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Zeta functions of singular curves over finite fields

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Abstract. Let X be a complete, geometrically irreducible, algebraic curve defined over a finite field $\mathbb{F}_q$ and let $\zeta(X,t)$ be its zeta function [Ser1]. If X is a singular curve, two other zeta functions exist. The first is the Dirichlet series $Z(\mathrm{Ca}(X),t)$ associated to the effective Cartier divisors on X; the second is the Dirichlet series $Z(\mathrm{Div}(X),t)$ associated to the effective divisors on X. In this paper we generalize F. K. Schmidt's results on the rationality and functional equation of the zeta function $\zeta(X,t)$ of a non-singular curve to the functions $Z(\mathrm{Ca}(X),t)$ and $Z(\mathrm{Div}(X),t)$ by means of the singular Riemann-Roch theorem.

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In [G] Galkin introduced a zeta function for orders in global fields which coincides with Schmidt's zeta function in the case of a non-singular curve but satisfies a functional equation only in the Gorenstein case. Green [Gr] obtained a functional equation by slightly modifying the definition of the zeta function of Galkin. However, this new zeta function is not uniquely determined by the curve. Finally, Stöhr [Sto] defines the zeta function $Z(\mathrm{Div}(X),t)$ which coincides with the zeta function of Galkin in the Gorenstein case and always satisfies a functional equation ([Sto, p. 133]). In [Z] we studied the local factors of the zeta function $Z(\mathrm{Ca}(X),t)$ using an adelic approach a lá Tate. As a consequence we obtained the rationality and a global functional equation of

$Z(\mathrm{Ca}(X),t)$([Z, thm. 5.3]). In this paper we generalize F. K. Schmidt's results on the rationality and functional equation of the zeta function $\zeta(X,t)$ of a non-singular curve to the zeta functions $Z(\mathrm{Ca}(X),t)$ and $Z(\mathrm{Div}(X),t)$ using an approach based on the singular Riemann-Roch theorem. This allows us to find the residue at $s=0$ of the zeta function $Z(\mathrm{Ca}(X),q^{-s})$ and $Z(\mathrm{Div}(X),t)$ (see thms. 2.1, 2.2, 3.1).

1. Preliminaries

In this section we present the basic facts about singular curves and zeta functions (cf. e.g. [Ros1], [Ros2], [Ser2], [Sto]).

Let X be a complete, geometrically irreducible, algebraic curve defined over a field k. We denote by $K=k(X)$ the function field of X over k, g will stand for its arithmetic genus, $\tilde{g}$ for its geometric genus, $\tilde{X}$ will be the normalization of X over k (also named the non-singular model of X) and $\pi:\tilde{X}\longrightarrow X$ will denote the normalization map. The regular surjective map π is birational. In particular, the function field of the smooth curve $\tilde{X}$ is K/k.

By a divisor of X we mean a coherent fractional ideal sheaf or, equivalently, a formal product

$$\mathcal{A}=\prod_{P\in X}\mathcal{A}_P,$$

where for each point P the P-component $\mathcal{A}_P$ (i.e. the stalk of $\mathcal{A}$ at P) is a non-zero fractional ideal of $\mathcal{O}_P$ and $\mathcal{A}_P=\mathcal{O}_P$ for all but finitely many points. Given two divisors $\mathcal{A}$ and $\mathcal{B}$, we define pointwise the product divisor $\mathcal{AB}$ and the quotient divisor $\mathcal{A}:\mathcal{B}$ by

$$(\mathcal{AB})_P:=\mathcal{A}_P\mathcal{B}_P,$$

and

$$(\mathcal{A}:\mathcal{B})_P:=\{z\in K\mid z\mathcal{B}_P\subseteq\mathcal{A}_P\}.$$

We denote by $\mathrm{Div}(X)$ the set of divisors of X. A divisor $\mathcal{A}$ is called a locally principal or a Cartier divisor if each $\mathcal{A}_P$ is a principal fractional ideal. The Cartier divisors form a multiplicative group having the structure divisor

$$\mathcal{O}:=\prod_{P\in X}\mathcal{O}_P,$$

as the identity. We denote by $\mathrm{Ca}(X)$ the group of the Cartier divisors. We define a partial order on $\mathrm{Div}(X)$ by

$$\mathcal{A}\leqslant\mathcal{B}\Longleftrightarrow\mathcal{A}_P\subseteq\mathcal{B}_P\quad\text{for all}\quad P\in X.$$

A divisor $\mathcal{A}$ is called effective if $\mathcal{A}\geq\mathcal{O}$. For our purposes, it is more convenient to work with the above ordering than with the usual one.

