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About Increments of Additive Functionals of Diffusion Processes

Sergio Alvarez-Andrade
Université de Technologie de Compiègne, FRANCE

Abstract. We study the increments of additive functionals of diffusion processes by using strong approximations and some properties of the local times.

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1. **Introduction**

Let $X = \{X_t : t \geq 0\}$ be a recurrent one-dimensional diffusion living on an interval $I \subset \mathbb{R}$, with $X_0 = 0$. Csáki and Salminen in [3], established strong approximations results for additive functionals of the form

$$Z_t = \int_0^t f(X_s)ds = \int_I f(x)L_t^x m(dx), \tag{1}$$

where $f(x)$, $x \in I$ is a locally integrable real valued function with the property $\int_I |f(x)|m(dx) \leq \infty$, L_t^x is the local time of X_t at every point $x \in I$ and m is the speed mesure of X.

Our aim is to establish the asymptotic behavior of the increments of additive functionnals defined as in (1) by using strong approximations results established in [3]. Our arguments are based on well known results for the increments of Brownian motion and those of local times.
In the sequel we consider $\log t = \log(\sup(t, e))$.

Let $A_u = A_u^0$ be the right continuous inverse of $L_t = L_t^0$ and S the scale function of X. If we put $t = A_u$ in (1), then we have

$$Z_{A_u} = \int_0^{A_u} f(X_s)ds = \int_I f(x)L_t^{A_u} m(dx),$$

and $\{Z_{A_u}, u \geq 0\}$ is a process with independent and stationary increments (see, Csáki *and al.,* (1992), Csáki and Csörgő (1995) and Csáki and Salminen (1996)). Moreover, we have

$$EZ_{A_u} = u\bar{f},$$

where

$$\bar{f} = \int_I f(x)m(dx), \tag{2}$$

and

$$VarZ_{A_u} = u\sigma^2,$$

where

$$\sigma^2 = 2 \iint\limits_{I\times I\cap\{xy>0\}} f(x)f(y)\min(|S(x)|, |S(y)|)m(dx)m(dy). \tag{3}$$

Two cases was considered in [3]: firstly, the case of positive recurrence (i.e. $\mu = EA_1 = m\{I\} < \infty$); secondly the case of null recurrence (i.e. $m\{I\} = \infty$).

The following strong approximation result was established in the case of positive recurrence.

Theorem 1. *Assume that*

$$E(A_1)^q < \infty \ \textit{for some}\ 1 < q \leq 2. \tag{4}$$

(i) If

$$E\left(\int_0^{A_1} |f(X_s)|ds\right)^{2+\delta} < \infty,$$

for some $\delta > 0$, then on a suitable probability space one can construct a diffusion process X_t and a standard Brownian motion $W(t)$ such that

$$|Z_t - \bar{f}L_t^0 - \frac{\sigma}{\sqrt{\mu}}W(t)| = O\left(t^\beta \log t\right) a.s., \tag{5}$$

as $t \to \infty$, where

$$\beta = \max(\frac{1}{2+\delta}, \frac{1}{2q}), \tag{6}$$

and σ^2 is defined as in (3).

(ii) If

$$E\left(\int_0^{A_1} |f(X_s) - \frac{\bar{f}}{\mu}|ds\right)^{2+\delta} < \infty, \tag{7}$$

for some $\delta > 0$, then on a suitable probability space one can construct a diffusion process X_t and a standard Brownian motion $W(t)$ such that

$$|Z_t - \bar{f}\frac{t}{\mu} - \frac{\sigma_1}{\sqrt{\mu}}W(t)| = O\left(t^{\beta}\log t\right) \ a.s.,$$

as $t \to \infty$, where β is defined by (6) and

$$\sigma_1^2 = 2 \iint\limits_{I\times I\cap\{xy>0\}} (f(x) - \frac{\bar{f}}{\mu})(f(y) - \frac{\bar{f}}{\mu}) \min(|S(x)|, |S(y)|)m(dx)m(dy).$$

In the case of null recurrence (i.e. $m\{I\}$ is infinite) under assumption that on a suitable probability space one can construct a diffusion process X and a stable process T_u of order α (cf. Samorodnitski and Taqqu (1994)) such that

$$|A_u - T_u| = O(u^k) \ a.s., \tag{8}$$

as $u \to \infty$ for some $0 \le k < 1/\alpha$ (see, [1]), the following result was established in [3].

Theorem 2. *Assume that X is a null recurrent diffusion process on an interval I, $0 \in I$ with local time L_t^x and such that (7) and (8) are both satisfies. Then on a suitable probability space one can construct a diffusion process X_t and a standard Brownian motion $W(u)$ and a non-decreasing stable process T_u of order α, such that W and T are independent and for $\epsilon > 0$ small enough, we have*

$$|Z_t - \bar{f}L_t^0 - \sigma W(V_t)| = O\left(t^{\alpha/2-\epsilon}\right) a.s.,$$

as $t \to \infty$, where V_t is the (continuous) inverse of T_u and Z_t, $\bar{f}$ and σ are defined by (1), (2) and (3) respectively.

