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Notable Finsler connections on a Finsler manifold

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Abstract. In this expository paper we present a comprehensive, invariant description of four important Finsler connections: the Berwald, Cartan, Chern–Rund and Hashiguchi connection. Following Grifone's theory [8], [9], our approach based on the Frölicher-Nijenhuis calculus of vector-valued forms and derivations [7], simplified by the technique of lifting vector fields to the tangent bundle. We give a fine analysis of the role of some axioms characterizing these connections, as well as explicit rules of calculations for the corresponding covariant derivatives.

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Resumen. En este artículo de divulgación se presenta una descripción geométrica de cuatro conecciones importantes dentro de una variedad de Finsler a saber: La de Cartan, la de Berwald, la de Chern-Rund y la de Hashiguchi. La técnica mostrada en la exposición para el cálculo de dichas conecciones es entre lo clásico y lo moderno como son las teorias de Grifone [8] [9] y el cálculo de formas diferenciales vectoriales de Frölicher–Nijenhuis [7]. En el presente artículo hacemos un análisis fino del papel de algunos axiomas que caracterizan dichas conecciones. Además se muestran los cálculos explícitos de las derivadas covariantes.

1. Introduction

There are four Finsler connections on a Finsler manifold which may be considered "natural" in some sense: the connections named Berwald, Cartan, Chern-Rund and Hashiguchi connection, respectively. (An important observation of M. Anastasiei [2] clarified that the Finsler connection introduced by H. Rund [20] is identical with the connection constructed by S.S. Chern [4] (for a recent account see [3]), so we use the terminology "Chern-Rund connection".) A satisfactory and truly aesthetical axiomatic description of Cartan's connection was first achieved by M. Matsumoto [13] in the sixties. A similar characterization of the Berwald connection is due to J. Grifone [9] and T. Okada [18] (in the seventies) and M. Abate [1] (lately). After the Cartan connection has been constructed, easy processes, baptized by M. Matsumoto "P^1-process" and "C-process" (see [15] and the concluding remark of this paper) yield the Chern-Rund, the Hashiguchi and the Berwald connection. The minimality of the axiom-systems of these notable Finsler connections has also been studied [16], by means of the classical tensor calculus.

In Matsumoto's global formulation [15] the framework for the Finsler connections is a principal bundle, the so-called Finsler bundle. In the exposition and applications of this theory the tools of the tensor calculus, adapted to the Finsler setting ([15], §6) play a dominant role. Another approach, developed in the most consistent and complete form by J. Grifone [8,9], works on the tangent bundle $\tau_{\mathcal{T}M}$.[1] Its distinctive feature is the systematic application of an *intrinsic calculus*, based on the *Frölicher-Nijenhuis formalism*. — Incidentally we remark that $\tau_{\mathcal{T}M}$ seems at first to be a too large arena for Finsler geometry.[2] Indeed, classically and from a vector bundle view-point Finsler connections, Riemann-Finsler metrics etc. live on the vertical bundle $\tau^v_{\mathcal{T}M}$ which is just the associated vector bundle of Matsumoto's Finsler bundle. Nevertheless, the larger arena is more convenient in many respects and, prescribing suitable conditions (see (FINS1) and (FINS2) in Definition 4), a faithful reconstruction of the classical picture is possible.

Well now, to make a long story short, in this paper we would like to link the above mentioned two approaches. Our presentation is based conceptually and technically on Grifone's work in the first place, but taking maximally into consideration Matsumoto's carefully elaborated theory and the results of his school.

[1]For basic notations see 1.1.

[2]Note that this seemingly wide scene can be significantly enlarged, preserving the fundamental geometric relations. With regard to this we refer to the work of R. Miron and his school [17].

What does this mean *in concreto*? — Let us see briefly some new or different features of our approach.

- From Okada's axioms we derive *intrinsically* the rules of calculation with respect to the *Berwald connection*, proving at the same time its unicity and existence. (We emphasize that the use of local coordinates will be completely avoided in this paper.) It turns out that Okada's axioms (O1)–(O2) guarantee the existence and the unicity of a Finsler connection $(\overset{\circ}{D}, h)$, while from the axioms (O3)–(O5) it follows that h is the *Barthel endomorphism*, described by Grifone's "fundamental theorem of Finsler geometry" ([8], II. 33 or Theorem B below). It becomes also clear that deleting axiom (O5), we have a free scope for important generalizations, see Remark 9.
- We present a more sophisticated version of the construction of the *Cartan connection*, which also puts into focus the main points. Our intrinsic proof adopts a "multiplied Christoffel process" and depends substantially on our generalization of the classial Cartan tensors and a delicate result concerning horizontal endomorphisms. In more detail, we suppose that the given horizontal endomorphism h is induced by a semispray. Then, in view of the cited result (Theorem A in our paper), the weak torsion (form) of h vanishes. This circumstance and Proposition 2 makes it possible to derive the rules of calculation (C1)–(C4).
- As to the *Chern-Rund* and the *Hashiguchi connection*, our statements are analogous. For example, (CHR1)–(CHR4) guarantee the existence and unicity of a Finsler connection $(\overset{R}{D}, h)$. If, in addition, (CHR5) is also required, then h becomes the Barthel endomorphism. The role of axioms (HSG1)–(HSG3) on the one hand and of the further axioms (HSG4)–(HSG6) on the other hand is similar.
- We establish an explicit relation among the covariant derivatives with respect to the Berwald, Cartan, Chern-Rund and Hashiguchi connections.

On the ground of these we also believe that our treatment puts some recent studies (see e.g. [1], [3]) into a general, transparent framework and makes them more applicable.

We conclude our overview with a few general and, partly, technical remarks. Because Grifone's theory and the Frölicher-Nijenhuis formalism are not so widely known, we shall review their rudiments in Section 1 in a relatively detailed form. This section also contains some novelties. We frequently found it advantageous to apply vertically and horizontally lifted vector fields. In connection with this, it proved useful to introduce the tension *form* and the strong torsion *form* in Lemma 1. (We note that the weak torsion form and the curvature form, whose practical role is similar, have already been constructed by Crampin [6]). Secondly, as we have already remarked, we had to define the so-called Cartan tensors under more general assumptions than classically. The important Proposition 2 concerning the second Cartan tensor is proved in section 2, using a Berwald-type Finsler connection.

1. Preliminaries

1.1. Notations and some basic facts. We try to make this paper self-contained in the framework of our possibilities. The interested reader can find all preparatory material required for a full understanding of the work in [11,12,8,9]. As we have just indicated, our approach and results depend considerably on Grifone's fundamental papers [8,9], so we adopt his terminology and conventions as far as feasible. The calculative apparatus which lie at the foundations of the theory was established by A. Frölicher and A. Nijenhuis. Their epoch-making paper [7] is still one of the best readable sources, for a nice recent account see [11].

(i) Throughout the paper the identity map of any set onto itself is denoted by 1.

(ii) We consider now once for all an n-dimensional ($n \in \mathbb{N}^+$), real, C^∞, connected, paracompact manifold M. $C^\infty(M)$ is the ring of real-valued smooth functions on M, $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M. $\Omega^k(M)$ ($k \in \mathbb{N}^+$) is the module of (scalar) k-forms on M, $\Omega^0(M) := C^\infty(M)$. $\Omega(M) := \oplus_{k=0}^n \Omega^k(M)$ is the graded algebra of differential forms, with multiplication given by the wedge product.

