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ON EXHAUSTIVE VECTOR MEASURES

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Abstract. Certain conditions related with the De Wilde and Valdivia closed graph theorems enable us to guarantee the exhaustivity of certain bounded vector measures. To obtain some of these conditions we show incidentally that the bounded countably valued scalar function space with the supremum norm is ultrabornological.

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In this paper 'space' means 'locally convex Hausdorff space over the real or complex field'. Unless other thing is stated, by 'topology' we understand 'Hausdorff locally convex topology'.

If s is a positive integer, then the countable family of subspaces $W = \{X_{m_1 m_2 \cdots m_p} \mid m_r \in \mathbb{N} \mid 1 \leq r \leq p \leq s\}$ of a space X is an s–net if $\{X_{m_1} \mid m_1 \in \mathbb{N}\}$ is an increasing covering of X and the sequence $\{X_{m_1 m_2 \cdots m_j} \mid m_j \in \mathbb{N}\}$ is an increasing covering of $X_{m_1 m_2 \cdots m_{j-1}}$, for $2 \leq j \leq s$. We write $W_s = \{X_{m_1 m_2 \ldots m_s} \mid m_r \in \mathbb{N}, 1 \leq r \leq s\}$. A space X is Γ_r, [11] (Λ_r, [12]) if given any quasi-complete (locally complete) subspace G of $X^*(\sigma(X^*, X))$, X^* being the algebraic dual of X, such that G meets the topological dual X' of X in a weak* dense subspace of X', then G contains X'.

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B_r-complete spaces are Γ_r, and reflexive BANACH spaces and FRÉCHET-SCHWARTZ spaces provide some examples of Λ_r-spaces.

By Σ we shall denote a σ-algebra of subsets of the set Ω. If μ is a mapping from Σ in a space X such that $\mu(A \cup B) = \mu(A) + \mu(B)$ for each disjoint pair (A, B) of elements of Σ, then μ is said to be a finite additive vector measure or simply a vector measure.

A vector measure μ is bounded if the set $\{\mu(A) \mid A \in \Sigma\}$ is bounded, and μ is called exhaustive if whenever we have a sequence $\{E_i \mid i \in \mathbb{N}\}$ of pairwise disjoint elements of Σ, then $\lim \mu(E_i) = 0$. It is well-known that each exhaustive vector measure is bounded, and the aim of this paper is to find out some conditions implying that bounded measures are exhaustive.

A vector measure μ is strongly additive if for every sequence $\{E_i : i \in \mathbb{N}\}$ of pairwise disjoint elements of Σ, $\sum\{\mu(E_i) \mid i \in \mathbb{N}\}$ converges in X. If, additionally, $\sum\{\mu(E_i) \mid i \in \mathbb{N}\} = \mu\left(\cup\{E_i \mid i \in \mathbb{N}\}\right)$, the measure μ is called countably additive. Obviously, each strongly additive measure is exhaustive, the converse being true when the space X is sequentially complete.

In the space $L^\infty(\Omega, \Sigma)$ of all bounded Σ-measurable scalar functions endowed with the supremum norm, we are going to consider the following subspaces:

- The subspaces $S_c(\Omega, \Sigma)$ of the functions of countable range. Note that a function $f \in S_c(\Omega, \Sigma)$ is determined by the bounded sequence of scalars $\{a_n \mid n \in \mathbb{N}\}$ formed by the elements of $f(\Omega)$ and the sets $f^{-1}(a_n)$.
- The subspace $L(\mathcal{A})$ of $S_c(\Omega, \Sigma)$ formed by those functions determined by all the elements of ℓ^∞ and the sequence $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ of pairwise disjoint elementes of Σ. Obviously $L(\mathcal{A})$ is isometric to ℓ^∞.
- The linear span of the characteristic funtions $e(E)$ of all $E \in \Sigma$. This space will be denoted either by $S(\Omega, \Sigma)$ or by $\ell_0^\infty(\Omega, \Sigma)$.

