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Some new results on common fixed points in certain topological spaces

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Abstract. The main purpose of this paper is to give some common fixed point theorems in F-type topological spaces.

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1. Introduction

In [1], Caristi proved that a selfmapping T of a complete metric space (X, d) has a fixed point if there exists a lower semi-continuous function $\phi : X \longrightarrow \mathbb{R}^+$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \forall x \in X.$$

This result was frequently used to prove existence theorems in fixed point theory. However, it is not hard to see that if the graph of T is closed and T satisfies the above inequality for arbitrary function ϕ, then T will have a fixed point x^* such that x^* is the limit of the sequence (x_n) defined by

$$\begin{cases} x_0 \in X, \\ x_{n+1} = Tx_n. \end{cases}$$

To support this remark, we give the following example. Let $X = [0, +\infty[$. Define T and ϕ by

$$Tx = \frac{1}{2}x, \quad \phi(x) = \begin{cases} x & \text{if } x \in [0, 1[, \\ 2x & \text{if } x \in [1, +\infty[. \end{cases}$$

Then we have $|x - Tx| = \frac{1}{2}x$ and

$$\phi(x) - \phi(Tx) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0,1[\\ \frac{3}{2}x & \text{if } x \in [1,2[\\ x & \text{if } x \in [2,+\infty[. \end{cases}$$

Therefore

$$|x - Tx| \leq \phi(x) - \phi(Tx) \text{ for all } x \in X.$$

It is easy to see that T has a closed graph and the function ϕ is not lower semi-continuous at 1 but $T0 = 0$.

On the other hand, Fang [4] introduced the concept of F-type topological space and gave a characterization of the kind of spaces. The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are all the special cases of F-type topological Spaces. Furthermore, Fang established a fixed point theorem in F-type topological spaces which extends Caristi's theorem in the following way:

Theorem 1.1 (Fang)**.** *Let (X, θ) be a sequentially complete F-type topological space generated by the family $\{d_\lambda, \lambda \in D\}$. Let $k : D \longrightarrow]0, +\infty[$ be a nonincreasing function and $\phi : X \longrightarrow \mathbb{R}^+$ be a lower semi-continuous function. Let T be a selfmapping of X such that*

$$d_\lambda(x, Tx) \leq k(\lambda)[\phi(x) - \phi(Tx)], \ \forall \lambda \in D, \ \forall x \in X.$$

Then T has a fixed point in X.

The aim of this paper is to give some common fixed point theorems in F-type topological spaces. To do this, we first recall the definition of this space as given in [4].

Definition 1.1 (Fang)**.** A topological space (E, θ) is said to be F-type topological space if it is Hausdorff and for each $x \in E$, there exists a neighborhood base $F_x = \{U_x(\lambda, t)/\lambda \in D, t > 0\}$, where $D = (D, \prec)$ denotes a directed set such that:

- (F_1) If $y \in U_x(\lambda, t)$, then $x \in U_y(\lambda, t)$;
- (F_2) $U_x(\lambda, t) \subset U_x(\mu, s)$ for $\mu \prec \lambda, t \leq s$;
- (F_3) $\forall \lambda \in D$, $\exists \mu \in D$ such that $\lambda \prec \mu$ and $U_x(\mu, t_1) \cap U_y(\mu, t_2) \neq \emptyset$ implies $y \in U_x(\lambda, t_1 + t_2)$;
- (F_4) $E = \cup_{t>0} U_x(\lambda, t)$, $\forall \lambda \in D$, $\forall x \in E$.

On the other hand, it is proved in [4] that for each F-type topological space (E, θ), there exists a family $M = \{d_\lambda, \lambda \in D\}$ of quasi-metrics on E satisfying:

1. $d_\lambda(x, y) = 0 \ \forall \lambda \in D$ iff $x = y$;
2. $d_\lambda(x, y) = d_\lambda(y, x) \ \forall \lambda \in D$;
3. $d_\lambda(x, y) \leq d_\mu(x, y)$ for $\lambda \prec \mu$;
4. $\forall \lambda \in D$, $\exists \mu \in D$ such that $\lambda \prec \mu$ and $d_\lambda(x, y) \leq d_\mu(x, z) + d_\mu(z, y)$ for all $x, y, z \in E$ such that $\theta_M = \theta$.

