Revista Colombiana de Matemáticas
Volumen **31** *(1997), páginas 71–76*

On the uniqueness of solutions in the class of increasing functions of a system describing the dynamics of a viscous weakly stratified fluid in three dimensional space

Andrei I. Giniatoullin
Universidad de los Andes, Bogotá, COLOMBIA

Abstract. We consider the Cauchy problem for a system of partial differential equations that describes the dynamics of a viscous weakly stratified fluid in three dimensional space. The existence of solutions of the problem follows from an explicit representation of the Fourier transform studied by the author in previous works. Here we prove the uniqueness of the weak solution of the problem in the class of growing functions.

Key words and phrases. Weak solution, class of uniqueness, partial differential equations, Cauchy problem, generalized functions.

1991 Mathematics Subject Classification. Primary 76026. Secondary 76099, 35Q35.

We consider the following system of partial differential equations:

$$
\begin{cases}
\rho_* \dfrac{\partial v_1}{\partial t} - \mu \Delta v_1 + \dfrac{\partial p}{\partial x_1} = f_1(x,t), \\
\rho_* \dfrac{\partial v_2}{\partial t} - \mu \Delta v_2 + \dfrac{\partial p}{\partial x_2} = f_2\left(x,t\right), \\
\rho_* \dfrac{\partial v_3}{\partial t} - \mu \Delta v_3 + g v_4 + \dfrac{\partial p}{\partial x_3} = f_3\left(x,t\right), \\
\dfrac{g}{N^2 \rho_*} \dfrac{\partial v_4}{\partial t} - v_3 = f_4(x,t), \\
\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0.
\end{cases}
\tag{1}
$$

Here, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\overrightarrow{v}(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ is a velocity field, $v_4(x,t)$ is the deviation of the fluid density caused by its motion, $p(x,t)$ is the pressure, $\rho_* = Const > 0$ is the mean initial density, $\mu > 0$ is the coefficient of viscosity, g is the gravitation constant, N is the Weisal-Brent frequency and $\widetilde{f}(x,t) = (f_1, f_2, f_3, f_4)$ is the vector of exterior mass forces acting on the fluid.

We examine system (1) in the domain $Q = \{(x,t) : x \in \mathbb{R}^3, 0 \le t \le T\}$ with $T > 0$ and the initial conditions:

$$\widetilde{v}(x,0) = \widetilde{v}_o(x), \quad x \in \mathbb{R}^3, \quad \widetilde{v} = (v_1, v_2, v_3, v_4)\,. \tag{2}$$

System (1) describes the dynamics of a viscous weakly stratified fluid. This model assumes that the stationary distribution of the density is $\rho_* e^{-\beta x_3}$. The importance of this case is justified by the fact that the exponential distribution is of Boltzmann type in a uniform gravitation field.

The presence of the stratification equation $\frac{g}{N^2 \rho_*} \frac{\partial v_4}{\partial t} - v_3 = f_4(x,t)$ in system (1) constitutes the practical interest of the investigation of this problem.

In [MG] problem (1)–(2) was considered for $\widetilde{f} = \overrightarrow{v_o} = 0$ and a discontinuous initial condition with compact support for v_{4_0} describing an intrusion problem. The asymptotic behavior and the smoothness properties were studied. Here we prove the uniqueness of the solution of problem (1)–(2) for a wide class of increasing functions which includes, in particular, the case considered in [MG].

We introduce the following definitions.

Definition 1. *A weak solution of problem* (1)–(2) *is a pair* $(\widetilde{v}(x,t), p(x,t))$ *of locally bounded measurable functions which satisfy equations* (1) *and the initial conditions* (2) *in the sense of generalized functions. By a class of uniqueness of weak solutions of problem* (1)–(2) *we understand a class of pairs* $(\widetilde{v}, p)$ *of functions defined on* Q *such that if a solution of* (1)–(2) *exists and belongs to that class then it is the unique solution in that class. In other words, a uniqueness class is a class of pairs* $(\widetilde{v}, p)$ *such that if* $\widetilde{f}(x,t) = 0$ *almost everywhere in* Q *and* $\widetilde{v}_o(x) = 0$ *almost everywhere in* $\mathbb{R}^3$*, then* $\widetilde{v}(x,t) \equiv 0$ *and* $\nabla p(x,t) \equiv 0$ *almost everywhere in* Q*.*

