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Infinite sets of positive integers whose sums are free of powers

Florian Luca
Mathematical Institute, UNAM, Morelia, MÉXICO

Abstract. In this short note, we construct an infinite set S of positive integers such that for all $n \geq 1$ and any n distinct elements $x_1, \ldots, x_n$ of S the sum $\sum_{i=1}^{n} x_i$ is not a perfect power.

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Problem 5 proposed at the *Fourth Central-American and Carribean Mathematical Olympiad* (Mérida, México, July, 2002) asked the contenders to construct an infinite set $\mathcal{S}$ of positive integers such that the sum of any number of distinct elements of $\mathcal{S}$ is not a perfect square. It is fairly straightforward to check that an example of such a set $\mathcal{S}$ is provided by the set of all *Fermat numbers* $\{F_n\}_{n\geq 0}$, where $F_n := 2^{2^n} + 1$ for $n \geq 0$. In this note, we show how to construct an infinite set of positive integers S such that for every $n \geq 1$ and any n distinct elements $x_1, \ldots, x_n$ of S the sum $\sum_{i=1}^{n} x_i$ is not a perfect power. For any positive integer k let p_k be the kth prime number, and let $q > 1$ be any positive integer. The construction of our set S follows somewhat closely the example provided by the set of Fermat numbers mentioned above and is contained in the following:

Theorem. *For any positive integer m let $x_m = q^{p_1 p_2 \cdots p_m} + 1$. Then, there exists an effectively computable constant c_1 depending only on q, such that the set $S := \{x_n \mid n \geq c_1\}$ has the property that any sum of some distinct elements of S is not a perfect power.*

Proof. Assume that $t \geq 1$ and that $n_1 < n_2 < \cdots < n_t$ are such that $x_{n_1} + \cdots + x_{n_t} = y^l$ for some positive integers y and l with $l \geq 2$. Clearly, $y > 1$ and we may assume that l is prime. We first show that there exists an effectively computable constant c_1 depending only on q such that $l < c_1$. We may assume that $t \geq 2$, because for $t = 1$ the above equation reduces to $q^{p_1 p_2 \cdots p_{n_1}} + 1 = y^l$ and since $p_1 = 2$ the above equation is a particular case of the Catalan equation $x^m + 1 = y^l$ with m even and it is known that this equation has no positive integer solution (x, y, m, l) with $l > 1$ (see [1]). We also assume that $n_t \geq 4$. We write

$$y^l = x_{n_1} + \cdots + x_{n_t} = q^{p_1 p_2 \cdots p_{n_t}} + z, \tag{1}$$

where

$$1 \leq t \leq z \leq n_t + \sum_{i=1}^{n_t - 1} q^{p_1 \cdots p_i} < 2n_t q^{p_1 \cdots p_{n_t - 1}} < (\sqrt{q})^{p_1 p_2 \cdots p_{n_t}}. \tag{2}$$

Indeed, the right most inequality appearing at (2) above is equivalent to

$$q^{p_1 p_2 \cdots p_{n_t - 1}\left(\frac{p_{n_t}}{2} - 1\right)} \geq 2n_t, \tag{3}$$

and since $n_t \geq 4$ and $q \geq 2$ the above inequality will be satisfied provided that

$$2^{15(p_{n_t} - 2)} \geq (2n_t)^2, \tag{4}$$

and inequality (4) can be immediately shown to hold by induction on $n_t \geq 4$.

At this point, we recall the following result due to Shorey and Stewart (see [3]).

Lemma. (Shorey-Stewart, [3]). *Let $(u_n)_{n \geq 0}$ be a sequence of positive integers such that there exists a positive integer $q > 1$ and a constant δ with $0 < \delta < 1$ such that the inequality*

$$0 < |u_n - q^n| < q^{\delta n} \tag{5}$$

holds for all positive integers n. Then, there exists a computable constant c_2 depending only on q and δ such that if $u_n = y^k$ with y and k integers and $|y| > 1$, then $k < c_2$.