(iii) $\pi : TM \to M$ is the tangent bundle of M, it is also denoted by τ_M. $\pi_0 : \mathcal{T}M \to M$ is the subbundle of τ_M constituted by the nonzero tangent vectors to M. The kernel of the tangent map $T\pi$ (or $T\pi_0$) yields a canonical subbundle τ_{TM}^v of τ_{TM} (or $\tau_{\mathcal{T}M}^v$ of $\tau_{\mathcal{T}M}$), called the *vertical subbundle*. The sections of the bundles τ_{TM}^v and $\tau_{\mathcal{T}M}^v$ are said to be *vertical vector fields*; the $C^\infty(TM)$-module constituted by them is denoted by $\mathfrak{X}^v(TM)$ and $\mathfrak{X}^v(\mathcal{T}M)$, respectively. The *vertical lift* of a vector field $X \in \mathfrak{X}(M)$ is denoted by X^v, while X^c stands for the *complete lift* of X. It is well-known that for any vector fields $X, Y \in \mathfrak{X}(M)$

$$(1) \qquad [X^v, Y^v] = 0, \qquad [X^v, Y^c] = [X, Y]^v, \qquad [X^c, Y^c] = [X, Y]^c.$$

(iv) In the geometry of the tangent bundle two canonical objects play a dominant role: the *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ and the *vertical endomorphism*

$J : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$. We have:

$$\text{(2)} \qquad \operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM), \qquad J^2 = 0;$$

$$\text{(3)} \qquad JX^c = X^v, \qquad [C, X^v] = -X^v \qquad (X \in \mathfrak{X}(M)).$$

1.2. Vector forms and derivations. [7,10,11]

(i) $\Psi^k(M)$ $(k \in \mathbb{N}^+)$ denotes the $C^\infty(M)$-module of vector k-forms on M. It can be regarded as the module of k-linear skew-symmetric maps $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$; $\Psi^0(M) := \mathfrak{X}(M)$. If K is a vector 1-form, its *adjoint operator* $K^* : \Omega(M) \to \Omega(M)$ is defined by the formula

$$\text{(4)} \qquad K^*\omega(X_1, \ldots, X_k) := \omega(K(X_1), \ldots, K(X_k))$$
$$(\omega \in \Omega^k(M),\ k \in \mathbb{N}^+; \qquad X_i \in \mathfrak{X}(M),\ 1 \leq i \leq k).$$

(ii) A vector form $L \in \Psi^l(TM)$ $(l \in \mathbb{N}^+)$ is said to be *semibasic* if $\forall X \in \mathfrak{X}(TM)$: $i_{JX}L = 0$ and $J \circ L = 0$ (i_{JX} stands for the insertion operator induced by JX). In particular, a scalar form $\omega \in \Omega^k(TM)$ $(k \in \mathbb{N}^+)$ is called semibasic if $\forall X \in \mathfrak{X}(TM) : i_{JX}\omega = 0$.

(iii) Let us consider a vector k-form $K \in \Psi^k(M)$. In the Frölicher-Nijenhuis theory two derivations, denoted by i_K and d_K, are associated to K. Now we briefly recall their definitions.

- i_K is of degree $k-1$; $i_K \restriction C^\infty(M) := 0$; $i_K\omega = \omega \circ K$, if $\omega \in \Omega^1(M)$.
- $d_K := [i_K, d] := i_K \circ d - (-1)^{k-1} d \circ i_K$ is of degree k (d denotes the operator of the exterior derivative).

We get immediately that

$$\text{(5)} \qquad \forall f \in C^\infty(M) : d_K f = i_K df = df \circ K.$$

It can be proved that d_K is uniquely determined by its action over $C^\infty(M)$. In case of a vector 0-form $X \in \Psi^0(M) = \mathfrak{X}(M)$ i_X means the usual insertion operator, while d_X is the Lie derivative $\mathcal{L}_X$ with respect to X.

(iv) Suppose that $K \in \Psi^k(M)$, $L \in \Psi^l(M)$. The graded commutator of d_K and d_L is defined by the formula

$$[d_K, d_L] = d_K \circ d_L - (-1)^{kl} d_L \circ d_K.$$

A substantial result of the Frölicher-Nijenhuis theory states that there exists a unique vector form $[K, L] \in \Psi^{k+l}(M)$ such that

$$[d_K, d_L] = d_{[K,L]};$$

$[K, L]$ is said to be the *Frölicher-Nijenhuis bracket* of K and L. If K and L are vector 0-forms, i.e. vector fields on M, then $[K, L]$ reduces to the usual Lie bracket of vector fields.

(v) In the sequel we shall need the evaluation of the Frölicher-Nijenhuis bracket in some special cases. – Let $K, L \in \Psi^1(M)$. Then for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$(6) \qquad [K, Y](X) = [K(X), Y] - K[X, Y];$$

$$(7) \qquad \begin{aligned}[K, L](X, Y) = {} & [K(X), L(Y)] + [L(X), K(Y)] \\ & + K \circ L[X, Y] + L \circ K[X, Y] - K[X, L(Y)] \\ & - K[L(X), Y] - L[X, K(Y)] - L[K(X), Y];\end{aligned}$$

in particular

$$(8) \qquad \begin{aligned}\frac{1}{2}[K, K](X, Y) = {} & [K(X), K(Y)] + K^2[X, Y] \\ & - K[X, K(Y)] - K[K(X), Y].\end{aligned}$$

$N_K := \frac{1}{2}[K, K]$ is said to be the *Nijenhuis-torsion* of K.

(vi) The vertical endomorphism J can obviously be interpreted as a vector 1-form of $\Psi^1(TM)$. We have:

$$(9) \qquad N_J := \frac{1}{2}[J, J] = 0, \qquad [J, C] = J.$$

Remark 1. In the sequel we shall introduce vector k-forms ($k \in \mathbb{N}^+$) over TM or $\mathcal{T}M$. Their **differentiability will be required only over** $\mathcal{T}M$, unless otherwise stated.

1.3. Semisprays and horizontal endomorphisms.

Definition 1. A mapping $S : TM \to TTM$ is said to be a *semispray* on M if it satisfies the following conditions:

(SPR1) S is a vector field of class C^1 on TM.

(SPR2) S is smooth over $\mathcal{T}M$.

(SPR3) $JS = C$.

If, in addition,

(SPR4) $[C, S] = S$ (i.e. S is homogeneous of degree 2) then S is called a *spray*.

Definition 2. A *horizontal endomorphism* on M is a vector 1-form $h \in \Psi^1(TM)$ satisfying:

(HE1) h is smooth over $\mathcal{T}M$.

(HE2) h is a projector, i.e. $h^2 = h$.

(HE3) $\operatorname{Ker} h = \mathfrak{X}^v(\mathcal{T}M)$.

Having a horizontal endomorphism h, one can construct the following important geometric data and structures.

(i) *Horizontal lifting*:

$$X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}(\mathcal{T}M).$$

(ii) *Tension*: $H := [h, C] \in \Psi^1(TM)$. If H vanishes, we say that h satisfies the *homogeneity condition*.

(iii) *Weak torsion*: $t := [J, h] \in \Psi^2(TM)$.

(iv) *Strong torsion*: $T := i_{S_0} t + H \in \Psi^1(TM)$, S_0 is an arbitrary semispray on M.

(v) *Curvature tensor*: $R := -N_h = -\frac{1}{2}[h, h]$.

(vi) *Almost complex structure*: $F \in \Psi^1(TM)$, $F^2 = -1$; F is characterized by the relations

$$F \circ h = -J, \qquad F \circ J = h. \tag{10}$$

(vii) The (module) direct sum decomposition

$$\mathfrak{X}(TM) = \mathfrak{X}^v(TM) \oplus \mathfrak{X}^h(TM), \tag{11}$$

$\mathfrak{X}^h(TM) := \operatorname{Im} h$ is the submodule of *horizontal vector fields*.

For details, we refer to [8]. From (11), one can easily deduce the next

Local basis property. *If* $(X_1, \ldots, X_n)$ *is a local basis of* $\mathfrak{X}(M)$, *then* $(X_1^v, \ldots, X_n^v, X_1^h, \ldots, X_n^h)$ *is a local basis for* $\mathfrak{X}(TM)$.

