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# Important result on the first passage time and its integral functional for a certain diffusion process

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Abstract. In this paper we consider the general birth and death process and study some important results on the first passage time and its integral functional, in particular we derive the asymptotic distribution of the integral functional. Finally we derive the asymptotic regression equation of first passage time on its integral functional.

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Resumen. En este artículo se consideran el proceso general de nacimiento y muerte y se estudian algunos resultados importantes entre el primer tiempo de paso y su funcional integral. En particular, se deriva la distribución asintótica del funcional integral. Finalmente, se deriva la correspondiente ecuación de regresión asintótica del primer tiempo de cruce sobre su funcional integral.

## 1. Introduction

Many important results related to random variable and their integrals have been studied from point of views of different authors. For example, Puri (1966), (1968) has investigated the joint distribution of the number of survivors $X(t)$ in the process and the associated integral $Y(t) = \int_0^t X(z)dz$. In particular he has obtained limiting results as $t \to \infty$. McNeil (1970) has derived the distribution of the integral functional $W_x \int_0^{\tau_x} g(X(t))dt$, where $\tau_x$ is the first passage time to the origin in general birth-death process with $X(0) = x$ and $g$ is an arbitrary function. Also Gani and McNeil (1971) have studied the joint distribution of both $\{X(t), Y(t)\}$ and $\{T_x, W_x\}$ for general birth-death processes.

Also in particular, they have studied the joint distribution of $\{T_x, W_x\}$ for a diffusion process whose backward equation is given by

$$\frac{df}{dt} = \beta x^{\alpha} \frac{d^2 f}{dx^2} \ ,$$

where

$$f(y; x, t) = P(y < X(t) \leq y + dy \mid X(0) = x) \ .$$

Functionals of the form arise naturally in traffic and storage theory.

In the present paper, we consider the most general diffusion process in which the drift condition is also exists. We derive the asymptotic $W_x$ distribution of and some asymptotic results on the joint distribution of $\{T_x, W_x\}$. Also, we derive the asymptotic regression equation (prediction equation) of $T_x$ on $W_x$. We follow the analysis of Gani and McNeil (1971).

## 2. Joint Distribution of $\{T_x, W_x\}$

Consider the birth-death diffusion process whose backward Kolmogorov equation is

$$\frac{df}{dt} = ax\frac{df}{dx} + \beta x^{\alpha}\frac{d^2 f}{dx^2} \ , \tag{1}$$

where

$$f(y;x,t) = P(y < X(t) \leq y + dy \mid X(0) = x) \ .$$

In order to insure that the first passage time is finite, we take $\alpha < 2$, $\beta$ is the diffusion condition and $a$ is the drift condition.

Now let

$$M_x(\theta,\phi) = E(\mathrm{e}^{-\theta T_x - \phi W_x}) \tag{2}$$

where $R_e(\theta)$, $R_e(\phi) > 0$. Note that $M_x$ is the joint Laplace transform of $\{T_x, W_x\}$. This satisfies the equation

$$(\theta + \phi x)M_x(\theta,\phi) = axM_x'(\theta,\phi) + \beta x^{\alpha}M_x''(\theta,\phi) \ , \tag{3}$$

where $M_x'$ and $M_x''$ indicate first and second derivative with respect to $x$ respectively. This can be used to obtain the asymptotic regression equation of $T_x$ on $W_x$, when $g(x) = x$.

Define

$$K_x(\phi) := -\frac{d}{d\theta}M_x(\theta,\phi) = \int_0^{\infty} \mathrm{e}^{-\phi u}E[T_x \mid W_x = u]dP(W_x \leq u) \ . \tag{4}$$

Letting $\theta = 0$, then $K_x(\phi)$ satisfies the differential equation

$$K_x''(\phi) + \left(\frac{a}{\beta}\right)x^{1-\alpha}K_x' - \left(\frac{\phi}{\beta}\right)x^{1-\alpha}K_x(\phi) = -\frac{1}{\beta}x^{-\alpha}M_x(0,\phi) \ . \tag{5}$$