The degree of a divisor is uniquely defined by the following properties:

(i) $\deg(\mathcal{O}) = 0$.

(ii) $\deg(\mathcal{B}) - \deg(\mathcal{A}) = \sum_{P\in X} \dim_k(\mathcal{B}_P/\mathcal{A}_P)$ whenever $\mathcal{B} \geq \mathcal{A}$.

We observe that in general $\deg(\mathcal{A}\mathcal{B}) \neq \deg(\mathcal{A})\deg(\mathcal{B})$ (cf. [Ha, sect. 1]). However, if at least one of the divisors $\mathcal{A}$ or $\mathcal{B}$ is a Cartier divisor then the equality holds.

For each non-zero rational function $z \in K^*$, let $\operatorname{div}(z)$ be its principal divisor, i.e.,

$$\operatorname{div}(z) := \prod_{P\in X} z^{-1}\mathcal{O}_P.$$

We denote by Prin (X) the subgroup of principal divisors of X.

Let

$$L(\mathcal{A}) := \bigcap_{P\in X} \mathcal{A}_P = \{z \in K \mid \operatorname{div}(z)\mathcal{A} \geqq \mathcal{O}\}$$

be the $k-$vector space of global sections of $\mathcal{A}$ (also denoted by $H^0(X,\mathcal{A})$). We denote the dimension of the above $k-$vector space by $\ell(\mathcal{A})$ (also denoted by $h^0(X,\mathcal{A})$).

The Riemann-Roch theorem for function fields was generalized by Rosenlicht to curves with singularities (cf. e.g. [Ros1], [Sto]).

Theorem 1.1. *(Riemann-Roch theorem for singular curves). Each divisor $\mathcal{A}$ of X satisfies*

$$\ell(\mathcal{A}) = \deg(\mathcal{A}) + 1 - g + \ell(\mathcal{C} : \mathcal{A}),$$

where $\mathcal{C}$ denotes the canonical divisor of X.

The local duality theorem also generalizes to singular curves (cf. e.g. [Sto, thm. 1.5])

Theorem 1.2. *(Local duality). Let $\mathcal{A}$, $\mathcal{B}$ be divisors of X such that $\mathcal{A} \geqq \mathcal{B}$. Then for each point P we have the k-isomorphism*

$$(\mathcal{C}_P : \mathcal{B}_P)/(\mathcal{C}_P : \mathcal{A}_P) \xrightarrow[\cong]{} Hom_k(\mathcal{A}_P/\mathcal{B}_P, k).$$

As a consequence of the theorem of local duality, we get the reciprocity formula (cf. e.g. [Sto, sect. 1.7]).

Corollary 1.3. (Reciprocity formula). For each divisor $\mathcal{A}$,

$$\mathcal{C} : (\mathcal{C} : \mathcal{A}) = \mathcal{A}.$$

From now on, we understand point to mean closed point. Let P be a point of X and $\mathcal{O}_P$ be the local ring of X at P. Let $Q_1, Q_2, \ldots, Q_d$ be the points of $\tilde{X}$ lying over P, i.e., $\pi^{-1}(P) = \{Q_1, \ldots, Q_d\}$, and let $\mathcal{O}_{Q_1}, \ldots, \mathcal{O}_{Q_d}$ be the corresponding local rings at these points. Since the function fields of X and $\tilde{X}$ are the same and $\widetilde{X}$ is a non-singular curve, the local rings $\mathcal{O}_{Q_1}, \ldots, \mathcal{O}_{Q_d}$ are valuation rings of K/k over $\mathcal{O}_P$. The integral closure of $\mathcal{O}_P$ in K is $\widetilde{\mathcal{O}}_P = \bigcap_{Q\in\pi^{-1}(P)} \mathcal{O}_Q$.

The degree of singularity of X at P is defined as

$$\delta_P = \dim_k(\tilde{\mathcal{O}}_P/\mathcal{O}_P).$$

By Theorem. 1 in [Ros1], $\delta_P < \infty$. The total degree of singularity of X is defined as

$$\delta = \sum_{P\in X} \delta_P.$$

The degree of singularity δ_P remains invariant under completion. The total degree of singularity δ remains invariant under separable constant extensions (cf. [Ros1, thm. 12]).

