ and

$$\begin{aligned}\Delta f = \mathrm{div}\nabla f &= \frac{-\|A\|^2 f_w}{1+f_w} - \frac{|\nabla f_w|^2}{(1+f_w)^2} = -\frac{1}{2}\Big(\frac{2\|A\|^2 f_w(f_w+1) + 2|\nabla f_w|^2}{(1+f_w)^2}\Big)\\ &= -\frac{1}{2}\Big(\frac{(2f_w + f_w^2 + f_w^2 + 1 - 1 + |w|^2 - |w|^2 + l_w^2 - l_w^2)\|A\|^2 + 2|\nabla f_w|^2}{(1+f_w)^2}\Big)\\ &= -\frac{1}{2}\Big(\|A\|^2 + \frac{(|w|^2-1)\|A\|^2 - (|w|^2 - f_w^2 - l_w^2)\|A\|^2 - l_w^2\|A\|^2 + 2|\nabla f_w|^2}{(1+f_w)^2}\Big)\\ &= -\frac{1}{2}\Big(\|A\|^2 + \frac{(|w|^2-1)\|A\|^2 - |w^T|^2\|A\|^2 - l_w^2\|A\|^2 + 2|A(w^T)|^2}{(1+f_w)^2}\Big).\end{aligned}$$

Since $\int_M \Delta f = 0$, then the lemma follows ☑

Lemma 3.2. *Let $\phi : M \longrightarrow S^n$ be a smooth map, $g \in B^{n+1}$ and $\{e_i\}_{i=1}^{n+1}$ be an orthonormal basis of $\mathbf{R}^{n+1}$. If we define $h_i : M \longrightarrow \mathbf{R}$ by $h_i(m) = \langle F_g(\phi(m)), e_i\rangle$ and $s_i : M \longrightarrow \mathbf{R}$ by $s_i(m) = \langle \phi(m), e_i\rangle$, then*

$$\sum_{i=1}^{n+1} |\nabla h_i|^2(m) = \frac{1-|g|^2}{(1+\langle \phi(m), g\rangle)^2} \sum_{i=1}^{n+1} |\nabla s_i|^2(m).$$

Proof. Let $\{v_i\}_{i=1}^{n-1}$ be an orthonormal basis of T_mM. We have that

$$\begin{aligned}|\nabla h_i|^2(m) &= \sum_{j=1}^{n-1} \big(v_j(h_i)\big)^2 \\ &= \sum_{j=1}^{n-1} \big(\langle (dF_g)_{\phi(m)}(d\phi(v_j)), e_i\rangle\big)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n+1} |\nabla h_i|^2(m) &= \sum_{i=1}^{n+1}\sum_{j=1}^{n-1} \big(\langle (dF_g)_{\phi(m)}(d\phi(v_j)), e_i\rangle\big)^2 \\ &= \sum_{j=1}^{n-1} \|(dF_g)_{\phi(m)}(d\phi(v_j))\|^2 \\ &= \sum_{j=1}^{n-1} \frac{1-|g|^2}{(1+\langle \phi(m), g\rangle)^2}\|d\phi(v_j)\|^2 \\ &= \sum_{j=1}^{n-1} \frac{1-|g|^2}{(1+\langle \phi(m), g\rangle)^2}\sum_{i=1}^{n+1}\big(v_j(s_i)\big)^2 \\ &= \frac{1-|g|^2}{(1+\langle \phi(m), g\rangle)^2}\sum_{i=1}^{n+1} |\nabla s_i|^2(m).\end{aligned}$$

☑

Theorem 3.1. *Let M be a compact oriented minimal hypersurface immersed in S^n by first eigenfunctions of the laplacian. Denote by $\{\kappa_i(m)\}_{i=1}^{n-1}$ the eigenvalues of the shape operator at $m \in M$. If M is not totally geodesic and $\kappa_i^2(m) \le \frac{\|A\|^2(m)}{2} = \frac{1}{2}\sum_{j=1}^{n-1}\kappa_j^2(m)$ for every $m \in M$ and every $i \in \{1, \dots, n-1\}$, then $\int_M \|A\|^2 \ge (n-1)|M|$ with equality only if M is isometric to a Clifford hypersurface.*

Proof. Let $\nu : M \longrightarrow S^n$ be the Gauss map. We will start the proof verifying that the map ν satisfies the hypothesis of the Lemma 2.2. For any $b \in S^n$ let us take a vector $w_0 \in S^n$ such that $\langle w_0, b\rangle = 0$, since $\nu^{-1}(b) \subset f_{w_0}^{-1}(0)$ and f_{w_0} satisfies and elliptic equation then the nodal set $f_{w_0}^{-1}(0)$ has measure 0 in M