This observation will be frequently used. The following relations will also be useful:

$$h \circ J = 0, \qquad J \circ h = J. \tag{12}$$

$$\text{If } v := 1 - h, \text{ then } J \circ F = v, \quad F \circ v = h \circ F. \tag{13}$$

$$\forall X \in \mathfrak{X}(M) : JX^h = X^v. \tag{14}$$

Lemma 1 and definition. *The vector forms H, t, T and the curvature tensor R are all semibasic, so they are completely determined by the following mappings:*

- $\eta : X \in \mathfrak{X}(M) \mapsto \eta(X) := H(X^h)$ – *tension form*
- $\tau : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \tau(X, Y) := t(X^h, Y^h)$ – *weak torsion form, briefly torsion form*
- $\theta : X \in \mathfrak{X}(M) \mapsto \theta(X) := T(X^h)$ – *strong torsion form*
- $\rho : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \rho(X, Y) := R(X^h, Y^h)$ – *curvature form. Explicitly,* $\forall X, Y \in \mathfrak{X}(M)$:

$$\eta(X) = [X^h, C]. \tag{15}$$

$$\tau(X, Y) = [X^h, Y^v] - [Y^h, X^v] - [X, Y]^v. \tag{16}$$

$$\theta(X) = v[hS_0, X^v] + X^h - X^c \ (S_0 \textit{ is an arbitrary semispray on } M). \tag{17}$$

$$\rho(X, Y) = -v[X^h, Y^h]. \tag{18}$$

Proof. A routine calculation shows that the considered vector forms are all semibasic, indeed. Then the local basis property guarantees that H, t, T and R are completely determined by η, τ, θ and ρ, respectively. It remains to verify (15)–(18). We are going to check only (15) and (16). (The proof of (18) is similar. To obtain (17), much more calculation is needed and we shall not apply this formula in the present paper.)

- $$\begin{aligned}\eta(X) &:= H(X^h) := [h, C]X^h \overset{(6)}{=} [hX^h, C] - h[X^h, C] \\ &= [X^h, C] - h[X^h, C] \overset{(11)}{=} v[X^h, C].\end{aligned}$$

Here the vector field $[X^h, C]$ is vertical, because

$$\begin{aligned}X^v &\overset{(14)}{=} JX^h \overset{(9)}{=} [J, C]X^h \overset{(6)}{=} [JX^h, C] - J[X^h, C] \\ &= [X^v, C] - J[X^h, C] \overset{(3)}{=} X^v - J[X^h, C]\end{aligned}$$

and hence $J[X^h, C] = 0$. So we obtain the relation $\eta(X) = [X^h, C]$.

- $$\begin{aligned}\tau(X, Y) &:= t(X^h, Y^h) := [J, h](X^h, Y^h) \overset{(7)}{=} [JX^h, hY^h] \\ &+ [hX^h, JY^h] + J \circ h[X^h, Y^h] + h \circ J[X^h, Y^h] \\ &- J[X^h, hY^h] - J[hX^h, Y^h] - h[X^h, JY^h] \\ &- h[JX^h, Y^h] \overset{(12),(14)}{=} [X^v, Y^h] + [X^h, Y^v] \\ &- J[X^h, Y^h] - h[X^h, Y^v] - h[X^v, Y^h].\end{aligned}$$

$[X^h, Y^v]$ and $[X^v, Y^h]$ are obviously vertical, so the last two terms vanish. Since

$$J[X^h, Y^h] = J(h[X^h, Y^h] + v[X^h, Y^h]) = J[X, Y]^h = [X, Y]^v,$$

we have the desired formula. ☑

The key relation between the semisprays and the horizontal endomorphisms is given by the following important result, discovered independently by M. Crampin [5,6] and J. Grifone [8].

Theorem A.

(i) *If S is a semispray on the manifold M, then*

$$h := \frac{1}{2}(1 + [J, S]) \tag{19}$$

is a horizontal endomorphism on M, whose weak torsion vanishes. If, in addition, S is a spray, then h satisfies the homogeneity condition, i.e. the tension of h also vanishes.

(ii) *Suppose, conversely, that h is a horizontal endomorphism on M. h arises from a semispray according to (19) if and only if its weak torsion form vanishes.* ☑

1.4. Finsler manifolds. To avoid any confusion arising from the different conventions, we have still to proceed with some very basic definitions and facts.

Definition 3. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E) (or simply M) is said to be a *Finsler manifold* with *energy function* E if the following conditions are satisfied:

(F1) $\forall\, a \in \mathcal{T}M : E(a) > 0;\ E(0) = 0.$

(F2) E is of class C^1 on TM and smooth over $\mathcal{T}M$.

(F3) $CE = 2E$, i.e. E is homogeneous of degree 2.

(F4) The form $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$, called the *fundamental form*, is nondegenerate.

Remark 2. Keeping the notations of the definition, we have the following useful identities:

$$i_C\omega = d_J E, \qquad \mathcal{L}_C\omega = \omega, \qquad i_J\omega = 0. \tag{20}$$

Lemma 2 and definition. *If (M, E) is a Finsler manifold with fundamental form ω, then the mapping*

$$\text{(21)} \qquad \bar{g} : \mathfrak{X}^v(\mathcal{T}M) \times \mathfrak{X}^v(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \quad (JX, JY) \to \bar{g}(JX, JY) := \omega(JX, Y)$$

is a well-constructed, nondegenerate, symmetric bilinear form (over $C^\infty(\mathcal{T}M)$), which is said to be the Riemann-Finsler metric *of (M, E).* ☑

Lemma 3 and definition. *On any Finsler manifold (M, E) there is a spray $S : TM \to TTM$ which is uniquely determined over $\mathcal{T}M$ by the formula*

$$i_S\omega = -dE.$$

This spray is called the canonical spray *of the Finsler manifold.* ☑

In his book [19], B. O'Neill refers to the fundamental lemma of semi-Riemannian geometry as "the miracle of semi-Riemannian geometry". Well now, the "*first miracle of Finsler geometry*" is the following essential result, due to J. Grifone [8].

Theorem B. *On a Finsler manifold (M, E) there is a unique horizontal endomorphism h such that*

(BT1) $d_hE = 0$ *("h is conservative").*

(BT2) *The strong torsion of h vanishes.*

h is given by the formula

$$h = \frac{1}{2}(1 + [J, S]),$$

where S is the canonical spray. ☑

We shall call the horizontal endomorphism guaranteed by Theorem B the *Barthel endomorphism* of the Finsler manifold.

Lemma 4. *Let us consider the Riemann-Finsler metric $\bar{g}$ and suppose that h is a horizontal endomorphism on M. If, as above, $v := 1 - h$, then*

$$\text{(22)} \qquad g : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \quad (X, Y) \to g(X, Y) := \bar{g}(JX, JY) + \bar{g}(vX, vY)$$

is a well-defined (pseudo-) Riemannian metric on $\mathcal{T}M$, called the prolongation *of $\bar{g}$ along h.* ☑

Remark 3. It can be checked immediately that

$$\text{(23)} \qquad \forall X, Y \in \mathfrak{X}(\mathcal{T}M) : g(hX, JY) = 0,$$

$$\text{(24)} \qquad g(C, C) = \bar{g}(C, C) = 2E.$$

Lemma 5 and definition. *Let (M,E) be a Finsler manifold and consider the prolonged metric g given by* (22). *There exists a unique tensor*

$$\mathcal{C} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$$

satisfying:

(i) $J \circ \mathcal{C} = 0$;

(ii) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : g(\mathcal{C}(X,Y), JZ) = \frac{1}{2}(\mathcal{L}_{JX} J^* g)(Y, Z).$

$\mathcal{C}$ has the following properties:

- *it is semibasic;*
- *if $\mathcal{C}_\flat(X,Y,Z) := g(\mathcal{C}(X,Y), JZ)$, then $\mathcal{C}_\flat$ is symmetric;*
- *$\mathcal{C}^0 := i_{S_0}\mathcal{C} = 0$, where S_0 is an arbitrary semispray.*

$\mathcal{C}$ (as well as $\mathcal{C}_\flat$) is called the first Cartan tensor *of the Finsler manifold.* ☑

Lemma 6 and definition. *Under the hypothesis of Lemma 5, there is a unique tensor field $\mathcal{C}' : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$ satisfying the following conditions:*

(i) $J \circ \mathcal{C}' = 0$

(ii) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : g(\mathcal{C}'(X,Y), JZ) = \frac{1}{2}(\mathcal{L}_{hX} g)(JY, JZ).$

Then $\mathcal{C}'$ is semibasic. $\mathcal{C}'$ (as well as its lowered tensor $\mathcal{C}'_\flat$) is said to be the second Cartan tensor *of the Finsler manifold, belonging to the horizontal endomorphism h.* ☑

2. Generalities on Finsler connections

Definition 4. Suppose that h is a horizontal endomorphism on M and D is a linear connection on the tangent manifold TM or on the manifold $\mathcal{T}M$.