We recall that the genus formula of a complete, geometrically irreducible, algebraic curve X is (see [H])

$$g = \tilde{g} + \sum_{P\in X} \delta_P,$$

where $\tilde{g}$ is the geometric genus of X.

The conductor ideal $\mathcal{F}_P$ of $\widetilde{\mathcal{O}}_P$ in $\mathcal{O}_P$ is defined as

$$\mathcal{F}_P = \{x \in K \mid x\widetilde{\mathcal{O}}_P \subseteq \mathcal{O}_P\}.$$

This ideal is the largest common ideal of $\widetilde{\mathcal{O}}_P$ and $\mathcal{O}_P$. Furthermore, since $\delta_P < \infty$, then $\mathcal{F}_P \neq 0$. On the other hand, $\widetilde{\mathcal{O}}_P$ is a Dedekind domain with a finite number of maximal ideals. Thus, $\mathcal{F}_P$ is an $\widetilde{\mathcal{O}}_P$-principal ideal. The degree of the conductor ideal $\mathcal{F}_P$ is defined as

$$\deg \mathcal{F}_P := \dim_k(\widetilde{\mathcal{O}}_P/\mathcal{F}_P).$$

The degree of $\mathcal{F}_P$ is also invariant under completions and under separable constant extensions.

We say that a local ring $\mathcal{O}_P$ is Gorenstein if $\deg \mathcal{F}_P = 2\delta_P$. An algebraic curve is called a Gorenstein curve if all its local rings are Gorenstein.

The dualizing sheaf ω_X is locally free of rank 1 if and only if X is a Gorenstein curve (cf. [A-K, chap. VIII, prop. 1.16]). We denote by $\mathcal{C}$ the canonical divisor of a complete, geometrically irreducible, algebraic curve X. Then $\mathcal{C}_P$ is a principal ideal if and only if $\omega_{X,P}$ is free of rank 1. Summarizing, we have the following result of Rosenlicht.

Theorem 1.4. *Let X be a complete, geometrically irreducible, algebraic curve defined over k. The curve X is Gorenstein if and only if its canonical divisor is a Cartier divisor.*

We denote by $\mathrm{Ca}^0(X)$ the subgroup of degree zero divisors of $\mathrm{Ca}(X)$, by $\mathrm{Pic}\,(X)$ the quotient group $\mathrm{Ca}(X)/\mathrm{Prin}\,(X)$ and by $\mathrm{Pic}^{\,0}(X)$ the quotient group

$$\mathrm{Ca}^0(X)/\mathrm{Prin}\,(X).$$

Let $\bar{k}$ be an algebraic closure of k. $\mathrm{Pic}^{\,0}(X \bigotimes_k \bar{k})$ is the *generalized Jacobian* of $X \bigotimes_k \bar{k}$ (see [Ser2], [Ros2]).If k is a finite field $\mathrm{Pic}^{\,0}(\tilde{X})$ is the group of rational points of $\mathrm{Pic}^{\,0}(X \bigotimes_k \bar{k})$. The generalized Jacobian can be enlarged to a projective algebraic scheme, called the *compactified Jacobian*, which parametrizes the equivalence classes of zero degree divisors of $X \bigotimes_k \bar{k}$. The rational points of the compactified Jacobian correspond bijectively to equivalence classes of zero degree divisors of X.

2. Rationality and functional equation of $Z(\mathrm{Ca}(X), t)$

In this section we present a generalization of results of F. K. Schmidt on the rationality and on the functional equation of $\zeta(X,t)$ to the function $Z(\mathrm{Ca}(X),t)$ (cf. [Sti, chap. V]). The zeta function $Z(\mathrm{Ca}(X),t)$ is defined as follows:

$$Z(\mathrm{Ca}(X),t) := \sum_{\mathcal{A} \geq \mathcal{O}} t^{\deg(\mathcal{A})},$$

where $\mathcal{A}$ runs through all Cartier divisors on X and $t = q^{-s}$, $s \in \mathbb{C}, Re(s) > 0$. We shall see later on that $Z(\mathrm{Ca}(X),t)$ converges analytically and uniformly on the semiplane $Re(s) > 0$. This zeta function decomposes formally into an Euler product