For more details we refer to [4].

2. Main results

Theorem 2.1. *Let (X, θ) be a sequentially complete F-type topological space generated by the family $\{d_\lambda, \lambda \in D\}$. Let $k : D \longrightarrow]0, +\infty[$ be a nonincreasing function and $\phi : X \longrightarrow \mathbb{R}^+$ be a function. Let T and S be two selfmappings of X with sequentially complete graphs such that $TX \subset SX$ and*

$$\begin{aligned} \max\{d_\lambda(Sx, Tx), &d_\mu(Tx, STx), d_\beta(Sx, TSx)\} \\ &\leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx) - \phi(Tx)], \end{aligned} \tag{1}$$

for all $(\lambda, \mu, \beta) \in D^3$, for all $x \in X$. Then T and S have a common fixed point in X.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Choose $x_2 \in X$ such that $Tx_1 = Sx_2$. In general, choose $x_n \in X$ such that $Tx_{n-1} = Sx_n$. Let $(\lambda, \mu, \beta) \in D^3$. From (1), it follows

$$\begin{aligned} d_\lambda(Sx_n, Sx_{n+1}) = d_\lambda(Sx_n, Tx_n) &\leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Tx_n)] \\ &\leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Sx_{n+1}]. \end{aligned}$$

For all $(\lambda, \mu, \beta) \in D^3$, we consider the nonnegative real sequence (a_n) defined by

$$a_n = \max\{k(\lambda), k(\mu), k(\beta)\}\phi(Sx_n), \quad n = 1, 2, \cdots .$$

It is easy to see that (a_n) is nonincreasing and bounded bellow by 0. Hence it is a convergent sequence. On the other hand, for all $\lambda \in D$, there exists $\lambda_1 \in D$ such that $\lambda \prec \lambda_1$ and

$$d_\lambda(Sx_n, Sx_{n+m}) \leq d_{\lambda_1}(Sx_n, Sx_{n+1}) + d_{\lambda_1}(Sx_{n+1}, Sx_{n+m}).$$

For this λ_1, there exists $\lambda_2 \in D$ such that $\lambda_1 \prec \lambda_2$ and

$$d_{\lambda_1}(Sx_{n+1}, Sx_{n+m}) \leq d_{\lambda_2}(Sx_{n+1}, Sx_{n+2}) + d_{\lambda_2}(Sx_{n+2}, Sx_{n+m}).$$

Continuing in this fashion, there exists $(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}) \in D^{m-1}$ such that $\lambda \prec \lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_{m-1}$ and

$$\begin{aligned} d_\lambda(Sx_n, Sx_{n+m}) \leq d_{\lambda_1}(Sx_n, Sx_{n+1}) + d_{\lambda_2}(Sx_{n+1}, Sx_{n+2}) + \cdots \\ + d_{\lambda_{m-1}}(Sx_{n+m-1}, Sx_{n+m}). \end{aligned}$$

Hence

$$\begin{aligned} d_\lambda(Sx_n, Sx_{n+m}) \leq &\max\{k(\lambda_1), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Sx_{n+1})] + \\ &\max\{k(\lambda_2), k(\mu), k(\beta)\}[\phi(Sx_{n+1} - \phi(Sx_{n+2})] + \cdots + \\ &\max\{k(\lambda_{m-1}), k(\mu), k(\beta)\}[\phi(Sx_{n+m-1} - \phi(Sx_{n+m})]. \end{aligned}$$

Therefore, since the function k is nonincreasing, we have

$$d_\lambda(Sx_n, Sx_{n+m}) \leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Sx_{n+m})]$$

which implies that (Sx_n) is a Cauchy sequence. Since X is sequentially complete, there exists $u \in X$ such that $\lim_{n\to\infty} Sx_n = u$. Hence