In fact, Shorey and Stewart proved a version of the above Lemma for case in which $(u_n)_{n \geq 0}$ is a non-degenerate linearly recurrent sequence whose characteristic equation has one simple dominant root, but their argument can be easily modified to yield the above Lemma (see [2], for example).

From formula (1), inequality (2), and the above Lemma, we get that there exists a constant c_2 depending only on q such that $l < c_2$. Set $c_1 := c_2$ and assume that $n_1 > c_1$ holds in formula (1). In this case, $n_t > n_1 > c_1$, and since

l is a prime, it follows that $l = p_i$ with some $i < n_t$. With $N := \dfrac{p_1 p_2 \cdots p_{n_t}}{p_i}$, equation (1) can be rewritten as

$$z = q^{Np_i} - y^{p_i} = (q^N - y)(q^{N(p_i-1)} + q^{N(p_i-2)y} + \cdots + y^{p_i-1}). \qquad (5)$$

From inequality (2), we get that $q^N - y > 0$ and

$$q^{Np_i/2} > z = (q^N - y)(q^{N(p_i-1)} + q^{N(p_i-2)y} + \cdots + y^{p_i-1}) > q^{N(p_i-1)},$$

or

$$\frac{p_i}{2} > p_i - 1,$$

which is a contradiction. The Theorem is therefore proved. ☑

One may ask if the dual statement to our Theorem is also true, i.e., whether there exist infinite sets S of positive integers such that for all $n \geq 1$ and any distinct elements $x_1, \ldots,\ x_n$ of S the sum $\sum_{i=1}^n x_i$ is a perfect power. The answer here is no.

Proposition. *There does not exist an infinite set of positive integers S such that for all $n \geq 1$ and any n distinct elements $x_1,\ \ldots,\ x_n$ of S the sum $\sum_{i=1}^n x_i$ is a perfect power.*

Proof. Assume that there exists such a set. Let p be any fixed prime. Then, infinitely many of the elements of S are in the same congruence class modulo p. Discarding the remaining elements of S, we may assume that all the elements of S are in the same congruence class modulo p. We label the elements of S as $x_1 < x_2 < \cdots < x_n < \cdots$. For any $i \geq 1$, let

$$y_i := \sum_{s=1}^{p} x_{(i-1)p+s}. \qquad (5)$$

That is, $y_1 = x_1 + x_2 + \cdots + x_p,\ y_2 = x_{p+1} + x_{p+2} + \cdots + x_{2p}$, etc. Clearly, y_i is a multiple of p for all $i \geq 1$ and the set $S' := \{y_i \mid i \geq 1\}$ has the same property that the set S has, namely all the sums of some distinct elements of S' are perfect powers. The above argument shows that we may assume that all the elements of S are multiples of p.

Let $i \geq 1$ be arbitrary. Since both x_i and $x_1 + x_i$ are perfect powers, it follows that the equation

$$x_1 = u^m - v^n \qquad (6)$$

has infinitely many positive integer solutions (u, v, m, n) with $\min(m, n) \geq 2$ and p divides both u and v. With x_1 fixed and variables $u,\ v,\ m,\ n$ in the above equation (6), the fact that p divides u and v implies that there exists a constant c_3 (depending only on p and x_1) such that $\min(m, n) < c_3$. In particular, equation (6) has infinitely many positive integer solutions (u, v, m, n) with $v > 1$ and $2 \leq \min(m, n) < c_3$, which contradicts a classical result from exponential diophantine equations (see, Theorem 12.2 in [4], for example). ☑

A somewhat more direct proof of the above Proposition can be achieved via a direct application of Faltings's Theorem. The above Proposition shows that if S is a set of positive integers such that all the sums of some distinct elements of S is a perfect power, then the cardinality of S is finite. We suspect that the cardinality of S is *uniformly bounded*, that is that there exists an absolute constant c_4 such that if S is a set of positive integers having the property that all the sums of some distinct elements of S is a perfect power, then the cardinality of S is bounded by c_4, and we leave this as a conjecture for the reader.

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Mathematical Institute, UNAM
Ap. Postal 61-3 (Xangari), CP 58 089
Morelia, Michoacán
MEXICO

e-mail: fluca@matmor.unam.mx