When Ω is the set $\mathbb{N}$ of natural numbers and Ω is the σ-algebra of all subsets of $\mathbb{N}$, we shall write ℓ_0^∞ instead of $\ell_0^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

If $\mu : \Sigma \to X$ is a vector measure, the linear mapping S from $S(\Omega, \Sigma)$ into X such that $S(e(E)) = \mu(E)$ for each $E \in \Sigma$ will be called the linear mapping associated to μ. If μ is bounded, we have that S is continuous. In fact, if $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_n| \leq 1$, $E_1, E_2, \ldots, E_n \in \Sigma$ and p is a continuous seminorm on X we have that

$$p\Big(S\big(\lambda_1 e(E_1) + \lambda_2 e(E_2) + \ldots + \lambda_n e(E_n)\big)\Big) \leq \sup\big\{p(\mu(E)) \mid E \in \Sigma\big\}$$

and continuity of S follows immediately from this inequality and the following result given by M. VALDIVIA in [13], Lemmas 1 and 2:

(*a*) *if B stands for the closed unit ball of $\ell_0^\infty(\Omega, \Sigma)$, then the absolutely convex hull of $\{e(E) \mid E \in \Sigma\}$ contains $\frac{1}{4}B$.*

The next result is an extension, due to L. DREWNOWSKI, [4] and [5], of a well known result of H. P. ROSENTHAL for Banach spaces [10].

(1) "(*b*)" *Let T be a linear continuous mapping from ℓ^∞ into a Hausdorff topological vector space X and let $\{e_n \mid n \in \mathbb{N}\}$ be the sequence of the unit vectors of ℓ^∞. If X has a neighbourhood of the origin U such that the set $\{n \in \mathbb{N} \mid Te_n \notin U\}$ is infinite, then X contains a copy of ℓ^∞.*

We use this in order to prove our first theorem (see also [8], Theorem 1, Corollary A).

Theorem 1. *Let μ be a bounded vector measure defined in Σ with values in a sequentially complete topological vector space X. If X does not contain any copy of ℓ^∞, then μ is strongly additive.*

Proof. We know that the linear mapping $S : S(\Omega, \Sigma) \to X$ associated with μ is continuous. The metrizability of $L^\infty(\Omega, \Sigma)$ and the sequential completeness of X enables us to extend S to a continuous linear mapping $\overline{S} : L^\infty(\Omega, \Sigma) \to X$.

If $\{A_i \mid i \in \mathbb{N}\}$ is a sequence of pairwise disjoint elements of Σ, $A_0 = \Sigma \setminus \cup\{A_i \mid i \in \mathbb{N}\}, \mathcal{A}$ is the sequence $\{A_0, A_1, A_2, \dots\}$ and T is the restriction of $\overline{S}$ to $L(\mathcal{A})$ then, by property b), the sequence $(S(e(A_i)))$ converges to zero given that X does not contain a copy of ℓ^∞. Therefore the measure μ is exhaustive and given that X is sequentially complete, μ is strongly additive. □

Corollary 1. *Let $\mu : \Sigma \to X$ be a bounded measure and suppose that X is sequentially complete. If $f : X \to Y$ is a linear mapping with closed graph taking its values into a webbed space Y which does not contain a copy of ℓ^∞, then $f\mu$ is exhaustive.*

Proof. Let $\overline{S} : L^\infty(\Omega, \Sigma) \to X$ be the continuous linear mapping determined in Theorem 1. Then $f\overline{S} : L^\infty(\Omega, \Sigma) \to Y$ has a closed graph and, according to the De Wilde closed graph theorem [2], $f\overline{S} : L^\infty(\Omega, \Sigma) \to Y$ is continuous. Then, the conclusion is obtained as in Theorem 1, considering the restriction of $f\overline{S}$ to $L(\mathcal{A})$. □

In Theorem 4 of [13], M. Valdivia has obtained the following result:

(c) *Let μ be a bounded finite additive X-valued vector measure and let $\{F_n \mid n \in \mathbb{N}\}$ be an increasing sequence of Γ_r-spaces covering a space F. If f is a linear mapping defined in X with values in F which has closed graph, then there is a positive integer q such that $f\mu$ is a F_q-valued bounded finite additive measure on Σ.*

In [9], Theorem 16, B. Rodríguez-Salinas gives the following extension of result (c):

(d) *Let $\mu : \Sigma \to X$ be a bounded finite additive measure and let F be a space which has an increasing covering $\{F_n \mid n \in \mathbb{N}\}$ of subspaces such that for each n there is a topology τ_n on F_n, finer than that induced by F, under which $F_n(\tau_n)$ is a sequentially complete Γ_r-space which does not contain a copy of ℓ^∞. If $f : X \to F$ is a linear mapping with closed graph, then there is a positive integer n such that $f\mu : \Sigma \to F_n(\tau_n)$ is a strongly additive measure.*

Our next result follows from these two results and from [6], Theorem 1, as well as from the observation that in result (d) there is in F a topology τ_F weaker than the initial one such that $f : X \to F(\tau_F)$ is continuous and the map $f\mu : \Sigma \to F(\tau_F)$ is a bounded finite additive measure.