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = u.$$

We shall show that $\lim_{n\to\infty} STx_n = u$. Let $\mu \in D$. There exists $\mu_1 \in D$ such that $\mu \prec \mu_1$ and

$$d_\mu(STx_n, u) \leq d_{\mu_1}(STx_n, Tx_n) + d_{\mu_1}(Tx_n, u).$$

In view of (1), for all $\mu \in D$ we have

$$d_\mu(Tx_n, STx_n) \leq a_n - a_{n+1}$$

which implies that $\lim_{n\to\infty} d_\mu(Tx_n, STx_n) = 0 \; \forall \mu \in D$. Therefore $\lim_{n\to\infty} STx_n = u$. Similarly, we show that $\lim_{n\to\infty} TSx_n = u$. Now we can show that u is a common fixed point of T and S. We have $\lim_{n\to\infty} STx_n = u$ and $\lim_{n\to\infty} Tx_n = u$. Therefore since the graph of S is sequentially closed, we conclude that $Su = u$. On the other hand, we have $\lim_{n\to\infty} TSx_n = u$ and $\lim_{n\to\infty} Sx_n = u$. Therefore since the graph of T is sequentially closed, we obtain $Tu = u$. ☑

Setting $\lambda = \mu = \beta$ and $S = Id_X$, we have the following result which gives a generalization of our earlier remark.

Corollary 2.1. *Let (X, θ) be a sequentially complete F-type topological space generated by the family $\{d_\lambda, \lambda \in D\}$. Let $k : D \longrightarrow]0, +\infty[$ be a nonincreasing function and $\phi : X \longrightarrow \mathbb{R}^+$ be a function. Let T be a selfmapping of X such that*

(1) $d_\lambda(x, Tx) \leq k(\lambda)[\phi(x) - \phi(Tx)], \;\; \forall \lambda \in D, \;\; \forall x \in X$;
(2) T has a sequentially closed graph.

Then T has a fixed point in X.

Taking $\lambda = \mu = \beta$ and $T = Id_X$, we get the following result.

Corollary 2.2. *Let (X, θ) be a sequentially complete F-type topological space generated by the family $\{d_\lambda, \lambda \in D\}$. Let $k : D \longrightarrow]0, +\infty[$ be a nonincreasing function and $\phi : X \longrightarrow \mathbb{R}^+$ be a function. Let S be a surjective selfmapping of X such that:*

(1) $d_\lambda(x, Sx) \leq k(\lambda)[\phi(Sx) - \phi(x)], \;\; \forall \lambda \in D, \; \forall x \in X$;
(2) S has a sequentially closed graph.

Then S has a fixed point in X.

In the setting of metric space, we have the following

Corollary 2.3. *Let T and S be two selfmappings of a complete metric space (X, d). Let $\phi : X \longrightarrow \mathbb{R}^+$ be a function such that:*

(1) $\max\{d(Sx, Tx), d(Tx, STx), d(Sx, TSx)\} \leq \phi(Sx) - \phi(Tx), \;\; \forall x \in X$;
(2) $TX \subset SX$;

(3) T and S have a sequentially closed graphs.
Then T and S have a common fixed point in X.

Proof. Take an arbitrary directed set D and let

$$d_\lambda(x, y) = d(x, y) \quad \forall x, y \in X, \ \ \forall \lambda \in D.$$

Taking $k(\lambda) = 1$ for all $\lambda \in D$, it is easy to see that all conditions of Theorem 2.1 are satisfied and the conclusion follows from this theorem immediately. ☑

As an example let $X = [0, +\infty[$ and consider $S, T : X \longrightarrow X$ defined as follows:

$$Sx = \begin{cases} \tan x & \text{if } x \in [0, \pi/2[, \\ x & \text{if } x \in [\pi/2, +\infty[\end{cases}$$

and $Tx = \arctan x, \ \ \forall x \in X$.