[2], therefore $|\nu^{-1}(b)| = 0$. Since ν satisfies the hypothesis of the Lemma 2.2, we can find $g \in B^{n+1}$ such that

$$\int_M (F_g \circ \nu) = (0, \dots, 0)$$

The equality above implies that the functions $h_i = \langle F_g(\nu(m)), e_i\rangle$ are perpendicular to the constant function, i.e. $\int_M h_i = 0$. By the Lemma 2.1 we have that

$$\sum_{i=1}^{n+1} |\nabla h_i|^2 \geq (n-1) \sum_{i=1}^{n+1} \int_M h_i^2 = (n-1)|M|$$

with equality only if $-\Delta h_i = (n-1)h_i$. On the other hand by the Lemma 3.2 we have that

$$\sum_{i=1}^{n+1} |\nabla h_i|^2 = \frac{1-|g|^2}{(1+\langle \nu(m), g\rangle)^2} \sum_{i=1}^{n+1} |\nabla f_{e_i}|^2 = \frac{1-|g|^2}{(1+\langle \nu(m), g\rangle)^2} \|A\|^2.$$

Therefore, using Lemma 3.1 we get that

$$\begin{aligned}(n-1)|M| &\leq \int_M \frac{(1-|g|^2)\|A\|^2}{(1+\langle \nu(m), g\rangle)^2} \\ &= \int_M \|A\|^2 - \int_M \frac{|g^T|^2\|A\|^2 + l_g^2\|A\|^2 - 2|A(g^T)|^2}{(1+f_g)^2}\end{aligned}$$

with equality only if $-\Delta h_i = (n-1)h_i$. Notice that the hypothesis on the eigenvalues of the shape operator A implies that the expression

$$\int_M \frac{|g^T|^2\|A\|^2 + l_g^2\|A\|^2 - 2|A(g^T)|^2}{(1+f_g)^2}$$

is positive unless $g = 0$. Therefore, we have that $\int_M \|A\|^2 \geq (n-1)|M|$. Moreover if $\int_M \|A\|^2 = (n-1)|M|$ then $g = 0$; therefore, for $i = 1, \dots, n+1$ we have that $h_i = f_{e_i}$ and

$$(n-1)h_i = (n-1)f_{e_i} = -\Delta h_i = -\Delta f_{e_i} = \|A\|^2 f_{e_i}$$

The equality above implies that $\|A\|^2 \equiv n-1$. Therefore, M is isometric to a Clifford hypersurface. ☑

Corollary 3.2. *If $M \subset S^3$ is a compact minimal torus immersed by first eigenfunctions, then M is a Clifford torus.*

Corollary 3.3. *Let M be a compact oriented minimal hypersurface immersed in S^n by first eigenfunctions. If M is not totally geodesic and the Ricci and scalar curvatures satisfy the inequality*

$$R \leq \frac{n-3}{n-1} + \frac{2\mathrm{Ric}(v)}{n-1} \quad \textit{for any } v \in TM \textit{ with } |v| = 1 \tag{$\star$}$$

then $\int_M \|A\|^2 \geq (n-1)|M|$ *with equality only if* M *is isometric to a Clifford hypersurface.*

Proof. Let $\{\kappa_i\}_{i=1}^{n-1}$ be the eigenvalues of A. Let $\{v_i\}_{i=1}^{n-1}$ be an orthonormal basis of T_mM such that $A(v_i) = \kappa_i v_i$ for $i = 1, \dots, n-1$. By the Gauss equation we have that for $i \neq j$ the sectional curvature $k(v_i, v_j) = 1 + \kappa_i \kappa_j$. Using this expression we get

$$\kappa_i^2 = \kappa_i(-\sum_{i\neq j} \kappa_j) = (n-2) - \sum_{i \neq j} k(v_i, v_j) = (n-2) - (n-2)\mathrm{Ric}(v_i) \qquad (1)$$

From the equation above we obtain that $\|A\|^2 = (n-1)(n-2)(1-R)$. Using the hypothesis of the corollary we get

$$\begin{aligned}
\kappa_i^2 &= (n-2) - (n-2)\mathrm{Ric}(v_i) \\
&\leq (n-2) + (n-2)\big(\frac{n-3}{2} - \frac{n-1}{2}R\big) \\
&= (n-2) + \frac{(n-2)(n-3)}{2} - \frac{(n-1)(n-2)}{2} + \frac{\|A\|^2}{2} \\
&= \frac{\|A\|^2}{2}.
\end{aligned}$$

Therefore the hypothesis of Theorem 3.1 is satisfied and the corollary follows. ☑

4. Minimal hypersurfaces with constant ricci curvature

Let us start by classifying the minimal hypersurfaces on spheres with constant ricci curvature. Using the equation (1) in §3 we have that the ricci curvature of any minimal hypersurface on S^n can not be greater than 1, moreover if the ricci curvature is constant, then we have that the principal curvatures of M must also be constant and they can only take the values $\pm\sqrt{(n-2)(1-\mathrm{Ric})}$. This observation forces M to be an isoparametric hypersurface with either one principal curvature or two principal curvatures, i.e. M is either an equator or a Clifford hypersurface. A direct computation shows that the only Clifford hypersurfaces with constant ricci curvature are those of the form $S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2}) \times S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2})$ with n odd. Therefore, if the ricci curvature of a minimal hypersurface in S^n is constant, this constant must be either 1 or $\frac{n-3}{n-2}$. Let $\epsilon(n) = \frac{2n-3}{(n-2)(2n-1)}$. We will show that if $M \subset S^n$ is a minimal hypersurface immersed by first eigenvalues with $|\mathrm{Ric}_m(v) - \frac{n-3}{n-2}| \leq \epsilon(n)$ for all $(m, v) \in TM$, $|v| = 1$, and with $\int_M |A|^2 = \int_M (n-1)$ (or equivalent with $\int_M R = \int_M \frac{n-3}{n-2}$), then, n must be odd and M must be isometric to the Clifford hypersurface $M_{\frac{n-1}{2},\frac{n-1}{2}}$. Namely we have the following theorem:

Theorem 4.1. *Let $M \subset S^n$ be a minimal compact hypersurface immersed by first eigenfunctions. If the average of the scalar curvature is $\frac{n-3}{n-2}$ and for every unit vector v in TM*

$$|\mathrm{Ric}(v) - \frac{n-3}{n-2}| \leq \epsilon(n) = \frac{2n-3}{(n-2)(2n-1)}$$

then n must be odd and M must be isometric to $S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2}) \times S^{\frac{n-1}{2}}(\frac{\sqrt{2}}{2})$.

Proof. By Theorem 3.1, it is enough to show that if $\kappa_1, \dots \kappa_{n-1}$ denote the principal curvatures of M, then $\kappa_i \leq \frac{1}{2}|A|^2$. Let $\{v_1, \dots, v_{n-1}\}$ be an orthonormal bases of T_mM such that $A_m(v_i) = \kappa_i(m)v_i$. The bounds on the ricci curvature imply the same kind of bounds for the scalar curvature R, namely we have that $|R - \frac{n-3}{n-2}| \leq \epsilon(n)$, since $\|A\|^2 = (n-1)(n-2)(1-R)$ then we get that

$$|A|^2 \geq (n-1)(n-2)(1 - \frac{n-3}{n-2} - \epsilon(n)) = 2\frac{n-1}{2n-1} \tag{2}$$

On the other hand, using the equation (1) in §3 and the bounds on the ricci curvature we get,

$$\begin{aligned}
\kappa_i^2 &= (n-2)(1 - \mathrm{Ric}(v_i)) \\
&\leq (n-2)(1 + \epsilon(n) - \frac{n-3}{n-2}) \\
&\leq 4\frac{n-1}{2n-1} \\
&\leq 2|A|^2.
\end{aligned}$$

In the last inequality we have used the equation (2). ☑

Notice that by the corollary 3.3, we can remove the condition on the ricci curvature when $n = 3$. The following example shows that the condition on the first eigenvalue of the laplacian is necessary.

Example 4.1. Let us consider the following family of minimal genus zero surfaces studied by Lawson in [6]. For any pair of relative prime integers r and s, let us define:

$$T_{rs} = \{\phi(x,y) = (\cos rx \cos y, \sin rx \cos y, \cos sx \sin y, \sin sx \sin y)\} : \; x, y \in \mathbf{R}\}$$

The immersion ϕ satisfies that $|\phi_x|^2 = E = r^2 \cos y + s^2 \sin y$, $\langle \phi_x, \phi_y \rangle = 0$ and $|\phi_y|^2 = 1$, here ϕ_x and ϕ_y denote the partial derivatives with respect to x and y respectively. A direct computation shows that the vector

$$\nu(x,y) = \frac{r^2 - s^2}{rs\sqrt{E}} \sin y \cos y \phi_x + \frac{\sqrt{E}}{rs} \phi_{xy}$$

defines a unit normal vector of T_{rs} as a submanifold of S^3 because $|\nu| = 1$ and $\langle \nu, \phi_x\rangle = \langle \nu, \phi_y\rangle = \langle \nu, \phi\rangle = 0$. We also have that

$$A(\phi_x) = -\nu_x = \frac{rs}{\sqrt{E}}\phi_y,$$
$$A(\phi_y) = -\nu_y = \frac{rs}{\sqrt{E^3}}\phi_x.$$

If we write the matrix of the linear transformation $A : T_mT_{rs} \to T_mT_{rs}$ in the orthonormal base $\{E^{-\frac{1}{2}}\phi_x, \phi_y\}$ we can deduce that the principal curvatures of T_{rs} at $\phi(x, y)$ are $\pm a_{rs}(x, y)$ where $a_{rs} = rsE^{-1}$. Notice that if $r = s+1$, then a_{rs} goes uniformly to 1 when r goes to infinity. Since $a^2 = (1 - \mathrm{Ric})$, then the ricci curvature of T_{rs} goes uniformly to zero when $r = s + 1$ goes to infinity.

Remark 4.1. *For $n = 3$ the Clifford torus is the only minimal surface with constant ricci curvature equal to $\frac{n-3}{n-2} = 0$. The example above shows that there exist infinitely many immersed minimal surfaces in S^3 with ricci curvature arbitrarily close to zero and with zero average of the scalar curvature. By the corollary 3.3, the first eigenvalue of the laplacian of these examples is less than 2. Therefore, at least for the case $n = 3$, we have that the condition on the first eigenvalue of the Theorem 4.1 is necessary.*

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(Recibido en octubre de 2004)

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