Theorem 2. *Let $\mu : \Sigma \to X$ be a bounded measure defined in a space X containing an s-net W such that every $H \in W_s$ has a topology τ_H finer than the induced by the original topology of X and such that $H(\tau_H)$ is a Γ_r-space which does not contain any copy of ℓ^∞. Then there is a $G \in W_s$ such that the mapping $f\mu : \Sigma \to G(\tau_G)$ is exhaustive.*

Proof. Let $S : S(\Omega,\Sigma) \to X$ be the continuous linear mapping associated with μ. By Theorem 1 of [6] there is a $G \in W_s$ such that $S^{-1}(G)$ is a dense and barrelled subspace of $S(\Omega,\Sigma)$. But according to Theorems 1 and 14 of [11] the restriction of S to $S^{-1}(G)$ has a continuous extension $\overline{S} : L^\infty(\Omega,\Sigma) \to G(\tau_G)$ which agrees with S in $S(\Omega,\Sigma)$. Now the conclusion follows exactly as in Theorem 1, changing X by $G(\tau_G)$. □

If $G(\tau_G)$ is sequentially complete then obviously $f\mu : \Sigma \to G(\tau_G)$ is strongly additive.

Corollary 2. *Let $\mu : \Sigma \to X$ be a bounded measure defined in a space X that contains an s-net W such that every $H \in W_s$ has a topology τ_H finer than the induced by the original topology of X and such that $H(\tau_H)$ is a Γ_r-space. If f is a linear mapping with closed graph from X into a webbed space Y which does not contain a copy of ℓ^∞, then $f\mu$ is exhaustive.*

Proof. In Theorem 2 we have obtained the continuous linear mapping $\overline{S} : L^\infty(\Omega,\Sigma) \to G(\tau_G)$. Then, $f\overline{S} : L^\infty(\Omega,\Sigma) \to Y$ has closed graph and is continuous according to De Wilde closed graph theorem. The conclusion follows as in Theorem 1, T being now the restriction of $f\overline{S}$ to $L(\mathcal{A})$. □

If the webbed space is sequentially complete then $f\mu$ is strongly additive.

Examples.

1. *Let E be a Banach space having a separable strong dual $E'(\beta(E',E))$ and let $\mu : \Sigma \to E'(\sigma(E',E))$ be a bounded measure.*

   $E'(\beta(E',E))$ is a webbed space and also a Γ_r–space and since it is separable, it cannot contain any copy of ℓ^∞. Clearly, $E'(\sigma(E',E))$ is also a Γ_r-space. Hence, if $\mu : \Sigma \to E'(\sigma(E',E))$ is a bounded measure, the preceding corollary and the completeness of $E'(\beta(E',E))$ imply that $\mu : \Sigma \to E'(\beta(E',E))$ is strongly additive.

2. *Let now E be a Fréchet space such that $E'(\beta(E',E))$ does not contain a copy of ℓ^∞. Let $\{V_n, n \in \mathbb{N}\}$ be a fundamental system of neighbourhoods of zero in E, and let $E_n(\tau_n)$ denote the Banach space generated by V_n^0, with its Minkowski functional as norm. The topology τ_n is finer than the induced by $\sigma(E',E)$ and $E_n(\tau_n)$ is a Γ_r-space. The immersion $i : E'(\sigma(E',E)) \to E'(\beta(E',E))$ has obviously closed graph. Now, if the measure $\mu : \Sigma \to E'(\sigma(E',E))$ is bounded, Corollary 2 implies that $\mu : \Sigma \to E'(\beta(E',E))$ is strongly additive.*

Our aim now is to debilitate the conditions imposed to X in the last Corollary. To compensate we have to strengthen the conditions imposed to the bounded vector measure μ.

