https://doi.org/10.15446/recolma.v57nSupl.112449

Revista Colombiana de Matemáticas
Volumen 57(2023) páginas 77-89

A note about Simpson's Inequality via weighted generalized integrals

Una nota sobre la desigualdad de Simpson mediante integrales generalizadas pesadas

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Abstract. In this work we establish a Simpson-type identity and several Simpson-type inequalities for generalized weighted integrals operators.

Key words and phrases. Simpson integral inequality, integral operators weighted, (α, m)-convex functions.

2010 Mathematics Subject Classification. 26D15, 26D10, 41A55.

Resumen. En este trabajo establecemos una identidad de tipo Simpson y varias desigualdades de tipo Simpson para operadores integrales pesados generalizados.

Palabras y frases clave. Desigualdad integral de Simpson, operadores integrales pesados, funciones (α, m)-convexas.

1. Introduction

Within Mathematical Sciences today, one of the notions that most attracts the attention of researchers is that of a convex function ([20]). Its theoretical overlaps and its multiple applications have made it the center of various works and investigations, with which it has been expanded in multiple directions. In [18] you can find a broad panorama of this development.

Definition 1.1. The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f\left(tx + (1-t)\,y\right) \leq t f\left(x\right) + (1-t)\, f\left(y\right),$$

$\forall\ x, y \in [a, b]$ and $t \in [0, 1]$.

In [15] the author introduces the class of functions $(\alpha, m)-$convex, as follows:

The function $f : [0, \infty] \to \mathbb{R}$ is said to be $(\alpha, m)-$ convex, where $(\alpha, m) \in (0, 1] \times (0, 1]$, if for each $a, b \in [0, \infty)$ and $t \in [0, 1]$, we have:

$$f(ta + m(1-t)b) \leq t^{\alpha} f(a) + m(1-t)^{\alpha} f(b).$$

The following inequality is known as Simpson's integral inequality:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4, \tag{1}$$

where the mapping $f : [a, b] \to \mathbb{R}$ is assumed to be four times continuously differentiable on the interval (a, b) and $\|f^{(4)}\|_{\infty} = \sup_{\tau \in (a,b)} |f^{(4)}(\tau)| < \infty$.

Many researchers in the field of inequalities of the last few decades have refined, extended, and obtained new inequalities of the Simpson type for various classes of convex functions ([2, 3, 5, 6, 7, 10, 12, 13, 15, 19, 21, 23, 24] and references therein).

To encourage comprehension of the subject, we present the definition of Riemann-Liouville Fractional Integral (with $0 \leq a_1 < t < a_2 \leq \infty$). The first is the Classic Riemann-Liouville Fractional Integrals.

Definition 1.2. Let $\phi \in L_1[a_1, a_2]$. Then the Riemann-Liouville Fractional Integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):

$$^{\alpha}I_{a_1^+} \phi(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x-t)^{\alpha-1} \phi(t)\, dt, \quad x > a_1 \tag{2}$$

and

$$^{\alpha}I_{a_2^-} \phi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{a_2} (t-x)^{\alpha-1} \phi(t)\, dt, \quad x < a_2. \tag{3}$$

Next we present the Weighted Integral Operators, which will be the basis of our work.

Definition 1.3. Let $f \in L([a, b])$ and let k be a continuous and positive function, $k : [0, 1] \to [0, +\infty)$, with first order derivatives piecewise continuous on I. Then the Weighted Fractional Integrals are defined by (right and left respectively):

$$^{q}J_{a^+}^{k} f(t) = \int_a^t k'\left(\frac{t-\tau}{t-a}\right) f(\tau) d\tau \tag{4}$$

and

$$^{p}J_{b^-}^{k} f(t) = \int_t^b k'\left(\frac{\tau-t}{b-t}\right) f(\tau) d\tau \tag{5}$$

***Remark* 1.4.** To have a clearer idea of the amplitude of the Definition 1.3, let's consider some particular cases of the kernel k':

(1) Putting $k'(t) \equiv 1$, we obtain the Classical Riemann Integral.