(i) The pair (D, h) is said to be a *Finsler connection* on M if it satisfies the following two conditions:

(FINS1) D is *reducible*: $Dh = 0$.

(FINS2) D is *almost complex*: $DF = 0$.

(F is the almost complex structure associated with h.)

(ii) The covariant differential $DC \in \Psi^1(TM)$ is said to be the *deflection map*. If (D,h) is a Finsler connection, then

$$h^*(DC) : X \in \mathfrak{X}(TM) \mapsto DC(hX) = D_{hX}C$$

is called the *h-deflection*, while $v^*(DC)$ $(v = 1 - h)$ is the *v-deflection*.

Remark 4. The "Finsler connections" just introduced are the very same as the "normal *d*-connections" in [17] and are *in essence* the same as Matsumoto's pair connections ([15] Def. 9.1).

Remark 5. We get immediately from (FINS1) that the covariant derivatives of a vertical vector field are vertical while the covariant derivatives of a horizontal vector field remain horizontal. (FINS1) also implies (cf. [17,15]) that the torsion tensor field $\mathbb{T}$ of D is completely determined by the following mappings:

$$\begin{aligned}
\mathbb{A}(X,Y) &:= h\mathbb{T}(hX,hY) - (h)h - \text{torsion},\\
\mathbb{B}(X,Y) &:= h\mathbb{T}(hX,vY) - (h)hv - \text{torsion},\\
\mathbb{R}^1(X,Y) &:= v\mathbb{T}(hX,hY) - (v)h - \text{torsion},\\
\mathbb{P}^1(X,Y) &:= v\mathbb{T}(hX,vY) - (v)hv - \text{torsion},\\
\mathbb{S}^1(X,Y) &:= v\mathbb{T}(vX,vY) - (v)v - \text{torsion}.
\end{aligned}$$

Remark 6. It is easy to see that any Finsler connection (D,h) has the following properties:

- $DJ = 0$.
- D is completely determined by its action over $\mathfrak{X}(TM) \times \mathfrak{X}^v(TM)$, namely: for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$D_{vX}hY = FD_{vX}JY, \tag{25}$$

$$D_{hX}hY = FD_{hX}JY. \tag{26}$$

- The curvature tensor field $\mathbb{K}$ of D can be described by the following three mappings:

$$\begin{aligned}
\mathbb{R}(X,Y)Z &:= \mathbb{K}(hX,hY)JZ - h\text{-curvature},\\
\mathbb{P}(X,Y)Z &:= \mathbb{K}(hX,JY)JZ - hv\text{-curvature},\\
\mathbb{Q}(X,Y)Z &:= \mathbb{K}(JX,JY)JZ - v\text{-curvature}.
\end{aligned}$$

Definition 5. (cf. [9,15,21]) The mapping

$$D^i : \mathfrak{X}^v(TM) \times \mathfrak{X}^v(TM) \to \mathfrak{X}^v(TM), \quad (JX, JY) \mapsto D^i_{JX}JY = [J, JY]X$$

is said to be the *intrinsic* or the *flat v-connection* in the vertical bundle τ^v_{TM}.

Remark 7.

- It is easy to check that D^i is well-defined: $JX' = JX$ implies that $D^i_{JX'}JY = D^i_{JX}JY$.
- D^i is a very simple but excessively important example of the so-called *pseudoconnections*; for their role played in the foundation of the theory of Finsler connections we refer to [21] and [22].
- Applying the property $[J, J] = 0$, a simple calculation yields the evaluated formula

$$D^i_{JX}JY = J[JX, Y]. \tag{27}$$

Definition 6. A Finsler connection $(\overset{\circ}{D}, h)$ is said to be of *Berwald type*, if it satisfies the following conditions:

$$\overset{\circ}{D} \restriction \mathfrak{X}^v(TM) \times \mathfrak{X}^v(TM) = D^i, \quad \text{i.e.} \quad J^*\overset{\circ}{D} = D^i. \tag{Brw1}$$

$$\forall X, Y \in \mathfrak{X}(TM) : \overset{\circ}{D}_{hX}JY = v[hX, JY]. \tag{Brw2}$$

Corollary 1. *If $(\overset{\circ}{D}, h)$ is a Finsler connection of Berwald type, then for any vector fields $X, Y \in \mathfrak{X}(TM)$,*

$$\overset{\circ}{D}_{vX}hY = h[vX, Y], \tag{Brw3}$$

$$\overset{\circ}{D}_{hX}hY = hF[hX, JY]. \tag{Brw4}$$

Proof. This follows immediately by (25) and (26). ☑

Proposition 1. *Suppose that $(\overset{\circ}{D}, h)$ is a Finsler connection of Berwald type.*

(i) The h-deflection of $(\overset{\circ}{D}, h)$ coincides with the tension of h.

(ii) The torsion tensor field $\overset{\circ}{\mathbb{T}}$ of $\overset{\circ}{D}$ can be represented in the form

$$\overset{\circ}{\mathbb{T}} = F \circ t + R \tag{28}$$

(t is the weak torsion, R is the curvature of h; see 1.3).

Proof.

(i) Let $X \in \mathfrak{X}(M)$ be arbitrary and let us choose a semispray S_0. Then, taking into account the proof of Lemma 1, we get:

$$\overset{\circ}{D}_{X^h} C = \overset{\circ}{D}_{X^h} J S_0 \overset{\text{(Brw2)}}{=} v[X^h, C] = \eta(X) = H(X^h).$$

This proves our first claim.

(ii) Let $X, Y \in \mathfrak{X}(M)$ be arbitrary. Since $[X^h, Y^v]$ is vertical, it follows that

$$\begin{aligned}\overset{\circ}{\mathbb{T}}(X^h, Y^v) &= \overset{\circ}{D}_{X^h} Y^v - \overset{\circ}{D}_{Y^v} X^h - [X^h, Y^v] \overset{\text{(Brw2,3)}}{=} \\ &= v[X^h, Y^v] - h[Y^v, X^h] - [X^h, Y^v] = 0.\end{aligned}$$

Similarly,

$$\overset{\circ}{\mathbb{T}}(X^v, Y^v) = \overset{\circ}{D}_{X^v} Y^v - \overset{\circ}{D}_{Y^v} X^v - [X^v, Y^v] \overset{\text{(Brw1),(1)}}{=} 0.$$

So we have to evaluate $\overset{\circ}{\mathbb{T}}$ only on a pair of form (X^h, Y^h). We obtain:

$$\begin{aligned}\overset{\circ}{\mathbb{T}}(X^h, Y^h) &\overset{\text{(Brw4), (13)}}{=} Fv[X^h, Y^v] - Fv[Y^h, X^v] - [X^h, Y^h] \\ &= F([X^h, Y^v] - [Y^h, X^v]) - [X, Y]^h - v[X^h, Y^h] \\ &\overset{(10)}{=} F([X^h, Y^v] - [Y^h, X^v] - J[X, Y]^h) - v[X^h, Y^h] \\ &\overset{\text{(14),Lemma 1}}{=} (F \circ t + R)(X^h, Y^h).\end{aligned}$$

This proves what we wanted. ☑

Corollary 2. *Hypothesis as in Proposition 1.*

(i) *The h-deflection of $(\overset{\circ}{D}, h)$ vanishes if and only if h satisfies the homogeneity condition.*

(ii) *The $(h)h$ torsion $\overset{\circ}{\mathbb{A}}$ of $\overset{\circ}{D}$ is related to the weak torsion of h by the formula*

$$\overset{\circ}{\mathbb{A}} = F \circ t,$$

consequently the vanishing of $\overset{\circ}{\mathbb{A}}$ is equivalent with the vanishing of the weak torsion t. In this case the torsion of $\overset{\circ}{D}$ coincides with the curvature of h.