Given a continuous seminorm p on the space X. The p-variation $|\mu|_p$ of the vector measure μ with values in X is defined by

$$|\mu|_p(E) = \sup\left\{\sum_{i=1}^{n} p\big(\mu(E_i)\big) \mid \{E_1, E_2, \ldots, E_n\} \in \mathcal{P}(E)\right\},$$

where $\mathcal{P}(E)$ denotes the family of all finite partitions of E by elements of Σ. The vector measure μ is said of bounded variation if $|\mu|_p(\Omega) < \infty$ for every continuous seminorm p of X. If the space X is sequentially complete and μ is a X-valued vector measure of bounded variation, then μ is strongly additive because $\Sigma\{p(\mu(E_i)) \mid i \in \mathbb{N}\} \leq |\mu|_p(\Omega)$. There are countably additive vector measures which are not of bounded variation ([3], pp. 7-8).

Lemma. *$S_c(\Omega, \Sigma)$ is ultrabornological.*

Proof. Let $\mathcal{P}_c$ be the family of all countable partitions $\mathcal{A}$ of Ω formed by elements of Σ. We are going to prove that $S_c(\Omega, \Sigma)$ is the locally convex hull of the family $\{L(\mathcal{A}) \mid \mathcal{A} \in \mathcal{P}_c\}$. Let V be an absolutely convex subset of $S_c(\Omega, \Sigma)$ such that $V \cap L(\mathcal{A})$ is a neighbourhood of zero in $L(\mathcal{A})$ for every $\mathcal{A} \in \mathcal{P}_c$. In order to prove that V is a neighbourhood of the origin in $S_c(\Omega, \Sigma)$, one can easily show that it suffices, because of result (b), to prove that there exists a $\lambda > 0$ such that $\lambda e(E) \subset V$, for every $E \in \Sigma$.

Suppose that this property is not true, and let $n_1 \in \mathbb{N}$ be such that $e(\Omega) \in (n_1/6)V$. Then there must be some $A_1 \in \Sigma$ such that $e(A_1) \notin (4n_1/3)V$. But, since $e(A_1) = e(\Omega) - e(\Omega \setminus A_1)$, it follows that $e(\Omega \setminus A_1) \notin n_1 V$. So we have shown that $e(A_1) \notin n_1 V$ and $e(\Omega \setminus A_1) \notin n_1 V$.

Let $n_2 > 2n_1$ be such that $e(A_1) \in (n_2/12)V$. Then

$$e(\Omega \setminus A_1) = e(\Omega) - e(A_1) \in (n_1/6)V + (n_2/12)V \subset (n_2/6)V.$$

So we have obtained that $e(A_1) \in (n_2/6)V$ and $e(\Omega \setminus A_1) \in (n_2/6)V$.

Since $e(E) = e(E \cap A_1) + e(E \cap (\Omega \setminus A_1))$ for every $E \in \Sigma$, we deduce that either V does not absorb the family $\{e(E) \mid E \subset A_1,\ E \in \Sigma\}$ or it does not absorb the family $\{e(E) \mid E \subset \Omega \setminus A_1, E \in \Sigma\}$.

We may suppose that V does not absorb $\{e(E) \mid E \subset \Omega \setminus A_1,\ E \in \Sigma\}$. In this moment we have $e(A_1) \notin n_1 V, e(\Omega \setminus A_1) \in (n_2/6)V$ and there is in $\Omega \setminus A_1$ a subset $A_2 \in \Sigma$ such that $e(A_2) \notin (4n_2/3)V$. We can obviously repeat the preceding argument, and proceeding by induction we can obtain a sequence $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ of pairwise disjoint subsets of Σ and a sequence $n_1 < n_2 < \cdots$ of positive integer numbers such that $e(A_j) \notin n_j V$, contradicting that $L(\mathcal{A}) \cap V$ is a neighbourhood of zero in $L(\mathcal{A})$. $\square$

Theorem 3. *Let $\mu : \Sigma \to X$ be a countably additive measure of bounded variation and suppose that every series in the space X that is subseries convergent is bounded multiplier convergent. Let Y be a webbed space that does not contain any copy of ℓ^∞ and let f be a linear mapping from X into Y with closed graph. Then $f\mu$ is an exhaustive vector measure.*

Proof. Since μ is countably additive we have that, given a sequence $\{E_n \mid n \in \mathbb{N}\}$ of pairwise disjoint elements of Σ, the series $\sum\{\mu(E_i) \mid i \in \mathbb{N}\}$ is subseries convergent and, consequently, by the hypotheses imposed on the space X, the series $\sum\{\mu(E_i) \mid i \in \mathbb{N}\}$ is bounded multiplier convergent. Then for every $\{a_n \mid n \in \mathbb{N}\} \in \ell^\infty$, the series $\sum\{a_i\mu(E_i) \mid i \in \mathbb{N}\}$ is also subseries convergent.