(2) If $k'(t) = \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the Riemann-Liouville Fractional Integral right, and left can be obtained similarly.

(3) With convenient kernel choices k' we can get the k-Riemann-Liouville Fractional Integral right and left of ([16]), the right-sided fractional integrals of a function ψ with respect to another function h on $[a, b]$ (see [1]), the right and left integral operator of [8], the right and left sided generalized fractional integral operators of [22] and the integral operators of [9] and [11], can also be obtained from above Definition by imposing similar conditions to k'.

Of course there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers.

The main purpose of this paper is to establish several integral inequalities of Simpson type using the Definition 1.3 of weighted integral.

2. Results

The following result will be fundamental in our work.

Lemma 2.1. *Let $0 < m \leq 1$; $f : [ma, b] \to \mathbb{R}$ be a differentiable function, $a < b$ with $a \in \mathbb{R}$, $b > 0$. If $f \in L^1([ma, b])$ and $k' \geq 0$ then we have:*

$$\begin{aligned} & k(1)f(w) + \frac{k(0)}{b-ma}\left[(w-ma)f\left(\frac{ma+w}{2}\right) + (b-w)f\left(\frac{w+b}{2}\right)\right] \\ & - \frac{2}{b-ma}\left[{}^{b}J^{k}_{w^+}f\left(\frac{w+b}{2}\right) + {}^{ma}J^{k}_{w^-}f\left(\frac{ma+w}{2}\right)\right] \\ & = \frac{(w-ma)^2}{2(b-a)}\int_0^1 k(t)f'\left(\frac{1-t}{2}ma + \frac{1+t}{2}w\right)dt \\ & - \frac{(b-w)^2}{2(b-a)}\int_0^1 k(t)f'\left(\frac{1+t}{2}w + \frac{1-t}{2}b\right)dt. \end{aligned} \tag{6}$$

Proof. Integrating by parts and changing the variable, we have

$$\begin{aligned}
&\int_0^1 k(t)f'\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)dt=\frac{2}{-ma+w}\left[k(1)f(w)-k(0)f\left(\frac{ma+w}{2}\right)\right]\\
&-\frac{2}{w-ma}\int_0^1 k'(t)f\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)dt\\
&=\frac{2}{-ma+w}\left[k(1)f(w)-k(0)f\left(\frac{ma+w}{2}\right)\right]-\frac{4}{(w-ma)^2}\,{}^{ma}J^k_{w-}f\left(\frac{ma+w}{2}\right),
\end{aligned} \tag{7}$$

in the above we use the following fact

$$\begin{aligned}
&\int_0^1 k(t)f'\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)dt\\
=&\frac{2}{w-ma}\int_{\frac{ma+w}{2}}^{w} k'\left[\frac{z-\frac{ma+w}{2}}{\frac{-ma+w}{2}}\right]f(z)dz\\
=&\frac{2}{w-ma}\int_{\frac{ma+w}{2}}^{w} k'\left[\frac{z-\frac{ma+w}{2}}{w-\frac{ma+w}{2}}\right]f(z)dz.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 k(t)f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)dt\\
=&\frac{2}{w-b}\left[k(1)f(w)-k(0)f\left(\frac{w+b}{2}\right)\right]+\frac{4}{(w-b)^2}\,{}^{b}J^k_{w+}f\left(\frac{w+b}{2}\right).
\end{aligned} \tag{8}$$