Proof. It is enough to note that for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$\begin{aligned}\overset{\circ}{\mathbb{A}}(X, Y) := h\overset{\circ}{\mathbb{T}}(hX, hY) &\overset{(28)}{=} hFt(hX, hY) + hR(hX, hY) \\ &\overset{(13)}{=} F \circ t(hX, hY),\end{aligned}$$

because t and R are both semibasic. ☑

Corollary 3. *If* $(\overset{\circ}{D}, h)$ *is a Finsler connection of Berwald type with vanishing h-deflection, then the weak torsion of h is homogeneous* of degree 0, i.e.* $[C, t] = -t$.

Proof. $[C, t] = [C, [J, h]]$. The graded Jacobi identity gives the relation

$$0 = [C, [J, h]] + [J, [h, C]] - [h, [C, J]].$$

Since $(\overset{\circ}{D}, h)$ has no h-deflection, the second term vanishes by Corollary 2/(i), while the third terms gives $-t$ by (9). Thus we obtain the desired formula. ☑

Proposition 2. *Let* (M, E) *be a Finsler manifold and* $(\overset{\circ}{D}, h)$ *a Berwald-type Finsler connection on* M. *Consider the second Cartan tensor* $\mathcal{C}'$ *belonging to* h.

(i) *For any vector fields* $X, Y, Z \in \mathfrak{X}(M)$,

$$2\mathcal{C}'_b(X^c, Y^c, Z^c) = \left(\overset{\circ}{D}_{X^h} g\right)(Y^v, Z^v) = \left[Y^v, [X^h, Z^v]\right]E + Y^v\left[Z^v(X^h E)\right].$$

(ii) *If h is conservative and its torsion form τ vanishes, then $\mathcal{C}'_b$ is totally symmetric.*

Proof.

(a) Let us note first that for any vector fields $X, Y \in \mathfrak{X}(M)$, the relation

$$(*) \qquad g(X^v, Y^v) = \bar{g}(X^v, Y^v) = X^v(Y^v E)$$

holds. Indeed,

$$\begin{aligned} g(X^v, Y^v) &= \bar{g}(X^v, Y^v) \overset{(3)}{=} \bar{g}(JX^c, JY^c) \\ &\overset{(21)}{=} \omega(X^v, Y^c) \overset{(F4)}{=} d(d_J E)(X^v, Y^c) \\ &= X^v\left(d_J E(Y^c)\right) - Y^c\left(d_J E(X^v)\right) - d_J E[X^v, Y^c] \\ &\overset{(5)}{=} X^v\left(dE(JY^c)\right) - Y^c\left(dE(JX^v)\right) - dE\left(J[X^v, Y^c]\right) \\ &\overset{(1),(2),(3)}{=} X^v\left(dE(Y^v)\right) = X^v(Y^v E). \end{aligned}$$

* For the definition of homogeneity of vector forms see [8], I. 4.

(b) Now we prove (i). For any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
2\mathcal{C}'_b(X^c, Y^c, Z^c) &= (\mathcal{L}_{hX^c} g)\,(JY^c, JZ^c) \overset{(3)}{=} (\mathcal{L}_{X^h} g)\,(Y^v, Z^v) \\
&= \Big(\overset{\circ}{D}_{X^h} g\Big)(Y^v, Z^v) = X^h g(Y^v, Z^v) - g\big([X^h, Y^v], Z^v\big) \\
&\quad - g\big(Y^v, [X^h, Z^v]\big) \overset{(*)}{=} X^h\big[Y^v(Z^v E)\big] \\
&\quad - [X^h, Y^v](Z^v E) - g\big([X^h, Z^v], Y^v\big) \\
&= Y^v\big[X^h(Z^v E)\big] - [X^h, Z^v](Y^v E) \\
&= Y^v\big([X^h, Z^v]E + Z^v(X^h E)\big) - [X^h, Z^v](Y^v E) \\
&= \big[Y^v, [X^h, Z^v]\big]E + Y^v\big[Z^v(X^h E)\big].
\end{aligned}$$

(c) Suppose that h is a conservative horizontal endomorphism with vanishing weak torsion. Then for any vector field $X \in \mathfrak{X}(M)$,

$$0 = (d_h E)\,(X^c) \overset{(5)}{=} dE(hX^c) = dE(X^h) = X^h E,$$

and hence (from (i)),

$$2\mathcal{C}'_b(X^c, Y^c, Z^c) = \big[Y^v, [X^h, Z^v]\big]E.$$

Now, using the Jacobi identity and the condition $\tau = 0$, we obtain: for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
0 &= \big[Y^v, [X^h, Z^v]\big] + \big[X^h, [Z^v, Y^v]\big] + \big[Z^v, [Y^v, X^h]\big] \\
&\overset{(1)}{=} \big[Y^v, [X^h, Z^v]\big] + \big[Z^v, [Y^v, X^h]\big] \overset{(16)}{=} \big[Y^v, [X^h, Z^v]\big] \\
&\quad + \big[Z^v, -[Y^h, X^v]\big] + \big[Z^v, -[X, Y]^v\big] \\
&\overset{(1)}{=} \big[Y^v, [X^h, Z^v]\big] - \big[Z^v, [Y^h, X^v]\big];
\end{aligned}$$

therefore $[Y^v, [X^h, Z^v]] = [Z^v, [Y^h, X^v]]$. This means that

$$\mathcal{C}'_b(X^c, Y^c, Z^c) = \mathcal{C}'_b(Y^c, Z^c, X^c).$$

The further symmetries of $\mathcal{C}'_b$ can be shown in the same manner. ☑

3. The Berwald connection of a Finsler manifold

Theorem 1. *(cf.* [1,9,18]*) Let a Finsler manifold (M, E) be given and suppose that h is a horizontal endomorphism on M.*

(i) *There is a unique Finsler connection $(\overset{\circ}{D}, h)$ on M such that*

(O1) *The $(v)hv$ torsion $\overset{\circ}{\mathbb{P}}{}^1$ of $\overset{\circ}{D}$ vanishes:* $\boxed{\overset{\circ}{\mathbb{P}}{}^1 = 0}$.

(O2) *The $(h)hv$ torsion $\overset{\circ}{\mathbb{B}}$ of $\overset{\circ}{D}$ vanishes:* $\boxed{\overset{\circ}{\mathbb{B}} = 0}$.

Then $(\overset{\circ}{D}, h)$ is of Berwald type, so the covariant derivatives with respect to $\overset{\circ}{D}$ can be explicitly calculated by (BRW1)–(BRW4).

