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by Mohamed A. Khamsi
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ON THE STABILITY OF THE FIXED POINT PROPERTY IN l_p SPACES

Mohamed A. Khamsi

The University of Texas at El Paso

Abstract. We prove in this work that for any $p > 1$ there exists an improved constant c_p, such that if $d(X, l_p) < c_p$ then X has the fixed point property.

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1. Introduction

Let X be a Banach space and let K be a nonempty weakly compact convex subset of X. We will say that K has the fixed point property (*f.p.p.* for short) if every nonexpansive $T : K \to K$ (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in K$) has a fixed point, *i.e.*, there exists $x \in K$ such that $T(x) = x$. We will say that X has the *f.p.p.* if every weakly compact convex subset of X has the *f.p.p.*

The fixed point property, as stated above, originated in four papers which appeared in 1965. They mainly assert that the presence of a geometric property, called "normal structure", implies the *f.p.p.*, [10]. A number of abstract results were brought to light, along with important discoveries related both to the structure of the fixed point sets and to techniques for approximating fixed points. The first negative result to the existence part of the theory goes to Alspach, cf. [2], who gave an example of a weakly compact convex subset K of L^1 and an isometry $T : K \to K$ which fails to have a fixed point. This example showed that further assumptions are needed in addition to weak compactness,

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and at the same time it set the stage for MAUREY's surprising discovery, [12] (see also [8], [11]).

For more on *f.p.p.*, one can consult [1], [5].

2. NOTATIONS, DEFINITIONS AND BASIC FACTS.

Let K be a nonempty weakly compact convex subset of a Banach space X. Suppose that $T: K \to K$ is nonexpansive. By Zorn's lemma, K contains a closed nonempty convex subset K_0 which is minimal for T. This means that $T(K_0) \subset K_0$ and that no strictly smaller closed nonempty convex subset of K_0 is invariant under T. A classical argument shows that any closed nonempty convex subset of K which is invariant under T contains an approximate fixed point sequence (*a.f.p.s.*) (x_n), *i.e.*, $\lim_{n\to\infty} \|x_n - T(x_n)\|_X = 0$.

The following lemma, cf. [4], [7], proved to be fundamental for the study of the *f.p.p.*

Lemma 1. *Suppose K_0 is a minimal weakly compact convex set for T and (x_n) is an a.f.p.s. for T. Then for all $x \in K_0$ we have*

$$\lim_{n\to n\infty} \|x_n - x\| = diam(K_0).$$

Since we will be using MAUREY's technique in proving our main result, we recall some basic definitions and facts.

Definition 1. *Let X be a Banach space and let $\mathcal{U}$ be a free ultrafilter over N. The ultraproduct $\widetilde{X}$ of X is the quotient space of*

$$l_\infty(X) = \big\{(x_n) \mid x_n \in X \text{ for all } n \in N \text{ and } \|(x_n)\|_\infty = \sup_n \|x_n\| < \infty\big\},$$

by

$$\mathcal{N} = \big\{(x_n) \in l_\infty(X) \mid \lim_{n\to\mathcal{U}} \|x_n\| = 0\big\}.$$

For $(x_n) \in l_\infty(X)$, we will denote $(x_n) + \mathcal{N}$ by $(x_n)_\mathcal{U} \in \widetilde{X}$. Clearly we have

$$\|(x_n)_\mathcal{U}\|_{\widetilde{X}} = \lim_{n\to\mathcal{U}} \|x_n\|.$$

It is also clear that X is isometric to a subspace of $\widetilde{X}$ by the mapping $x \to (x, x, \dots)$. Hence, we will see X as a subspace of $\widetilde{X}$ and we will write $\tilde{x}, \tilde{y}, \tilde{z}$ for the general elements of $\widetilde{X}$ and x, y, z for those of X.