Suppose $g(r) \in C([0;\infty))$*,* $g(r) > 0$*,* $g(r_1) < g(r_2)$ *for* $r_1 < r_2$*,* $g(r)/r \to 0$ *as* $r \to \infty$*, and also* $\varphi(r) \in C([0;\infty))$*,* $\varphi(r) > 0$*,* $\varphi(r_1) < \varphi(r_2)$ *for* $r_1 < r_2$*. Furthermore assume that the following condition holds:* $\int_1^\infty \frac{r}{\varphi(r)} dr = \infty$*. Then we define* the class of uniqueness $K_{\varphi,g}$ *of solutions of problem (1)-(2) by the inequalities*

$$\left|p(x,t)\right| \le g(|x|)\,, \ \left|\overrightarrow{v}(x,t)\right| \le C_1 \exp\{\varphi(|x|)\}\,, \ \left|v_4(x,t)\right| \le C_2 \exp\{C_3 |x|^2\}$$

holding almost everywhere in Q, where C_i are some fixed positive constants, $i = 1, 2, 3$.

We shall demonstrate the following result.

Theorem 1. *The weak solution of problem* (1)–(2) *is unique in the class* $K_{\varphi,g}$.

Proof. We have to verify that $\widetilde{v}_o = 0$ implies $\widetilde{v} = 0$, $\nabla p = 0$ .We first ascertain that it is sufficient to prove the statement for $(\widetilde{v}, p) \in C^\infty(\mathbb{R}^4)$. Let $(\widetilde{v}(x,t), p(x,t))$ be a weak solution of (1)–(2) for $\widetilde{f} = \widetilde{v}_o = 0$ and extend $\widetilde{v}$ and p to $\mathbb{R}^4 \setminus Q$ by $\widetilde{v}(x,t) = \widetilde{v}(x,T)$, $p(x,t) = p(x,T)$ for $t > T$, $\widetilde{v}(x,t) = p(x,t) = 0$ for $t < 0$. We denote by $p^h(x,t)$, $v^h(x,t)$ the averaging of the functions $p(\xi,\tau)$, $v(\xi,\tau)$ over the ball $|\xi - x|^2 + |\tau - t|^2 \le h^2$ by means of the kernel $\omega\left(\frac{x-\xi}{h}, \frac{t-\tau-h}{h}\right)$, where $\omega(x,t) \in C_o^\infty\left(\mathbb{R}^4\right)$ and $0 < h < 1$. Here we understand "averaging" as the respective convolution integrals of $p(\xi,\tau)$ and $v(\xi,\tau)$ with a function in C_o^∞, in this case with the above kernel.

Taking the averaging kernel as a test function we obtain that the functions $\widetilde{v}^h$, p^h belong to C^∞ for every $h \in (0,1)$ and satisfy (1)–(2) in the "ordinary" sense. It is also evident that if $(\widetilde{v}, p)$ belongs to $K_{\varphi,g}$ then also $\left(\widetilde{v}^h, p^h\right)$ belongs to the class K_{φ_1,g_1} where $\varphi_1(r) \equiv \varphi(r+1)$, $g_1(r) \equiv g(r+1)$. Thus, we may assume $(\widetilde{v}, p) \in C^\infty$. Let $\widetilde{f} = 0$, $\widetilde{v}_o = 0$. From (1)–(2) we get that in Q either

$$\begin{cases} \left(\rho_* \dfrac{\partial}{\partial t} - \mu\Delta\right)(\Delta p)_t + N^2 \rho_* \left(\dfrac{\partial^2 p}{\partial x_1^2} + \dfrac{\partial^2 p}{\partial x_2^2}\right) = 0 \\ \Delta p\Big|_{t=0} = (\Delta p)_t\Big|_{t=0} = 0 \end{cases} \tag{3}$$

or

$$\begin{cases} \left(\rho_* \dfrac{\partial}{\partial t} - \mu\Delta\right)\Delta p = 0 \\ \Delta p\Big|_{t=0} = 0. \end{cases} \tag{4}$$