(ii) *If $(\overset{\circ}{D}, h)$ satisfies the further conditions*

(O3) $d_h E = 0$ *(i.e., h is conservative),*

(O4) *the h-deflection $h^*(\overset{\circ}{D}C)$ vanishes,*

(O5) *the $(h)h$ torsion $\overset{\circ}{\mathbb{A}}$ of $\overset{\circ}{D}$ vanishes,*

then h is just the Barthel-endomorphism of the Finsler manifold.

Proof.

Step 1. We assume that $(\overset{\circ}{D}, h)$ is a Finsler connection, satisfying (O1)–(O2). First we show that (O1) and (O2) imply (BRW2) and (BRW3). Indeed:

<u>(O1) ⇒ (BRW2)</u> Let $X, Y \in \mathfrak{X}(TM)$ be arbitrary. Then

$$0 \overset{(O1)}{=} \overset{\circ}{\mathbb{P}}{}^1(X,Y) := v\overset{\circ}{\mathbb{T}}(hX, JY) = v(\overset{\circ}{D}_{hX} JY - \overset{\circ}{D}_{JY} hX - [hX, JY],$$
$$= \overset{\circ}{D}_{hX} JY - v[hX, JY] \qquad \text{(by Remark 5), thus we get (BRW2).}$$

<u>(O2) ⇒ (BRW3)</u> Indeed, $\forall X, Y \in \mathfrak{X}(TM)$:

$$0 \overset{(O2)}{=} \overset{\circ}{\mathbb{B}}(Y,X) = h\overset{\circ}{\mathbb{T}}(hY, JX) = h(\overset{\circ}{D}_{hY} JX - \overset{\circ}{D}_{JX} hY - [hY, JX])$$
$$= -\overset{\circ}{D}_{JX} hY - h[hY, JX],$$

hence $\overset{\circ}{D}_{JX}hY = h[JX, hY]$. Replacing X by FX, this is just (Brw3). Having these formulas, now we derive (Brw4) and (Brw1). For any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$0 \overset{\text{(Fins2)}}{=} \overset{\circ}{D}F(JY, hX) = \overset{\circ}{D}_{hX}FJY - F\overset{\circ}{D}_{hX}JY$$
$$\overset{(10)}{\Rightarrow} \overset{\circ}{D}_{hX}hY = F\overset{\circ}{D}_{hX}JY \overset{\text{(Brw2)}}{=} Fv[hX, JY] = hF[hX, JY],$$

so (Brw4) is valid;

$$0 = \overset{\circ}{D}F(JY, JX) = \overset{\circ}{D}_{JX}FJY - F\overset{\circ}{D}_{JX}JY \overset{(10)}{=}$$
$$= \overset{\circ}{D}_{JX}hY - F\overset{\circ}{D}_{JX}JY,$$

therefore

$$\overset{\circ}{D}_{JX}JY = -F\overset{\circ}{D}_{JX}hY \overset{\text{(Brw3)}}{=} -Fh[JX, Y] \overset{(10)}{=} J[JX, Y],$$

thus we have obtained (Brw1).

Step 2. Suppose that (O3)–(O5) are also satisfied. Then (O5) implies by Corollary 2/(ii) the vanishing of the weak torsion of h, and from (O4) it follows that the tension H also vanishes. Thus the strong torsion

$$T := i_{S_0}t + H = 0.$$

This, together with (O3), means that h is indeed the Barthel endomorphism.

Step 3. The method of the existence proof is the following. We consider the Barthel endomorphism h of (M, E) (its existence is guaranteed by Theorem B) and *define* $\overset{\circ}{D}$ by the formulas (Brw1)–(Brw4). Then (O3) is satisfied automatically, while (O4) and (O5) follow by Corollary 2. It remains to check that $\overset{\circ}{D}$ is indeed a linear connection and (O1), (O2) are also satisfied. This can be done by an easy straightforward calculation so we omit the details. ☑

Remark 8. The connection just described is said to be the *Berwald connection* of the Finsler manifold of (M, E). Axioms (O1)–(O5) were formulated by T. Okada [18] with a slight difference. Based on these axioms, Okada calculated the connection parameters by means of classical tensor calculus. For another, but closely related, approach see [9]. Recently, M. Abate [1] also characterized the Berwald connection. Similarly to Okada's, his proof is not coordinate-free.

Remark 9. In his paper [14], M. Matsumoto raises the following question: keeping the axioms (O1)–(O4), is there some possibility to revive the $(h)h$-torsion tensor field? He gives a criterion for the existence of a "*Berwald connection with nonvanishing* $(h)h$*-torsion*" or "$B\Gamma T$*-connection*" in the following local form:

$$y^r\left(\frac{\partial}{\partial y^k}T_j{}^i{}_r - \frac{\partial}{\partial y^j}T_k{}^i{}_r\right) = 0; \tag{$**$}$$

here $T_j{}^i{}_k$ are the components of the $(h)h$-torsion with respect to a suitable local basis of $\mathfrak{X}(\mathcal{T}M)$. Now we are in a position to present a brief discussion of this problem with the help of our intrinsic tools.

- Because of Corollary 2/(ii), the desired horizontal endomorphism must be different from the Barthel endomorphism and, by Theorem A/(ii), it certainly *does not arise from a semispray*.
- According to Corollary 3, the 0-homogeneity of the weak torsion is forced, so

  $$[C, t] = -t \tag{29}$$

  gives a necessary condition. (Making some effort, it can be shown that (29) implies the *1-homogeneity* of the $(h)h$-torsion $\overset{\circ}{\mathbb{A}} \overset{\text{Cor.2}}{=} F \circ t$.)
- A further necessary condition can be obtained as follows: Applying (5.8) of [7] and (2), we have:

  $$\begin{aligned} 0 &= [J,[J,h]] + [J,[h,J]] + [h,[J,J]] \\ &= 2[J,[J,h]] = 2[J,t], \end{aligned}$$

  hence

  $$[J, t] = 0. \tag{30}$$

- An easy but a little lengthy calculation shows that in local coordinates (29) and (30) lead to $(**)$, so we have obtained an intrinsic formulation of Matsumoto's criterion. This formulation seems to be fruitful for further study of such important topics as generalized Berwald spaces, Wagner spaces and so on; for remarkable recent results see [23].

4. The Cartan connection on a Finsler manifold

Theorem 2. *(cf.* [9,15]*) Let a Finsler manifold (M, E) be given and suppose that h is a horizontal endomorphism on M,* arising from a semispray. *Let g be the prolongation of $\bar{g}$ along h and $\mathcal{C}'$ the second Cartan tensor belonging to h.*

(i) *There is a unique Finsler connection (D, h) on M (D is given on $\mathcal{T}M$) such that*

(M1) *D is metrical:* $\boxed{Dg = 0}$.

(M2) *The $(v)v$-torsion $\mathbb{S}^1$ of D vanishes:* $\boxed{\mathbb{S}^1 = 0}$.

(M3) *The $(h)h$-torsion $\mathbb{A}$ of D vanishes:* $\boxed{\mathbb{A} = 0}$.

(M4) *The h-deflection of (D, h) vanishes:* $\boxed{h^*DC = 0}$.

The covariant derivatives with respect to D can be explicitly calculated by the following formulas:

$$\boxed{\begin{array}{ll} \text{(C1)} & D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y) \\ \text{(C2)} & D_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y) \\ \text{(C3)} & D_{vX}hY = h[vX, Y] + F\mathcal{C}(FX, Y) \\ \text{(C4)} & D_{hX}hY = hF[hX, JY] + F\mathcal{C}'(X, Y) \end{array}}$$

$$(X, Y \in \mathfrak{X}(\mathcal{T}M)).$$

(ii) *If, in addition,*

(M5) *h is homogeneous:* $\boxed{H = [h, C] = 0}$

then h coincides with the Barthel endomorphism of the Finsler manifold.

Proof. The plan of attack is the very same as that of Theorem 1. We are going to show the unicity of (D, h) and to check that h is the Barthel endomorphism. The existence proof will be omitted again.

Unicity.