$$Z(\mathrm{Ca}(X),t) = \prod_{P \in X} Z(\mathcal{O}_P, t) = \prod_{P \in X} \left(\sum_{I \supseteq \mathcal{O}_P} t^{\dim_k (I/\mathcal{O}_P)} \right), \qquad (2.1)$$

and where I runs through all fractional principal ideals of $\mathcal{O}_P$, such that $I \supsetneqq \mathcal{O}_P$. By comparing product (2.1) with $\zeta(\tilde{X},t) = Z(\mathrm{Ca}(\tilde{X}),t)$, we see that it converges on the semiplane $Re(s) > 0$. On the other side, the correspondence $I \longrightarrow I^{-1}$ is a bijection between the principal fractional ideals containing $\mathcal{O}_P$ and the principal ideals contained in $\mathcal{O}_P$. Furthermore, this correspondence preserves degrees, therefore, we have that

$$Z(\mathcal{O}_P, t) = \sum_{I \subseteq \mathcal{O}_P} t^{\dim_k (\mathcal{O}_P/I)}.$$

We observe that if P is a smooth point of X, then $Z(\mathcal{O}_P,t) = (1 - t^{\deg(P)})^{-1}$, where $\deg(P)$ is the cardinality of the residue field of P.

Theorem 2.1. *Let X be a complete, geometrically irreducible algebraic curve over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(\mathrm{Ca}(X), t)$ is a rational function. More precisely,*

$$Z(\mathrm{Ca}(X), t) = \frac{L(\mathrm{Ca}(X), t)}{(1-t)(1-qt)},$$

where $L(\mathrm{Ca}(X), t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $2g$, and $L(\mathrm{Ca}(X), 1) = \#\mathrm{Pic}^0(X)$, i.e. $L(\mathrm{Ca}(X), 1)$ is the number of rational points of the generalized Jacobian of $X \bigotimes_k \bar{k}$.

Proof.
In order to generalize the proof of F.K. Schmidt we need to prove the following three claims.

Claim 1.

$$\#\mathrm{Pic}^0(X) < \infty. \tag{2.2}$$

Claim 2. There exists a Cartier divisor of degree 1.

Claim 3. For any integer d, the number of divisor classes in $\mathrm{Pic}\,(X)$ *of degree d is independent of d and is equal to the cardinality of* $\mathrm{Pic}^0(X)$.

After this, we can follow the argument of F.K. Schmidt as in [Sti, chap. V]. It is important to note that in this argument it is irrelevant to know whether a canonical divisor is a Cartier divisor or not. We denote by $[\mathcal{A}]$ the linear equivalence class of a divisor $\mathcal{A}$ and by $\pi^*(\mathcal{A})$ the pullback of $\mathcal{A}$.

To establish claim 1, we first observe that $\mathrm{Pic}^0(\tilde{X})$ is a finite group (more precisely, the group of divisor classes of degree zero of K(cf. [Sti, chap. V]), so, it is sufficient to show that the kernel of the morphism

$$\begin{aligned} \pi_0^* : \mathrm{Pic}^0(X) &\longrightarrow \mathrm{Pic}^0(\tilde{X}) \\ \pi_0^*([\mathcal{A}]) &\longmapsto [\pi^*(\mathcal{A})], \end{aligned}$$

is a finite group.
We note that $[\mathcal{A}] \in \ker(\pi_0^*)$ if and only if $\pi^*(\mathcal{A})$ is a principal divisor of $\tilde{X}$, i.e.

$$\pi^*(\mathcal{A})_Q = z^{-1}\mathcal{O}_Q = \mathcal{A}_P\mathcal{O}_Q, \tag{2.3}$$

for some $z \in K^*$ and every point Q of $\widetilde{X}$ lying over P. Therefore

$$\mathcal{F}_P \subseteqq z\mathcal{A}_P \subseteqq \tilde{\mathcal{O}}_P.$$

Since $\deg \mathcal{F}_P < \infty$, the above relation implies that the kernel of π_0^* contains only a finite number of linear equivalence classes. The equivalence class of an effective Cartier divisor $\mathcal{A}$ contains $\frac{q^{\ell(\mathcal{A})-1}}{q-1}$ linearly equivalent Cartier divisors. Therefore the kernel of π_0^* is a finite group.