It is easy to see that T and S have closed graphs and $TX \subset SX$. Furthermore

$$|Sx - Tx| = \begin{cases} \tan x - \arctan x & \text{if } x \in [0, \pi/2[, \\ x - \arctan x & \text{if } x \in [\pi/2, +\infty[; \end{cases}$$

$$|Sx - TSx| = \begin{cases} \tan x - x & \text{if } x \in [0, \pi/2[, \\ x - \arctan x & \text{if } x \in [\pi/2, +\infty[\end{cases}$$

and

$$|Tx - STx| = x - \arctan x \ \ \forall x \in X.$$

Therefore

$$\max\{|Sx - Tx|, |Tx - STx|, |Sx - TSx|\} = \begin{cases} \tan x - \arctan x & \text{if } x \in [0, \pi/2[, \\ x - \arctan x & \text{if } x \in [\frac{\pi}{2}, +\infty[. \end{cases}$$

Consider the function ϕ defined on X by

$$\phi(x) = 2x.$$

We have

$$\phi(Sx) - \phi(Tx) = \begin{cases} 2(\tan x - \arctan x) & \text{if } x \in [0, \pi/2[, \\ 2(x - \arctan x) & \text{if } x \in [\pi/2, +\infty[. \end{cases}$$

Subsequently, we have

$$\max\{|Sx - Tx|, |Tx - STx|, |Sx - TSx|\} \leq \phi(Sx) - \phi(Tx), \quad \forall x \in X.$$

Therefore all conditions of Theorem 2.1 are verified and $T0 = S0 = 0$.

Corollary 2.4. *Let (X, θ) be a Hausdorff sequentially complete topological vectorial space and $\{U_\lambda, \lambda \in D\}$ be a balanced neighborhood base of 0 in X. Let $\phi : X \longrightarrow \mathbb{R}^+$ be a function and $k : D \longrightarrow]0, +\infty[$ be a nonincreasing function. Suppose further that two mappings $T, S : X \longrightarrow X$ satisfy the following conditions:*

(1) $\psi(x) = \phi(Sx) - \phi(Tx) \geq 0, \quad \forall x \in X$;

(2) for all $x \in X$ and for all $(\lambda, \mu, \beta) \in D^3$

$$\begin{cases} Tx - Sx & \in \max\{k(\lambda), k(\mu), k(\beta)\}\psi(x)U_\lambda, \\ Sx - TSx & \in \max\{k(\lambda), k(\mu), k(\beta)\}\psi(x)U_\mu, \\ Tx - STx & \in \max\{k(\lambda), k(\mu), k(\beta)\}\psi(x)U_\beta. \end{cases}$$

(3) $TX \subset SX$.

(4) T and S have a sequentially closed graphs.

Then T and S have a common fixed point in X.

Proof. As in [4], we define a partial order on D as follows:

$$\lambda \prec \mu \Longleftrightarrow U_\mu \subset U_\lambda.$$

Then X is an F-type topological space generated by the family $\{d_\lambda : \lambda \in D\}$ where

$$d_\lambda(x, y) = inf\{t > 0 | x - y \in tU_\lambda\}, \ \ \forall x, y \in X, \ \forall \lambda \in D.$$

Therefore $\forall(\lambda, \mu, \beta) \in D^3$ and $\forall x \in X$, we have the following:

$$\max\{d_\lambda(Sx, Tx), d_\mu(Tx, STx), d_\beta(Sx, TSx)\} \leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx) - \phi(Tx)]$$

The conclusion follows immediately from Theorem 2.1. ☑

3. **Applications**

Let $(D_1, \prec_{D_1})$ and $(D_2, \prec_{D_2}))$ be directed sets.

Theorem 3.1. *Let (E, θ_1) (resp. (F, θ_2)) be a sequentially complete F-type topological space generated by the family $\{d_\lambda, \lambda \in D_1\}$(resp. $\{d_\mu, \mu \in D_2\}$). Let $v : E \longrightarrow F$ be a function with sequentially closed graph . Let $k_1 : D_1 \longrightarrow \mathbb{R}^+$ and $k_2 : D_2 \longrightarrow \mathbb{R}^+$ be two nonincreasing functions. Let $\phi : E \longrightarrow \mathbb{R}^+$ and $\psi : F \longrightarrow \mathbb{R}^+$ be two arbitrary functions. Let T and S be selfmappings of E with sequentially closed graphs such that $TE \subset SE$ and*