(a) First we derive (C1). For the sake of simplicity, we use vertically lifted vector fields, so let $X, Y, Z \in \mathfrak{X}(M)$ be arbitrary. From (M1) it follows that

$$\begin{aligned} X^v g(Y^v, Z^v) &= g(D_{X^v} Y^v, Z^v) + g(Y^v, D_{X^v} Z^v), \\ Y^v g(Z^v, X^v) &= g(D_{Y^v} Z^v, X^v) + g(Z^v, D_{Y^v} X^v), \\ -Z^v g(X^v, Y^v) &= -g(D_{Z^v} X^v, Y^v) - g(X^v, D_{Z^v} Y^v). \end{aligned}$$

Adding these three equations and using that from (M2)

$$D_{X^v}Y^v - D_{Y^v}X^v = [X^v, Y^v] \overset{(1)}{=} 0 \qquad \textit{and so on,}$$

we get the relation

$$g(2D_{X^v}Y^v, Z^v) = X^v g(Y^v, Z^v) + Y^v g(Z^v, X^v) - Z^v g(X^v, Y^v). \tag{31}$$

From the definition of the tensor $\mathcal{C}$ (Lemma 5)

$$\begin{aligned} 2g(\mathcal{C}(X^h, Y^h), JZ^h) = X^v g(Y^v, Z^v) - g(J[X^v, Y^h], Z^v) \\ - g(Y^v, J[X^v, Z^h]). \end{aligned}$$

$[X^v, Y^h]$ and $[X^v, Z^h]$ are vertical hence $J[X^v, Y^h] = J[X^v, Z^h] = 0$. So we have:

$$2g(\mathcal{C}(X^h, Y^h), JZ^h) = X^v g(Y^v, Z^v).$$

In the same way,

$$\begin{aligned} 2g(\mathcal{C}(Y^h, Z^h), JX^h) &= Y^v g(Z^v, X^v), \\ -2g(\mathcal{C}(Z^h, X^h), JY^h) &= -Z^v g(X^v, Y^v). \end{aligned}$$

Adding again the last three equations and taking into account the symmetry of $\mathcal{C}$, we get

$$\begin{aligned} g(2\mathcal{C}(X^h, Y^h), Z^v) = X^v g(Y^v, Z^v) + Y^v g(Z^v, X^v) \\ - Z^v g(X^v, Y^v). \end{aligned} \tag{32}$$

From (31) and (32) it follows that

$$D_{X^v}Y^v = \mathcal{C}(X^h, Y^h),$$

which is just (C1) in the case $X := X^h$, $Y := Y^h$. If $f \in C^\infty(\mathcal{T}M)$, then

$$D_{X^v} fY^v \overset{(14)}{=} D_{JX^h} fJY^h = (X^v f)Y^v + f\mathcal{C}(X^h, Y^h);$$

on the other side

$$J[JX^h, fY^h] = J[X^v, fY^h] = J[(X^v f)Y^h] = (X^v f)Y^v,$$

therefore (C1) holds in the general case too.

(b) Next we show that h *is conservative*. Let $X \in \mathfrak{X}(M)$ be an arbitrary vector field. Applying conditions (M1) and (M4) we obtain:

$$0 \overset{\text{(M1)}}{=} (D_{X^h} g)(C, C) = X^h g(C, C) - 2g(D_{X^h} C, C)$$
$$\overset{\text{(M4)}}{=} X^h g(C, C) \overset{(24)}{=} 2X^h E = 2dE(X^h) = 2d_h E(X^c).$$

This means that $d_h E = 0$, indeed. Since h arises from a semispray, in view of Theorem A/(ii) its torsion form τ vanishes. Thus we can conclude by Proposition 2 that *the second Cartan tensor $\mathcal{C}'$ belonging to h is symmetric.*

(c) Now we are in a position to derive (C2). As in the preceding reasoning, we begin with the usual "Christoffel process". $\forall X, Y, Z \in \mathfrak{X}(M)$:

$$(33) \qquad \begin{cases} X^h g(Y^v, Z^v) = g(D_{X^h} Y^v, Z^v) + g(Y^v, D_{X^h} Z^v), \\ Y^h g(Z^v, X^v) = g(D_{Y^h} Z^v, X^v) + g(Z^v, D_{Y^h} X^v), \\ -Z^h g(X^v, Y^v) = -g(D_{Z^h} X^v, Y^v) - g(X^v, D_{Z^h} Y^v) \end{cases}$$

(since D is "h-metrical" as well). From the vanishing of the $(h)h$-torsion

$$D_{X^h} Y^h - D_{Y^h} X^h = h[X^h, Y^h] = [X, Y]^h,$$

hence

$$F D_{X^h} Y^h - F D_{Y^h} X^h = F h[X, Y]^c \overset{(10),(3)}{=} -[X, Y]^v.$$

Here $F D_{X^h} Y^h \overset{(26)}{=} -D_{X^h} J Y^h = -D_{X^h} Y^v$ and, similarly, $F D_{Y^h} X^h = -D_{Y^h} X^v$. So we obtain:

$$(34) \qquad \begin{cases} D_{X^h} Y^v - D_{Y^h} X^v = [X, Y]^v, \\ D_{Y^h} Z^v - D_{Z^h} Y^v = [Y, Z]^v, \\ -D_{Z^h} X^v + D_{X^h} Z^v = -[Z, X]^v. \end{cases}$$

Adding now both sides of (33) and using (34) it follows that

$$(35) \qquad \begin{aligned} g(2D_{X^h} Y^v, Z^v) = {} & X^h g(Y^v, Z^v) + Y^h g(Z^v, X^v) \\ & - Z^h g(X^v, Y^v) + g([X, Y]^v, Z^v) \\ & - g([Y, Z]^v, X^v) + g([Z, X]^v, Y^v). \end{aligned}$$

In the next step we apply the Christoffel process to the tensor $\mathcal{C}'$ belonging to h. We get:

$$\begin{aligned} 2g(\mathcal{C}'(X^h,Y^h),Z^v) &= X^h g(Y^v,Z^v) - g([X^h,Y^v],Z^v) - g(Y^v,[X^h,Z^v]),\\ 2g(\mathcal{C}'(Y^h,Z^h),X^v) &= Y^h g(Z^v,X^v) - g([Y^h,Z^v],X^v) - g(Z^v,[Y^h,X^v]),\\ -2g(\mathcal{C}'(Z^h,X^h),Y^v) &= -Z^h g(X^v,Y^v) + g([Z^h,X^v],Y^v) + g(X^v,[Z^h,Y^v]). \end{aligned}$$

We add these three equations. Using the symmetry of $\mathcal{C}'$ (assured in (b)), the following relation drops out:

$$\begin{aligned} &g(2\mathcal{C}'(X^h,Y^h),Z^v) = X^h g(Y^v,Z^v) + Y^h g(Z^v,X^v)\\ &-Z^h g(X^v,Y^v) - g([X^h,Y^v]+[Y^h,X^v],Z^v) + g([Z^h,X^v]\\ &-[X^h,Z^v],Y^v) + g([Z^h,Y^v]-[Y^h,Z^v],X^v). \end{aligned} \tag{36}$$

Substituting the first three terms of the right side of (35) from (36), we have:

$$\begin{aligned} g(2D_{X^h}Y^v,Z^v) = g(2\mathcal{C}'(X^h,Y^h),Z^v)&\\ + g([X^h,Y^v]+[Y^h,X^v]+[X,Y]^v,Z^v)&\\ + g([X^h,Z^v]-[Z^h,X^v]-[X,Z]^v,Y^v)&\\ + g([Y^h,Z^v]-[Z^h,Y^v]-[Y,Z]^v,X^v).& \end{aligned} \tag{37}$$

By our assumption h *arises from a semispray* so in view Theorem A/(ii) the torsion form τ of h vanishes. Thus (by (16))

$$\begin{aligned} \tau(X,Y) &= [X^h,Y^v]-[Y^h,X^v]-[X,Y]^v = 0,\\ \tau(X,Z) &= [X^h,Z^v]-[Z^h,X^v]-[X,Z]^v = 0,\\ \tau(Y,Z) &= [Y^h,Z^v]-[Z^h,Y^v]-[Y,Z]^v = 0; \end{aligned}$$

consequently the third and the fourth term on the right side of (37) vanish, while in the second term

$$[X^h,Y^v]+[Y^h,X^v]+[X,Y]^v = 2[X^h,Y^v].$$

So at last we have achieved our aim: it follows that

$$g(2D_{X^h}Y^v,Z^v) = g(2\mathcal{C}'(X^h,Y^h)+2[X^h,Y^v],Z^v),$$

therefore
$$D_{X^h}Y^v = [X^h, Y^v] + \mathcal{C}'(X^h, Y^h).$$
This proves (C2) and, at the same time, the unicity assertion of the Theorem. ☑

Remark 10.