The equations in (3) and (4) are derived from system (1) by consecutive elimination of unknown functions. The initial conditions are obtained as follows. We apply the divergence operator to the first three equations of (1) and then take into account the fifth equation to obtain the relation

$$\Delta p + g \frac{\partial v_4}{\partial x_3} = 0, \quad \text{for all } (x,t) \in Q. \tag{5}$$

Also, from (2) one has for $t = 0$ that $\partial v_4 / \partial x_3 = 0$. This implies the initial condition $\Delta p|_{t=0} = 0$. From the fourth equation in (1) and the homogeneous initial conditions (2), we get

$$\left.\frac{\partial v_4}{\partial t}\right|_{t=0} = 0, \qquad \left.\frac{\partial^2 v_4}{\partial x_3 \partial t}\right|_{t=0} = 0.$$

Then, differentiating (5) with respect to t and taking into account the last relation, we obtain the initial condition $(\Delta p)_t|_{t=0} = 0$.

From the definition of $K_{\varphi,g}$ we conclude that $p(x,t)$ belongs for every t to the dual $S'\left(\mathbb{R}^3_x\right)$ of the Schwartz space $S\left(\mathbb{R}^3_x\right)$ of rapidly decreasing functions. In fact, its growth is limited by a power function of $|x|$ as $|x| \to \infty$ (see, for example, [V]). Therefore, the function $q(\xi,t)$ defined by $q(\xi,t) \equiv |\xi|^2 F_{\xi\to x}[p(x,t)]$ will belong to $S'\left(\mathbb{R}^3_\xi\right)$ (see [V]). Here F means Fourier transform.

From (4) we get that for $t \in [0,T]$ the following relation holds:

$$\begin{cases} \left(\rho_* \dfrac{\partial}{\partial t} + \mu |\xi|^2\right) q(\xi,t) = 0 \\ q|_{t=0} = 0. \end{cases}$$

The solution of the last problem is unique if and only if the associated adjoint problem

$$\begin{cases} \left(-\rho_* \dfrac{\partial}{\partial t} + \mu |\xi|^2\right) \psi(\xi,t) = 0, \quad t \in [0,t_o] \\ \psi(\xi,t_o) = \psi_o(\xi) \end{cases} \tag{6}$$

has a solution for any function $\psi_o \in S$ (see [GS]), where $t_o \in (0,T]$. But it is evident that the solution of problem (6) exists for any $\psi_o \in S$. In fact, $\exp\left\{\frac{\mu}{\rho_*}|\xi|^2(t-t_o)\right\}$ is a multiplier in S for $t \le t_o$. Therefore, $q(\xi,t) = |\xi|^2 p(\xi,t) \equiv 0$, and $p(\xi,t)$ may be different from zero only at the point $\xi = 0$. Then, from the theorem of structure of generalized functions with pointwise support [V] we obtain that the function $p(x,t)$ is a polynomial in x, and since $(\tilde{v},p)$ belongs to $K_{\varphi,g}$, then p increases with respect to x slower that the first degree of x. This means that p does not depend on x, and thus $\nabla p \equiv 0$.

Similarly, let us consider the corresponding problem related to (3),

$$\begin{cases} \left(\rho_* \dfrac{\partial}{\partial t} + \mu|\xi|^2\right) \dfrac{\partial}{\partial t} q(\xi,t) + \dfrac{N^2\rho_*}{|\xi|^2}(\xi_1^2 + \xi_2^2) q(\xi,t) = 0, \\ q|_{t=0} = q_t|_{t=0} = 0, \quad t \in [0,T] \end{cases} \tag{7}$$

and its associated adjoint problem

$$\begin{cases} \left(-\rho_* \dfrac{\partial}{\partial t} + \mu \left|\xi\right|^2\right) \left(-\dfrac{\partial}{\partial t}\right) \psi(\xi, t) + \dfrac{N^2 \rho_*}{\left|\xi\right|^2} \left|\xi'\right|^2 \psi(\xi, t) = 0, \\ \psi(\xi, t_o) = \psi_o(\xi), \quad \psi_t(\xi, t_o) = \psi_1(\xi), \quad t \in [0, t_o], \quad t_o \in (0, T]. \end{cases} \tag{8}$$