Let K and T be as described above. Define $\widetilde{K}$ and $\widetilde{T}$ by

$$\widetilde{K} = \big\{\tilde{x} \in \widetilde{X} \mid \text{there exists a representative } (x_n) \text{ of } \tilde{x} \text{ with } x_n \in K \text{ for } n \geq 1\big\}$$

and $\widetilde{T}(\tilde{x}) = (T(x_n))_\mathcal{U}$ for any $\tilde{x} \in \widetilde{K}$. Then $\widetilde{K}$ is a bounded closed convex subset of $\widetilde{X}$ and $\widetilde{T}(\widetilde{K}) \subset \widetilde{K}$. We remark that $\widetilde{K}$ is not minimal for T. Indeed, if $(x_n) \subset K$ is an *a.f.p.s.*

then $\widetilde{T}(\tilde{x}) = \tilde{x}$, where $\tilde{x} = (x_n)_{\mathcal{U}}$. On the other hand, if $\tilde{x} = (x_n)_{\mathcal{U}}$ with $x_n \in K$ and $\widetilde{T}(\tilde{x}) = \tilde{x}$, then there exists a subsequence (x'_n) of (x_n) which is an *a.f.p.s.* for T. Finally we recall that if K_0 is a minimal set for T and $\tilde{x}$ is a fixed point for T in $\widetilde{K}_0$, then for any $x \in K_0$ we have, from Lemma 1, that

$$\|\tilde{x} - x\|_{\widetilde{X}} = \operatorname{diam}(K_0).$$

The next lemma was proved by Maurey in [12].

Lemma 2. *Suppose $\tilde{x}$ and $\tilde{y}$ are two fixed points of $\widetilde{T}$ in $\widetilde{K}$. Then for every $r \in (0,1)$, there exists a fixed point $\tilde{z}$ of $\widetilde{T}$ so that*

$$\|\tilde{x} - \tilde{z}\| = r\|\tilde{x} - \tilde{y}\| \ \text{ and } \ \|\tilde{y} - \tilde{z}\| = (1-r)\|\tilde{x} - \tilde{y}\|.$$

3. Main result

Let $p \in (1, \infty)$ and consider the function defined on [0,1] by

$$\varphi_p(x) = \frac{1 + (1-x)^p}{x^p + (1-x)^p}.$$

Then $\sup_{x \in [0,1]} \varphi_p(x) = \varphi_p(x_p)$ where x_p is the only root of

$$(1 - x^{p-1})(1-x)^{p-1} - x^{p-1} = 0$$

in [0,1]. It can be easily proved that

$$\lim_{p \to 1} x_p = 2, \quad \lim_{p \to \infty} x_p = 2 \ \text{ and } \ \varphi_p(x_p) = \frac{1}{x_p^{p-1}}.$$

Also one can check that $x_p < \frac{1}{\frac{1}{2^{p-1}}}$.

Now recall that the Banach–Mazur distance between two isomorphic Banach spaces X and Y, denoted $d(X,Y)$, is infimum of $\|U\|\,\|U^{-1}\|$ taken over all bicontinuous linear operators U from X onto Y.

We now state and prove the main result of this work.

Main Theorem. *Let X be a Banach space such that*

$$d(X, l_p) < c_p = \varphi_p(x_p)^{\frac{1}{p}}$$

for some $p > 1$. Then X has the f.p.p.

Proof. It is enough to prove that $X = (l_p, |\cdot|)$ has the *f.p.p.*, where $|\cdot|$ is an equivalent norm to $\|\cdot\|_p$ satisfying

$$\|\cdot\|_p \le |\cdot| \le d\,\|\cdot\|_p$$

with $d < c_p$.

Assume on the contrary that X fails to have the *f.p.p.* Then there exist K, a nonempty weakly compact convex subset, and $T : K \to K$, a nonexpansive map, with no fixed point. Without any loss of generality, we can assume that K is minimal for T and that $\operatorname{diam}(K) = 1$. Classical arguments imply that K contains an *a.f.p.s.* (x_n) which can be assumed to be weakly convergent to 0. Passing to subsequences, if needed, we may suppose that there exist coordinate projections P_{F_n} on X (with respect to the canonical Schauder basis of l_p) such that

$$\begin{aligned}
&(1) \quad F_n \cap F_m = \varnothing \text{ for } n \neq m,\\
&(2) \quad \lim_{n\to\infty} |x_n - P_{F_n}(x_n)| = 0,\\
&(3) \quad \lim_{n\to\infty} |x_n - x_{n+1}| = 1.
\end{aligned}$$