This obsevation enables us to define a linear mapping U from $S_c(\Omega, \Sigma)$ into X by $Ug = \sum\{a_i\mu(E_i) \mid i \in \mathbb{N}\}$, where $\{a_n \mid n \in \mathbb{N}\} \in \ell^\infty$ and $g(E_i) = \{a_i\}$ for every $i \in \mathbb{N}$. The definition is correct because if $\{F_m \mid m \in \mathbb{N}\}$ is a refinement of $\{E_n \mid n \in \mathbb{N}\}$ we have by the subseries convergence that

$$\sum\{b_i\mu(F_i) \mid i \in \mathbb{N}\} = \sum\{a_i\mu(E_i) \mid i \in \mathbb{N}\},$$

since in the series $\sum\{b_i\mu(F_i) \mid i \in \mathbb{N}\}$ we may reorder and associate as needed. The same argument shows that U is linear.

The mapping U is continuous since taking any continuous seminorm p we have that

$$p(Ug) = p\Big(\sum\{a_i\mu(E_i) \mid i \in \mathbb{N}\}\Big) \leq |\mu|_p(\Omega) \times \|\{a_i \mid i \in \mathbb{N}\}\|_\infty = |\mu|_p(\Omega) \times \|g\|$$

Therefore $T = fU$ is a mapping with closed graph from $S_c(\Omega, \Sigma)$ into Y. Since $S_c(\Omega, \Sigma)$ is ultrabornological according to the previous lemma, then T is continuous by De Wilde closed graph theorem [2].

Finally, to prove that $f\mu$ is exhaustive, let $\{A_i \mid i \in \mathbb{N}\}$ be a sequence of pairwise disjoint elements of Σ, $A_0 := \Omega \smallsetminus \cup\{A_i \mid i \in \mathbb{N}\}$ and $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$. Since $L(\mathcal{A})$ is isometric to ℓ^∞, we have that $0 = \lim T(e(A_n)) = \lim f\mu(A_n)$. □

In the preceding theorem the space X has to verify that every subseries convergent series is bounded multiplier convergent or, briefly, BM–convergent. The next proposition gives a sufficient condition which implies this property.

Proposition 1. *Let $\sum\{x_n \mid n \in \mathbb{N}\}$ be subseries convergent in a space X containing an s-net W such that every $F \in W_s$ has a finer topology τ_F such that $F(\tau_F)$ is a Λ_r-space. Then there is a $G \in W_s$ such that $\sum\{x_n \mid n \in \mathbb{N}\}$ is a BM-convergent series in $G(\tau_G)$.*

Proof. The subseries convergence condition implies that the mapping K from $\ell_0^\infty(\sigma(\ell_0^\infty, \ell^1))$ into X given by $K(e(A)) = \sum\{x_n \mid n \in A\}$ is well-defined and linear. It is also continuous since given an $x^* \in X'$ we have that

$$\begin{aligned}(K^t x^*)e(A) &= x^*\Big(\sum\{x_n \mid n \in A\}\Big) = \sum\{x^* x_n \mid n \in A\} \\ &= \langle (x^* x_n, n \in \mathbb{N}), e(A)\rangle\end{aligned}$$

and therefore the sequence $(x^* x_n)$ is subseries convergent, which implies that $(x^* x_n) \in \ell^1$.

By [6], Theorem 1, there is a $G \in W_s$ such that $H = K^{-1}(G)$ is a barrelled and dense subspace of ℓ_0^∞. Then if M is a subset of ℓ^1 which is $\sigma(\ell^1, H)$ bounded, M is ℓ_0^∞-equicontinuous and, since H is dense in ℓ^∞, it is also ℓ^∞-equicontinuous. Thus we have that every $u \in \ell^\infty$ is bounded on M and hence ℓ^∞ is contained in the bounded closure of H respect to the dual pair $\langle H, \ell^1\rangle$. From Theorems 2 and 6 of [12], it follows that the restriction of K to H possesses a continuous linear extension $T : \ell^\infty(\sigma(\ell^\infty, \ell^1)) \to G(\sigma(G, G(\tau_G)'))$. Now, the continuity of K implies that T is also an extension of K.