Let Σ be a set of functions of S which vanish in some neighborhood of the point $\xi = 0$. It is easy to see that problem (8) is solvable for any $\psi_o \in \Sigma$, $\psi_1 \in \Sigma$. In fact, the functions

$$\exp\left\{ \left[\frac{\mu}{2\rho_*} \left|\xi\right|^2 \pm \sqrt{\frac{\mu^2}{4\rho_*^2} \left|\xi\right|^4 - N^2 \frac{\left|\xi'\right|^2}{\left|\xi\right|^2}} \right] (t - t_o) \right\} \tag{9}$$

are multipliers in Σ for $t \leq t_o$.

We justify the election of Σ as a base space by the fact that the multipliers (9) lose smoothness at the point $\xi = 0$.

According to [GS], for every function $\psi \in \Sigma$, $\langle q, \psi \rangle = \left\langle \left|\xi\right|^2 p, \psi \right\rangle = 0$ holds for every $t \in [0, T]$. Here $q(\xi, t)$ is the solution of (7) and $\langle \cdot, \cdot \rangle$ denotes a linear continuous functional over Σ. Thus, we have that for every t the function $p(\xi, t)$ may be different from zero only at the point $\xi = 0$. Therefore $p(x, t)$ is a polynomial in x (see [V]). On the other hand, $p(x, t)$ belongs to $K_{\varphi, g}$. So, we conclude as before that p does not depend on x, and thus $\nabla p \equiv 0$.

We use now this result to exclude the unknown function $v_3(x, t)$ from the fourth equation of system (1). In this way, we obtain that for $v_4(x, t)$ the following problem holds,

$$\begin{cases} \dfrac{\partial^2 v_4}{\partial t^2} - \dfrac{\mu N^2}{g} \dfrac{\partial \Delta v_4}{\partial t} + N^2 v_4 = 0 \\ v_4 \big|_{t=0} = \dfrac{\partial v_4}{\partial t} \bigg|_{t=0} = 0. \end{cases}$$

The reduced order of the last equation equals to 2 [GS], which implies that its solution is unique for $v_4 \in K_{\varphi, g}$ (see [GS]).

Now we may consider, without loss of generality, that $\nabla p = 0$ and $v_4 = 0$. Coming back to system (1) with zero initial conditions we obtain a parabolic system for the velocity field $\overrightarrow{v}(x, t)$ with zero initial data. The solution of that problem is unique for $\overrightarrow{v} \in K_{\varphi, g}$ as follows from [E], [T]. This concludes the proof. ☑

References

[MG] Maslennikova, V. N. & Giniatoullin, A. I., *On the intrusion problem in a viscous stratified fluid for three space variables*, Mat. Zametki **51** (1992) 69–77 (Russian). Translated in AMS Math. Notes **51** (1992) 374–379.

[V] Vladimirov, V. S. *Equations of Mathematical Physics*, Marcel Dekker, 1971.

[GS] Gelfand, I. M. & Shilov, G. E. *Generalized Functions Vol. 3, Theory of differential Equations*, Acad. Press, 1967.

[E] Eidelman, S. D. *Parabolic systems* (Russian), Nauka, Moscow, 1964.

[T] Täcklind, S. *Sur les classes quasianalytiques des solutions des équations aux dérivées partielles du type parabolique,* Nord Acta Regial, Sosietatis Scientiarum Upsaliensis **4** no. 10 (1937), 47.

(Recibido en agosto de 1995, revisado en diciembre de 1997)

Departamento de Matemáticas
Universidad de los Andes
Apartado Aéreo 4976
Bogotá, COLOMBIA
FAX : 57-1-284-18-90
e-mail: aginiato@zeus.uniandes.edu.co