Multiplying both sides of (7) and (8) by $\frac{(w-ma)^2}{2(b-a)}$ and $\frac{(b-w)^2}{2(b-a)}$, respectively, and subtracting the last from the first, we get

$$\begin{aligned}
&\frac{w-ma}{b-ma}\left[k(1)f(w)-k(0)f\left(\frac{ma+w}{2}\right)\right]-\frac{2}{(b-ma)}\,{}^{ma}J^k_{w-}f\left(\frac{ma+w}{2}\right)\\
&+\frac{b-w}{b-ma}\left[k(1)f(w)-k(0)f\left(\frac{w+b}{2}\right)\right]-\frac{2}{b-ma}\,{}^{b}J^k_{w+}f\left(\frac{w+b}{2}\right)\\
&=\frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t)f'\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)dt-\frac{(w-b)^2}{2(b-ma)}\int_0^1 k(t)f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)dt.
\end{aligned}$$

After rearranging and simplifying on the left side of the previous equality, the desired result is obtained. This completes the proof. ☑

***Remark* 2.2.** If we take $k(t)=\frac{t^\alpha}{2}-\frac{1}{5}$, we have the Lemma 2.1 of [14].

Let's call

$$S = k(1)f(w) + \frac{k(0)}{b-ma}\left[(w-ma)f\left(\frac{ma+w}{2}\right) + (b-w)f\left(\frac{w+b}{2}\right)\right] - \frac{2}{b-ma}\left[{}^{b}J^{k}_{w^{+}}f\left(\frac{w+b}{2}\right) +{}^{ma}J^{k}_{w^{-}}f\left(\frac{ma+w}{2}\right)\right]. \tag{9}$$

From this result, we obtain different Simpson-type inequalities, which are generalizations of several reported in the literature.

Based on the Lemma 2.1, we can obtain the following inequality.

Theorem 2.3. *Let $0 < m \leq 1$; $f : [ma, b] \to \mathbb{R}$ a differentiable function, $a < b$ with $a \in \mathbb{R}$, $b > 0$. If $f \in L^1([ma, b])$ is bounded, that is, $\|f'\|_\infty = \sup|f'(t)| < \infty$, so for $w \in [ma, b]$, we have*

$$|S| \leq \frac{(-ma+w)^2 + (w-b)^2}{2(b-ma)}\|f'\|_\infty \int_0^1 k(t)dt. \tag{10}$$

Proof. If we use the Lemma 2.1 and the absolute value properties, then we have

$$\begin{aligned} |S| &= \left|\frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t)f'\left(\frac{1-t}{2}ma + \frac{1+t}{2}w\right)dt - \frac{(w-b)^2}{2(b-ma)}\right. \\ &\qquad \left.\int_0^1 k(t)f'\left(\frac{1+t}{2}w + \frac{1-t}{2}b\right)dt\right| \qquad (11) \\ &\leq \frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t)\left|f'\left(\frac{1-t}{2}ma + \frac{1+t}{2}w\right)\right|dt + \frac{(w-b)^2}{2(b-ma)} \\ &\qquad \int_0^1 k(t)\left|f'\left(\frac{1+t}{2}w + \frac{1-t}{2}b\right)\right|dt \\ &\leq \frac{(-ma+w)^2 + (w-b)^2}{2(b-ma)}\|f'\|_\infty \int_0^1 k(t)dt. \end{aligned}$$

Thus, we get the desired result. ☑

***Remark* 2.4.** If we take $k(t) = \frac{t^\alpha}{2} - \frac{1}{5}$, we have the Theorem 3.1 of [14].

Theorem 2.5. *Let $0 < m \leq 1$; $f : [ma, b] \to \mathbb{R}$ a differentiable function, $a < b$ with $a \in \mathbb{R}$, $b > 0$. If $f' \in L^1([ma, b])$, then for $w \in [ma, b]$, we have*

$$|S| \leq k(0)\|f'\|_1 \tag{12}$$

where $\|f'\|_1 = \int_{ma}^{b}|f'(x)|dx < \infty$,

Proof. If the Lemma 2.1 is used and changing variables, we have

$$\begin{aligned}|S| \leq & \frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t)\left|f'\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)\right|dt+\frac{(w-b)^2}{2(b-ma)} \\ & \int_0^1 k(t)\left|f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)\right|dt \\ \leq & \frac{(-ma+w)}{(b-ma)}\int_{\frac{ma+w}{2}}^{w} k\left(\frac{x-\frac{m+w}{2}}{\frac{-ma+w}{2}}\right)\left|f'(x)\right|dx+\frac{(w-b)}{(b-ma)} \\ & \int_w^{\frac{w+b}{2}} k\left(\frac{\frac{w+b}{2}-x}{\frac{w+b}{2}}\right)\left|f'(x)\right|dx.\end{aligned}$$