The subsets (F_n) can be chosen to be successive intervals, and (3) follows from Lemma 1. Put $u_n = P_{F_N}(x_n)$ for all $n \in N$. Then for $z \in l_p$ we have

$$\|z\|_p^p + \|z - u_n - u_{n+1}\|_p^p = \|z - u_n\|_p^p + \|z - u_{n+1}\|_p^p. \tag{$*$}$$

Let $\widetilde{X}$ be an ultraproduct of X and $\widetilde{K}$, $\widetilde{T}$ be as defined in the previous section. Set $\tilde{x} = (x_n)_{\mathcal{U}}$ and $\tilde{y} = (x_{n+1})_{\mathcal{U}}$. Then

$$\tilde{x} = (u_n)_{\mathcal{U}} \quad \text{and} \quad \tilde{y} = (u_{n+1})_{\mathcal{U}}.$$

Relation $(*)$ translates into $\widetilde{X}$ as

$$\|\tilde{z}\|_p^p + \|\tilde{z} - \tilde{x} - \tilde{y}\|_p^p = \|\tilde{z} - \tilde{x}\|_p^p + \|\tilde{z} - \tilde{y}\|_p^p \tag{$**$}$$

for every $\tilde{z} \in \widetilde{X}$. Let $r \in (0,1)$, and let $\tilde{z}$ be a fixed point of $\widetilde{T}$, as given by Lemma 2, such that

$$|\tilde{z} - \tilde{x}| = r \quad \text{and} \quad |\tilde{z} - \tilde{y}| = 1 - r.$$

Then,

$$\|\tilde{z} - \tilde{x} - \tilde{y}\|_p^p \geq \frac{1}{d^p}|\tilde{z} - \tilde{x} - \tilde{y}|^p \geq \frac{1}{d^p}\big(1 - |\tilde{z} - \tilde{x}|\big)^p \geq \frac{1}{d^p}(1-r)^p.$$

Hence,

$$\frac{1}{d^p}|\tilde{z}|^p + \frac{1}{d^p}(1-r)^p \leq \|\tilde{z}\|_p^p + \|\tilde{z} - \tilde{x} - \tilde{y}\|_p^p \leq |\tilde{z} - \tilde{x}|^p + |\tilde{z} - \tilde{y}|^p = r^p + (1-r)^p.$$

Then

$$|\tilde{z}|^p \leq d^p\big(r^p + (1-r)^p\big) - (1-r)^p,$$

and since $|\tilde{z}| = 1$, we get $\varphi_p(r) \leq d^p$. Since r was arbitrary in (0,1), we deduce that

$$\sup_{r\in(0,1)} \varphi_p(r) = \varphi_p(x_p) \leq d^p,$$

which contradicts our assumption on d. The proof of the main theorem is therefore complete. □

Remarks

(1) It is known [3], that if $d(X, l_p) < 2^{\frac{1}{p}}$ then X has the normal structure property and therefore, via a Theorem of KIRK [10], has the *f.p.p.* If $d(X, l_p) = 2^{\frac{1}{p}}$ then BYNUM [3] has proved that X has the *f.p.p.* He also gave an example of a situation where X fails to have normal structure. For

$$p > \frac{\ln(2)}{\ln\left(\frac{\sqrt{33}-3}{2}\right)},$$

one has to use the result of LIN [11], to prove that if $d(X, l_p) < \frac{\sqrt{33}-3}{2}$ then X has the *f.p.p.* It is worth mentioning that

$$c_p \geq c_2 = \frac{1}{\sqrt{x_2}} = \left(\frac{3+\sqrt{5}}{2}\right)^{\frac{1}{2}}$$

for every $p > 1$ and $c_2 > \frac{\sqrt{33}-3}{2}$. Therefore we get through the main theorem an improvement to all the well known results.

(2) It is a surprising fact that the constants (c_p) do no decrease as p goes to ∞. On the contrary, for $p \geq 2$ the constants c_p increase to 2, which by itself project new light on the stability of the fixed point property (for the l_p spaces).

(3) For $p = 2$ the main theorem reduces to the main result of [6].

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Mohamed A. Khamsi
Department of Mathematical Sciences
The University of Texas at El Paso
El Paso, Texas 79968-0514, USA
e–mail: khamsi@math-1.sci.kuniv.edu.kw
mohamed@banach.math.ep.utexas.edu.co