$$\begin{aligned} \max\{&d_{\lambda_1}(Sx, Tx) + d_{\mu_1}(v(Sx), v(Tx)), d_{\lambda_2}(Sx, TSx) \\ &+ d_{\mu_2}(v(Sx), v(TSx)), d_{\lambda_3}(Tx, STx) + d_{\mu_3}(v(Tx), v(STx))\} \\ &\leq \max\{k_1(\lambda_1), k_1(\lambda_2), k_1(\lambda_3)\}[\phi(Sx) - \phi(Tx)] \\ &\quad + \max\{k_2(\mu_1), k_2(\mu_2), k_2(\mu_3)\}[\psi(v(Sx)) - \psi(v(Tx))], \end{aligned}$$

for all $x \in E$ and for all $(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3) \in D_1^3 \times D_2^3$. Then T and S have a common fixed point in E.

Proof. We define on $D = D_1 \times D_2$ a relation "$\prec_D$" as follows: $(\lambda_1, \mu_1) \prec_D (\lambda_2, \mu_2) \Longleftrightarrow \lambda_1 \prec_{D_1} \lambda_2$ and $\mu_1 \prec_{D_2} \mu_2$. For all $(\lambda, \mu) \in D$, we consider the function $\psi_{\lambda,\mu} : E \times E \longrightarrow \mathbb{R}^+$ defined by

$$\psi_{\lambda,\mu}(x, y) = d_\lambda(x, y) + d_\mu(v(x), v(y)).$$

Next we show that $\psi_{\lambda,\mu}$ is a quasi-metric on E:

(1) $\psi_{\lambda,\mu}(x,y)=0 \Longrightarrow d_\lambda(x,y)=0 \Longrightarrow x=y.$

(2) $\psi_{\lambda,\mu}(x,y)=\psi_{\lambda,\mu}(y,x)$, $\forall(\lambda,\mu)\in D.$

(3) Let $(\lambda,\alpha,\mu,\beta)\in {D_1}^2\times {D_2}^2$ such that $(\lambda,\mu)\prec_D(\alpha,\beta)$.Then, $\forall(x,y)\in E^2$, $d_\lambda(x,y)\le d_\alpha(x,y)$ and $d_\mu(v(x),v(y))\le d_\beta(v(x),v(y))$. Hence $\psi_{\lambda,\mu}(x,y)\le \psi_{\alpha,\beta}(x,y)$.

(4) Let $(\lambda,\mu)\in D_1\times D_2$. Then, $\exists(\alpha,\beta)\in D_1\times D_2$, such that $(\lambda,\mu)\prec_D(\alpha,\beta)$, $d_\lambda(x,y)\le d_\alpha(x,z)+d_\alpha(z,y)$ and $d_\mu(v(x),v(y))\le d_\beta(v(x),v(z))+d_\beta(v(z),v(y))$. Therefore, $\forall(\lambda,\mu)\in D_1\times D_2$, $\exists(\alpha,\beta)\in D_1\times D_2$, such that $(\lambda,\mu)\prec_D(\alpha,\beta)$ and $\psi_{\lambda,\mu}(x,y)\le\psi_{\alpha,\beta}(x,z)+\psi_{\alpha,\beta}(z,y)$, $\forall(x,y,z)\in E^3$.

Now we show that E, generated by the family $\{\psi_{\lambda,\mu}:(\lambda,\mu)\in D\}$ and which we denote by E', is sequentially complete. Let (x_n) be a cauchy sequence of E'. Then (x_n) (resp. $v(x_n)$) is a cauchy sequence in (E,θ_1) (resp. in (F,θ_2)), which implies that there exists $(x,y)\in E\times F$ such that $\lim\limits_{n\to\infty}x_n=x\in E$ and $\lim\limits_{n\to\infty}v(x_n)=y$. As the function v has a closed graph, we have $v(x)=y$. So, (x_n) converges in E' to x. Therefore E' is sequentially complete.