Since $0 \leq \frac{(-ma+w)}{(b-ma)} \leq 1$ and $0 \leq \frac{(w-b)}{(b-ma)} \leq 1$, then

$$\begin{aligned}|S| &\leq \int_{\frac{ma+w}{2}}^{w} k\left(\frac{x-\frac{m+w}{2}}{\frac{-ma+w}{2}}\right)\left|f'(x)\right|dx+\int_w^{\frac{w+b}{2}} k\left(\frac{\frac{w+b}{2}-x}{\frac{w-b}{2}}\right)\left|f'(x)\right|dx \quad (13)\\ &= k(0)\int_{\frac{ma+w}{2}}^{\frac{w+b}{2}-x}\left|f'(x)\right|dx \\ &\leq k(0)\int_{ma}^{b}\left|f'(x)\right|dx.\end{aligned}$$

Therefore, the proof is finished. ☑

***Remark* 2.6.** If we take $k(t)=\frac{t^\alpha}{2}-\frac{1}{5}$, we have the Theorem 3.2 of [14].

Theorem 2.7. *Let $0<m\leq 1$; $f:[ma,b]\to\mathbb{R}$ a differentiable function, $a<b$ with $a\in\mathbb{R}$, $b>0$. If $f'\in L^q([ma,b])$, with $1<q,p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, so for $w\in[ma,b]$ we have*

$$|S| \leq 2^{\frac{1}{q}}\left\|f'\right\|_q\left((b-ma)\int_0^1 |k(t)|^p\,dt\right)^{\frac{1}{p}}, \qquad (14)$$

where $\left\|f'\right\|_q=\left(\int_{ma}^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}$.

Proof. Using the Lemma 2.1 and Hölder's inequality we have that

$$\begin{aligned}|S| &\leq \frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t)\left|f'\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)\right|dt+\frac{(w-b)^2}{2(b-ma)}\\ &\qquad \int_0^1 k(t)\left|f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)\right|dt \qquad (15)\\ &\leq \frac{(-ma+w)^2}{2(b-ma)}\left(\int_0^1|k(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}ma+\frac{1+t}{2}w\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(w-b)^2}{2(b-ma)}\left(\int_0^1|k(-t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}w+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Making some proper substitutions

$$\begin{aligned}|S| &\leq \left(\int_0^1|k(t)|^p\,dt\right)^{\frac{1}{p}}\left[\frac{(-ma+w)^2}{2(b-ma)}\left(\frac{2}{-ma+w}\int_{\frac{ma+w}{2}}^{w}|f'(x)|^q\,dt\right)^{\frac{1}{q}}\right.\\ &\left.+\frac{(w-b)^2}{2(b-ma)}\left(\frac{2}{w-b}\int_w^{\frac{w+b}{2}}|f'(x)|^q\,dt\right)^{\frac{1}{q}}\right]\\ &=\left(\int_0^1|k(t)|^p\,dt\right)^{\frac{1}{p}}\left[\frac{(-ma+w)^{2-\frac{1}{q}}}{2^{1-\frac{1}{q}}(b-ma)}\left(\int_{\frac{ma+w}{2}}^{w}|f'(x)|^q\,dt\right)^{\frac{1}{q}}+\right.\\ &\qquad\left.\frac{(w-b)^{2-\frac{1}{q}}}{2^{1-\frac{1}{q}}(b-ma)}\left(\int_w^{\frac{w+b}{2}}|f'(x)|^q\,dt\right)^{\frac{1}{q}}\right] \qquad (16)\\ &\leq\left(\int_0^1|k(t)|^p\,dt\right)^{\frac{1}{p}}\left(2^{\frac{1}{q}-1}(b-ma)^{1-\frac{1}{q}}\right)\left[\left(\int_{\frac{ma+w}{2}}^{w}|f'(x)|^q\,dt\right)^{\frac{1}{q}}+\right.\\ &\qquad\left.\left(\int_w^{\frac{w+b}{2}}|f'(x)|^q\,dt\right)^{\frac{1}{q}}\right]\\ &\leq\left(\int_0^1|k(t)|^p\,dt\right)^{\frac{1}{p}}\left(2^{\frac{1}{q}}(b-ma)^{\frac{1}{p}}\right)\left(\int_{ma}^{b}|f'(x)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