Also, using $\theta = 0$ and the boundary conditions $M_0(0,\phi) = 1$ and $M_\infty(0,\phi) = 0$ in equation (3) , we get

$$M_x(0,\phi) = \left(1 + \frac{3a}{\sqrt{a^2 + 4\beta\phi}}\right)\mathrm{e}^{\frac{a}{2\beta} + \sqrt{\frac{a^2}{4\beta^2} + \frac{\phi}{\beta}}} +$$

$$\frac{3a}{\sqrt{a^2 + 4\beta\phi}}\mathrm{e}^{\frac{a}{2\beta} + \sqrt{\frac{a^2}{4\beta^2} + \frac{\phi}{\beta}}} \ .$$

Also the solution of the differential equation in (5), is given by

$$K_x(\phi) = \left(1 - \frac{\frac{1}{2\beta}x^1 M_x - \frac{a}{2\beta}}{\sqrt{\frac{a^2}{4\beta^2} + \frac{4\phi}{\beta}}}\right) \mathrm{e}^{\frac{a}{2\beta}\sqrt{\frac{a^2}{4\beta^2}+\frac{4\phi}{\beta}}} +$$

$$\frac{\frac{1}{2\beta}x^1 M_x - \frac{a}{2\beta}}{\sqrt{\frac{a^2}{4\beta^2} + \frac{4\phi}{\beta}}}\ \mathrm{e}^{\frac{a}{2\beta}\sqrt{\frac{a^2}{4\beta^2}+\frac{4\phi}{\beta}}} ,$$

where $M_x$ is the solution in equation (6).

The inversion of equation (6), using the behavior of the Laplace transforms

$$\lim_{\phi\to 0} \phi M_x(0,\phi) = \lim_{x\to\infty} f_{W_x}(u) ,$$

where $f_{W_x}(u)$ is the probability density function of $W_x$, for large $x$. Now after some manipulation we get,

$$f_{W_x} = \begin{cases} \dfrac{a^2 x}{4\beta}\,\mathrm{e}^{-\frac{a^2 x}{4\beta}u} & \text{if} \quad u \geq 0 \\ 0 \quad \text{if} & u < 0 \end{cases} \tag{6}$$

This implies that $W_x$ has an asymptotic exponential distribution with parameter $\dfrac{a^2 x}{4\beta}$.

Similarly, the inversion of equation (7), using the asymptotic behavior of the Laplace transforms

$$\lim_{\phi\to 0} \phi K_x(\phi) = \lim_{x\to\infty} E\left[T_x \mid W_x = u\right] f_{W_x}(u) ,$$

and after some manipulation for large values of $x$, we get

$$E\left[T_x \mid W_x = u\right] f_{W_x}(u) \approx \frac{a^2 x}{4\beta}\, u\, \mathrm{e}^{\frac{a^2}{16}u} \quad u \geq 0 . \tag{7}$$

Now dividing equation (9) into equation (8), we finally obtain the asymptotic regression equation of $T_x$ on $W_x$ . I.e.

$$E\left[T_x \mid W_x = u\right] \approx u\, \mathrm{e}^{\left(\frac{a^2}{4\beta} - \frac{a^2}{16}\right)} u , \quad u \geq 0 , \tag{8}$$

for large value of $x$ and $\alpha = 0$.

## References

[1] Gani,J. & McNeil, D.R. *Joint Distributions of Random Variables and their Integrals for Certain Birth-Death and Diffusion Process.* Adv. Appl. Prob. **3**, (1971), págs. 339–325.

[2] McNeil, D.R. *Integral functionals of birth and death processes and related limiting distributions.* Ann. Math. Statist. **41**, (1970), págs. 480–485.

[3] Puri, Prems S. *On the homogeneous birth and death process and its integral.* Biometrika **53**,(1966), págs. 61–71.

[4] Puri, Perm S. *Some further results on the birth and death process and its integral.* Proc. Camb. Phil. Soc. **64**, (1968), págs. 141–154.

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