The second claim can be reduced to the non-singular case using the fact that the morphism $\pi^* : \mathrm{Ca}(X) \longrightarrow \mathrm{Ca}(\tilde{X})$ is surjective and preserves degrees. The preservation of the degree of a Cartier divisor under π^* follows from the fact that $\dim_k \mathcal{O}_P/h\mathcal{O}_P = \dim_k \tilde{\mathcal{O}}_P/h\tilde{\mathcal{O}}_P$, for every non-zero rational function h of X. Thus, there exists a Cartier divisor of degree 1.
We observe that claim 1 and claim 2 imply that the number of effective Cartier divisors with a given degree is finite. Using the same argument as in the non-singular case, we prove that $Z(\mathrm{Ca}(X), t)$ converges absolutely and uniformly on the semiplane $Re(s) > 0$ (cf. [Sti, chap. V, prop. 1.6]).
The last claim follows from claim 2 and the fact that $\deg(\mathcal{A}\mathcal{B}) = \deg(\mathcal{A}) + \deg(\mathcal{B})$, for any two Cartier divisors $\mathcal{A}$ and $\mathcal{B}$ (see [Ha, sect. 1]). ☑

Corollary 2.2. *The quotient* $\frac{Z(\mathcal{O}_P,t)}{Z(\tilde{\mathcal{O}}_P,t)}$ *is a rational function of* t.

Proof. By taking a partial resolution of singularities of X, we may assume that P is the only singular point of X. Then

$$\frac{Z(\mathrm{Ca}(X),t)}{Z(\mathrm{Ca}(\tilde{X}),t)} = \prod_{Q\in\pi^{-1}(P)} (1 - t^{\deg(Q)})\ Z(\mathcal{O}_P, t) = \frac{Z(\mathcal{O}_P,t)}{Z(\widetilde{\mathcal{O}}_P,t)}.$$

The result thus follows from the previous theorem. ☑

Theorem 2.3. *Let X be a complete, geometrically irreducible, algebraic curve defined over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(\mathrm{Ca}(X), t)$ satisfies the functional equation*

$$Z(\mathrm{Ca}(X), t) = q^{g-1}t^{2g-2}Z\Big(\mathrm{Ca}(X), \frac{1}{qt}\Big) \tag{2.3}$$

if and only if X is a Gorenstein curve.

Proof. If X is a Gorenstein curve, the argument of Schmidt for the non-singular case, the reciprocity (Corollary 1.2), and the observations made in the proof of Lemma 2.1 imply the functional equation (2.3) (see [Sti, chap. V, prop. 1.13]). Conversely, if the zeta function $Z(\mathrm{Ca}(X), t)$ satisfies the functional equation (2.3) and $g \geqq 1$, then the Riemann-Roch theorem and (2.2) imply that

$$Z(\mathrm{Ca}(X), t) = Z_1(X, t) + Z_2(X, t)$$

with

$$Z_1(X,t) = \frac{1}{q-1} \sum_{0\leq\deg[\mathcal{A}]\leq 2g-2} q^{\ell(\mathcal{A})}t^{\deg(\mathcal{A})} = \frac{1}{q-1}\sum_{j=0}^{2g-2} a_j t^j \tag{2.4}$$

and

$$
\begin{aligned}
Z_2(X,t) &= \frac{1}{q-1}\sum_{\deg[\mathcal{A}]\geqq 2g-1} q^{\ell(\mathcal{A})}t^{\deg(\mathcal{A})} - \frac{1}{q-1}\sum_{\deg[\mathcal{A}]\geqq 0} t^{\deg(\mathcal{A})} \\
&= \frac{1}{q-1}\left(\#\mathrm{Pic}^0(X)q^g t^{2g-1}\frac{1}{1-qt} - \frac{\#\mathrm{Pic}^0(X)}{1-t}\right). \qquad (2.5)
\end{aligned}
$$

By hypothesis, $Z(\mathrm{Ca}(X),t)$ satisfies the functional equation (2.3). A direct verification shows that the function $Z_2(X,t)$ also satisfies the functional equation (2.3). Therefore, $Z_1(X,t)$ satisfies the functional equation (2.3). This implies that

$$a_{2g-2-j} = a_j q^{g-1-j}, \qquad j = 0,1,\ldots,2g-2.$$

On the other hand, $a_0 = 1$. Thus, $a_{2g-2} = q^{g-1}$. Since $g \geq 1$, a divisor class $[\mathcal{C}]$ with $\deg(\mathcal{C}) = 2g-2$ and $\ell(\mathcal{C}) = g$ must appear in the sum in (2.4). These properties characterize the canonical class. Hence, by theorem 1.4, X is Gorenstein. In the case $g = 0$, the genus formula implies $\delta = 0$, so that X is a non-singular curve. Thus, in this case, also X is Gorenstein. ☑

As a consequence of the functional equation (2.3), the degree of the polynomial $L(\mathrm{Ca}(X),t)$ is $2g$.