Given a vector $a = \{a_n \mid n \in \mathbb{N}\} \in \ell^\infty$ we write $a^{(n)} = (a_1, a_2, \dots, a_{n-1}, a_n, 0, 0, 0, \dots)$ for every $n \in \mathbb{N}$. Then $\lim a^{(n)} = a$ in $\sigma(\ell^\infty, \ell^1)$ and therefore $\sum\{a_n x_n \mid n \in \mathbb{N}\}$ is $\sigma(G, G(\tau_G)')$–convergent in G. Changing some of the a_n's by zeros we obtain that $\sum\{a_n x_n \mid n \in \mathbb{N}\}$ is weakly subseries convergent, and then the conclusion follows directly from the Orlicz–Pettis theorem. □

From the two former results we obtain the following corollary.

Corollary. *Let $\mu : \Sigma \to X$ be a countably additive measure with bounded variation and suppose that the space X contains an s-net W such that every $F \in W_s$ has a finer topology τ_F under which $F(\tau_F)$ is a Λ_r-space. Suppose that Y is a webbed space that does not contain a copy of ℓ^∞. If f is a linear mapping from X into Y with closed graph, then $f\mu$ is an exhaustive vector measure.*

If X and Y are two spaces, a linear mapping $f : X \to Y$ is said to be subcontinuous if given any series $\sum\{x_n \mid n \in \mathbb{N}\}$ which is subseries convergent, then $\sum\{f(x_n) \mid n \in \mathbb{N}\}$ converges to $f\left(\sum\{x_n \mid n \in \mathbb{N}\}\right)$.

In [9], B. Rodríguez–Salinas proves the following, which generalizes Theorem 1 in G. Bennett and N. J. Kalton [1].

(*e*) *Let X and Y be two spaces. Let us suppose that Y is a sequentially complete $\Gamma_r(\ell_0^\infty)$-space. If $f : X \to Y$ is a linear mapping with closed graph and Y does not contain a copy of ℓ^∞, then f is subcontinuous.*

This result has motivated our following theorem.

Theorem 4. *Let X and Y be two spaces. Let W be an s-net in Y such that every $L \in W_s$ has a finer topology τ_L such that $L(\tau_L)$ is a Γ_r-space sequentially complete which does not contain a copy of ℓ^∞. If $f : X \to Y$ is a linear mapping with closed graph, then f is subcontinuous.*

Proof. Let us assume that $\sum\{x_n \mid n \in \mathbb{N}\}$ is subseries convergent in X. In [9], Theorem 14, it is proved that the mapping $K : \ell_0^\infty \to X$ defined by

$$K(\{a_n \mid n \in \mathbb{N}\}) = \sum\{a_n x_n \mid n \in \mathbb{N}\}$$

is continuous. In fact, if sup $\{|a_n|, n \in \mathbb{N}\} \leq 1$ and $y' \in Y'$, then

$$\left| y'\big(K(\{a_n \mid n \in \mathbb{N}\})\big) \right| < \infty,$$

since $\sum\{y' x_n \mid n \in \mathbb{N}\}$ is absolutely convergent.

Consequently by [6], Theorem 1, there is a $G \in W_s$ such that $(fK)^{-1}(G)$ is barrelled and dense in ℓ_0^∞.

By [11], Theorems 1 and 14, there is a continuous extension $U : \ell^\infty \to G(\tau_G)$ of the restriction of fK to $(fK)^{-1}(G)$. Since f has closed graph, U coincides with fK in ℓ_0^∞. Then, exactly as we did in Theorem 1, but working with the restriction of U to $L(\mathcal{A})$ instead of T, we obtain that the $G(\tau_G)$–valued bounded measure μ defined in the σ-algebra of subsets of $\mathbb{N}$ by $\mu(A) = U(e(A))$ is strongly additive. Hence, taking $A_n = \{n\}$, we have that

$$\sum\Big\{\mu(\{n\}) \mid n \in \mathbb{N}\Big\} = \sum\Big\{f(x_n) \mid n \in \mathbb{N}\Big\}$$

converges in $G(\tau_G)$ and hence in Y. Thus by the closed graph condition,

$$\sum\{f(x_n), n \in \mathbb{N}\} = f\Big(\sum\{x_n \mid n \in \mathbb{N}\}\Big).$$

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