- The Finsler connection (D, h) described by the theorem is called the *Cartan connection* of the Finsler manifold (M, E). It is most closely related to the Levi-Civita connection of a Riemannian manifold so Theorem 2 can be referred to as the "*second miracle of Finsler geometry*".
- Axioms (M1)–(M5) were first formulated by M. Matsumoto; for a brief historical account and a comparison see his monograph [15]. Grifone's axioms in [8] are the same *in essence*. We would like to emphasize two important features of our treatment again.
  - The starting horizontal endomorphism is *not* supposed to be the Barthel endomorphism.
  - The fine logical connections among the axioms (M1)–(M5), the rules (C1)–(C4) and the (partly forced) properties of the given horizontal endomorphism became transparent.

5. The Chern-Rund and the Hashiguchi connection

After having proved Theorems 1 and 2, following the line of the preceding reasoning we can easily deduce the following results.

Corollary 4. *(cf.* [15,16]*) Let (M, E) be a Finsler manifold and h a horizontal endomorphism on M, arising from a semispray. Let us consider the Riemannian metric g given by (22).*

(i) *There is a unique Finsler connection $(\overset{R}{D}, h)$ on M, satisfying:*

(CHR1) $J^*\overset{R}{D} = D^i$,

(CHR2) $\overset{R}{D}$ *is h-metrical (i.e.,* $\forall X \in \mathfrak{X}(\mathcal{T}M) : \overset{R}{D}_{hX}g = 0$),

(CHR3) *the $(h)h$-torsion of $\overset{R}{D}$ vanishes,*

(CHR4) *the h-deflection $h^*(\overset{R}{D}C)$ of $\overset{R}{D}$ vanishes.*

The covariant derivatives with respect to $\overset{R}{D}$ and their connection with the previously described ones are given by the following table:

(R1)	$\overset{R}{D}_{JX}JY = J[JX,Y] = \overset{\circ}{D}_{JX}JY$
(R2)	$\overset{R}{D}_{hX}JY = v[hX,JY] + \mathcal{C}'(X,Y) = D_{hX}JY$
(R3)	$\overset{R}{D}_{vX}hY = h[vX,Y] = \overset{\circ}{D}_{vX}hY$
(R4)	$\overset{R}{D}_{hX}hY = hF[hX,JY] + F\mathcal{C}'(X,Y) = D_{hX}hY$

$$(X,Y \in \mathfrak{X}(\mathcal{T}M)).$$

(ii) *If, in addition,*

(CHR5) *h is homogeneous (i.e., $H = [h,C] = 0$),*

then h is the Barthel endomorphism of the Finsler manifold. ☑

Remark 11. The Finsler connection just described is called the *Chern-Rund connection* of the Finsler manifold (see the Introduction).

Corollary 5. *(cf.* [15,16]*) Let us consider a Finsler manifold (M,E) and the Riemannian metric g given by (22). Suppose that h is a horizontal endomorphism on M.*

(i) *There is a unique Finsler connection $(\overset{H}{D},h)$ such that*

(HSG1) *The $(v)hv$-torsion of $\overset{H}{D}$ vanishes:* $\boxed{\overset{H}{\mathbb{P}}{}^1 = 0}$.

(HSG2) *$\overset{H}{D}$ is v-metrical, i.e. $\forall X \in \mathfrak{X}(\mathcal{T}M)$:* $\boxed{\overset{H}{D}_{vX}g = 0}$.

(HSG3) *The $(v)v$-torsion of $\overset{H}{D}$ vanishes:* $\boxed{\overset{H}{\mathbb{S}}{}^1 = 0}$.

The covariant derivatives with respect to $(\overset{H}{D},h)$ and their relations to the Berwald and Cartan covariant derivatives are summarized in the following table:

(H1)	$\overset{H}{D}_{JX}JY = J[JX,Y] + \mathcal{C}(X,Y) = D_{JX}JY$
(H2)	$\overset{H}{D}_{hX}JY = v[hX,JY] = \overset{\circ}{D}_{hX}JY$
(H3)	$\overset{H}{D}_{vX}hY = h[vX,Y] + F\mathcal{C}(FX,Y) = D_{vX}hY$
(H4)	$\overset{H}{D}_{hX}hY = hF[hX,JY] = \overset{\circ}{D}_{hX}hY$

$$(X,Y \in \mathfrak{X}(\mathcal{T}M)).$$

(ii) *If* $(\overset{H}{D}, h)$ *satisfies the further conditions*

(HSG4) $d_h E = 0$;

(HSG5) *the* $(h)h$*-torsion of* $\overset{H}{D}$ *vanishes (i.e.,* $\overset{H}{\mathbb{A}} = 0$);

(HSG6) *the* h*-deflection* $h^*(\overset{H}{D}C)$ *of* $\overset{H}{D}$ *vanishes*

then h *is the Barthel endomorphism of* (M, E). ☑

Remark 12. The Finsler connection $(\overset{H}{D}, h)$ described by Corollary 5 is called the *Hashiguchi connection* of (M, E).

Remark 13. We have the following interesting relation between the $(v)hv$-torsion of the Cartan connection, the $(v)hv$-torsion of the Chern-Rund connection and the second Cartan tensor $\mathcal{C}'$:

$$\mathbb{P}^1(X, Y) = \overset{R}{\mathbb{P}}{}^1(X, Y) = \mathcal{C}'(X, FY) \qquad (X, Y) \in \mathfrak{X}(\mathcal{T}M)).$$

Indeed, for any vector fields $X, Y \in \mathfrak{X}(\mathcal{T}M)$,

$$\begin{aligned} \mathbb{P}^1(X, Y) &:= v\mathbb{T}(hX, vY) = v(D_{hX}vY - D_{vY}hX - [hX, vY]) \\ &= D_{hX}vY - v[hX, vY] \\ &\overset{(C2)}{=} v[hX, vY] + \mathcal{C}'(X, FY) - v[hX, vY] = \mathcal{C}'(X, FY), \end{aligned}$$

and $\overset{R}{\mathbb{P}}{}^1 = \mathbb{P}^1$ due to (R2). This observation yields the following "commutative diagram" (cf. [15], p. 120) and [22]):

$$\begin{array}{ccc} \textbf{Cartan connection } (D, h) & \xrightarrow{J^*D = D^i} & \textbf{Chern-Rund connection } (\overset{R}{D}, h) \\ {\scriptstyle h^*D = \overset{\circ}{D}}\Big\downarrow & & \Big\downarrow{\scriptstyle h^*\overset{R}{D} = \overset{\circ}{D}} \\ \textbf{Hashiguchi connection } (\overset{H}{D}, h) & \xrightarrow{J^*D = D^i} & \textbf{Berwald connection } (\overset{\circ}{D}, h) \end{array}$$

(h is the Barthel endomorphism).

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