Next, it is clear that

$$\begin{aligned}&\max\{k_1(\lambda_1),k_1(\lambda_2),k_1(\lambda_3),k_2(\mu_1),k_2(\mu_2),k_2(\mu_3)\}\\&=\max\{\max\{k_1(\lambda_1),k_1(\lambda_2),k_1(\lambda_3)\},\max\{k_2(\mu_1),k_2(\mu_2),k_2(\mu_3)\}\}\\&=\max\{\max\{k_1(\lambda_1),k_2(\mu_1)\},\max\{k_1(\lambda_2),k_2(\mu_2)\},\max\{k_1(\lambda_3),k_2(\mu_3)\}\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\max\{\psi_{\lambda_1,\mu_1}(Sx,Tx),\psi_{\lambda_2,\mu_2}(Sx,TSx),\psi_{\lambda_3,\mu_3}(Tx,STx)\}\\&\qquad\le\max\{k(\lambda_1,\mu_1),k(\lambda_2,\mu_2),k(\lambda_3,\mu_3)\}[f(Sx)-f(Tx)]\end{aligned}$$

where $f:E\longrightarrow\mathbb{R}^+$ and $k:D_1\times D_2\longrightarrow]0,+\infty[$ are defined by

$$f(x)=\phi(x)+\psi(v(x)),\quad\forall x\in E$$

and

$$k(\lambda,\mu)=\max\{k_1(\lambda),k_2(\mu)\},\quad\forall\lambda,\mu)\in D_1\times D_2.$$

It is clear that the function k is nonincreasing. In view of the Theorem 2.1, the conclusion follows immediately. ☑

When $\lambda_1=\lambda_2=\lambda_3=\lambda$, $\mu_1=\mu_2=\mu_3=\mu$ and $S=Id_E$ (resp. $T=Id_E$) , we get the following results.

Corollary 3.1. *Let (E,θ_1) (resp. (F,θ_2)) be a sequentially complete F-type topological space generated by the family $\{d_\lambda,\lambda\in D_1\}$(resp. $\{d_\mu,\mu\in D_2\}$). Let $v:E\longrightarrow F$ be a function. Let $k_1:D_1\longrightarrow\mathbb{R}^+$ and $k_2:D_2\longrightarrow\mathbb{R}^+$ be two* *nonincreasing functions. Let* $\phi : E \longrightarrow \mathbb{R}^+$ *and* $\psi : F \longrightarrow \mathbb{R}^+$ *be two arbitrary functions. Let* T *be a selfmapping of* E *such that:*

(1) $$d_\lambda(x, Tx) + d_\mu(v(x), v(Tx)) \le k_1(\lambda))[\phi(x) - \phi(Tx)] + k_2(\mu)[\psi(v(x)) - \psi(v(Tx))], \quad \forall x \in E,\ \forall(\lambda, \mu) \in D_1 \times D_2;$$

(2) T *and* v *have sequentially closed graphs.*

Then T *has a fixed point.*

Corollary 3.2. *Let* (E, θ_1) *(resp.* (F, θ_2)*) be a sequentially complete* F*-type topological space generated by the family* $\{d_\lambda, \lambda \in D_1\}$*(resp.* $\{d_\mu, \mu \in D_2\}$*). Let* $v : E \longrightarrow F$ *be a function. Let* $k_1 : D_1 \longrightarrow \mathbb{R}^+$ *and* $k_2 : D_2 \longrightarrow \mathbb{R}^+$ *be two nonincreasing functions. Let* $\phi : E \longrightarrow \mathbb{R}^+$ *and* $\psi : F \longrightarrow \mathbb{R}^+$ *be two arbitrary functions. Let* S *be a surjective selfmapping of* E *such that:*

(1) $$d_\lambda(x, Sx) + d_\mu(v(x), v(Sx)) \le k_1(\lambda))[\phi(Sx) - \phi(x)] + k_2(\mu)[\psi(v(Sx)) - \psi(v(x))], \quad \forall x \in E,\ \forall(\lambda, \mu) \in D_1 \times D_2;$$

(2) S *and* v *have a sequentially closed graphs.*

Then S *has a fixed point.*

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