2. Main results

Proposition 1. *Under conditions i) and ii) of Th.1, let h_T be a real function satisfying $h_T \to +\infty$, h_T/T is non-increasing and $\log(\frac{T}{h})/\log\log T = \infty$ as $T \to \infty$. Then we have*

$$\sup_{0\le t\le T-h_T} (Z_{t+h_T} - Z_t) = O(d_i(T)) \ a.s.,$$

where

$$d_1(T) = \sup(h_T,\ T^{\beta}\log T),$$

under i) and

$$d_2(T) = h_T/\mu + T^{\beta}\log T,$$

under ii).

Proposition 2. *Under conditions of Theorem 2, let $h_T = T^{\beta}$, we have*

$$\sup_{0\le t\le T-h_T} (Z_{t+h_T} - Z_t) = O(T^{\beta+\epsilon}) \ a.s. \tag{9}$$

3. Proofs

Proof of Proposition 1. (i) Put $h = h_T$, by (5), we have

$$|Z_{t+h} - Z_t| \le \bar{f}|L_{t+h} - L_t| + \frac{\sigma}{\sqrt{\mu}}|W(t+h) - W(t)| + O((t+h)^{\beta}\log(t+h)) \ a.s. \tag{10}$$

It is clear that we only study the first two terms of the right hand side of (10). We begin by stating that

$$|L_{t+h} - L_t| = O(d_0(t)) \ a.s.$$

Put $t + h = A_{u+h}$ in $|L_{t+h} - L_t|$. Under condition (4), we have

$$A_u = \mu u + O(u^{1/q}(\log u)^{1/2}) \ a.s., \tag{11}$$

as $u \to \infty$ and consequently

$$L_t = \frac{t}{\mu} + O(t^{1/q}(\log t)^{1/2}) \ a.s.,$$

as $t \to \infty$. By relations (10) and (11), we have as $u \to \infty$ that

$$L_{A_{u+h}} = L_{A_u+h} \ a.s.$$

By Proposition (2.1) of Csáki and Salminen (1996), we have

$$L_{A_u} = l_{a_u} \ a.s.,$$

where $l_t = l_t^0$ is the local time of the Wiener process $W(t)$ and a_t their inverse right continuous function, then we can consider

$$|L_{A_u+h} - L_{A_u}| = |l_{a_u+h} - l_{a_u}| \ a.s. \tag{12}$$

By using results for the increments of the local time process of the Wiener process (see, Csáki *et al.,* (1992)), we have

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t-h}(l_{s+h} - l_s)}{d_0(t)} = 1 \ a.s.,$$

where $d_0(t) = \sqrt{h(\log(t/h) + 2\log\log t)}$ and if $\log(t/h)/\log\log t = \infty$ then

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t-h}(l_{s+h} - l_s)}{\sqrt{h\log(t/h)}} = 1 \ a.s.$$

If in the right term of (12) we put $t = a_u$, then we have

$$|l_{t+h} - l_t| = O(d_0(t)) \ a.s. \tag{13}$$

This last result is sufficient for to get the announced result.
For the remaining term, by Theorem 1.2.1 of Csörgő and Révész (1981), we have

$$|W(t+h)-W(t)| = O(d_0(t)) \ a.s., \tag{14}$$

and by condition h/T is non-increasing when $T \to \infty$, we have that the $O(\ )$ term in (10) is an $O(T^\beta \log T)$. With this last result and by (13) and (14), we get the expected result. ☑

The proof of part ii) is close to the previous proof.

Proof of Proposition 2. In the same way as in Proposition 1, we have

$$|Z_{t+h}-Z_t| \leq \bar{f}|L_{t+h}-L_t| + \sigma|W(V_{t+h})-W(V_t)| + O(t^{\alpha/2-\epsilon}) \ a.s. \tag{15}$$

For to study the right term of (15), we recall some results given in Csáki and Salminen (1996) : for $0<\beta<1$ and $\epsilon>0$, the following relations are satisfied

$$\sup_{0\leq s\leq t}(V_{s+t^\beta}-V_s) = O(t^{\alpha\beta+\epsilon}) \ a.s., \tag{16}$$

$$\sup_{0\leq s\leq t}(L_{s+t^\beta}-L_s) = O(t^{\alpha\beta+\epsilon}) \ a.s., \tag{17}$$

$$|V_t-L_t| = O(t^{\alpha\beta+\epsilon}) \ a.s.$$

By (15), (16) and (17), we can deduce (9). ☑

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L.M.A.C., Université de Technologie de Compiègne
BP. 529, 60205 Compiègne cedex, France
e-mail: sergio.alvarez@utc.fr