Corollary 2.4. *The zeta function $Z(\mathcal{O}_P,t)$ satisfies the functional equation*

$$\frac{Z(\mathcal{O}_P,t)}{Z(\widetilde{\mathcal{O}}_P,t)} = q^{\delta_P}t^{2\delta_P}\frac{Z\left(\mathcal{O}_P,\frac{1}{qt}\right)}{Z\left(\widetilde{\mathcal{O}}_P,\frac{1}{qt}\right)}$$

if and only if $\mathcal{O}_P$ is Gorenstein.

3. Rationality and functional equation of $Z(\mathrm{Div}(X),t)$

In this section, we study the rationality and functional equation of the zeta function $Z(\mathrm{Div}(X),t)$ associated to the set of divisors $\mathrm{Div}(X)$. This zeta function is defined as follows:

$$Z(\mathrm{Div}(X),t) = \sum_{\mathcal{A}\geq\mathcal{O}} t^{\deg(\mathcal{A})},$$

where $\mathcal{A}$ runs through all effective divisors on X and $t = q^{-s}$, $s\in\mathbb{C}, Re(s) > 0$. This zeta function decomposes formally into the following Euler product

$$Z(\mathrm{Div}(X),t) = \prod_{P\in X} Z(\mathcal{O}_P,t) = \prod_{P\in X}\left(\sum_{I\supseteq\mathcal{O}_P} t^{\dim_k(I/\mathcal{O}_P)}\right),$$

where I runs trough all $\mathcal{O}_P$-fractional ideals containing $\mathcal{O}$.

Theorem 3.1. *Let X be a complete, geometrically irreducible algebraic curve over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(\mathrm{Ca}(X), t)$ is a rational function. More precisely,*

$$Z(\mathrm{Ca}(X), t) = \frac{L(\mathrm{Div}(X), t)}{(1-t)(1-qt)},$$

where $L(\mathrm{Div}(X), t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $2g$, and $L(\mathrm{Ca}(X), 1) = \#Cl^0(X)$, the number of classes of divisors of degree zero, i.e. $L(\mathrm{Div}(X), 1)$ is the number of rational points of the compactified Jacobian of $X \bigotimes_k \bar{k}$.

Proof. The proof is similar to the proof of theorem 2.1. In order to prove that

$$\#Cl^0(X) = \mathrm{Div}^0(X)/Prin(X) < \infty,$$

we observe that $\mathrm{Pic}^0(X)$ acts on $Cl^0(X)$ by multiplication and the quotient set is isomorphic to

$$\Big\{ \prod_{P \in X_{sing}} \mathcal{A}_P \mid \mathcal{A}_P \text{ is an ideal in } \mathcal{O}_P \text{ and } \mathcal{F}_P \subseteq \mathcal{A}_P \subseteq \widetilde{\mathcal{O}}_P \Big\}. \tag{3.1}$$

Since $\dim_k(\tilde{\mathcal{O}}_P/\mathcal{F}_P) < \infty$ and $\#\mathrm{Pic}^0(X) < \infty$, we conclude from (3.1) that $\#Cl^0(X) < \infty$. Now we can follow the proof of theorem 2.1. ☑

The zeta function $Z(\mathrm{Div}(X), t)$ satisfies the functional equation (2.3). The proof of this fact follows from the singular Riemann-Roch theorem and the reciprocity, by a similar reasoning to that of Schmidt (see [Sti, chap. V]). Stöhr defined $Z(\mathrm{Div}(X), t)$ and proved that $Z(\mathrm{Div}(X), t)$ satisfies the functional equation (2.4) (cf. [Sto, p. 133]).
The local factors of $Z(\mathrm{Div}(X), t)$ are rational functions (as in Corollary 2.2) and satisfy a functional equation (as in Corollary 2.4). The functional equation (2.3) implies that the degree of the polynomial $L(\mathrm{Div}(X), t)$ is $2g$.

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