And thus, we have obtained the desired inequality. ☑

***Remark* 2.8.** if we take $k(t)=\frac{t^\alpha}{2}-\frac{1}{5}$, we have the Theorem 3.3 of [14].

Theorem 2.9. *Let* $0<m\leq 1$; $f:\left[ma,\frac{b}{m}\right]\to\mathbb{R}$ *a differentiable function,* $0\leq a<b$ *such that* $f'\in L^1\left([ma,\frac{b}{m}]\right)$. *If* $|f'|^q$ *is a* $(\alpha,m)-$*convex function,*

with $(\alpha, m) \in (0,1]^2$ *for* $1 < q$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *so for* $w \in [ma, b]$ *we have*

$$|S| \leq \left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}} \left[\frac{(-ma+w)^2}{2(b-ma)}\left(\frac{m}{2^\alpha(\alpha+1)}\left|f'(a)\right|^q + \frac{2^{\alpha-1}-1}{2^\alpha(\alpha+1)}\left|f'(w)\right|^q\right)^{\frac{1}{q}}\right] + \left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}} \left[\frac{(b-w)^2}{2(b-ma)}\left(\frac{2^{\alpha-1}-1}{2^\alpha(\alpha+1)}\left|f'(w)\right|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right]. \tag{17}$$

Proof. Continuing from the equation (15) and using the fact that $|f'|^q$ is (α, m)-convex, we obtain (17). ☑

Corollary 2.10. *Considering the Theorem 2.9 we have the following cases:*

(1) Putting $w = ma$ *we have*

$$|S| \leq \frac{b-ma}{2}\left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}} \left(\frac{2^{\alpha-1}-1}{2^\alpha(\alpha+1)}\left|f'(ma)\right|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{18}$$

(2) Putting $w = \frac{ma+b}{2}$, *we have*

$$\begin{aligned}
|S| &\leq \frac{b-ma}{8}\left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}} \left[\left(\frac{2^{\alpha-1}-1}{2^\alpha(\alpha+1)}\left|f'\left(\frac{ma+b}{2}\right)\right|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'(a)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\quad + \frac{b-ma}{8}\left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}} \left[\left(\frac{2^{\alpha-1}-1}{2^\alpha(\alpha+1)}\left|f'\left(\frac{ma+b}{2}\right)\right|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-ma}{8}\left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}} \left[\left(\frac{m}{2^\alpha(\alpha+1)}\right)^{\frac{1}{q}}\left|f'(a)\right| \right. \\
&\quad \left. + 2\left(\frac{2^{\alpha-1}-1}{2^\alpha(\alpha+1)}\right)^{\frac{1}{q}}\left|f'\left(\frac{ma+b}{2}\right)\right| + \left(\frac{m}{2^\alpha(\alpha-1)}\right)^{\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|\right]
\end{aligned}$$

(3) Putting $w = b$, we have

$$|S| \leq \frac{b-ma}{2}\left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}}\left(\frac{m}{2^{\alpha}(\alpha+1)}\left|f'(a)\right|^q + \frac{2^{\alpha-1}-1}{2^{\alpha}(\alpha+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \tag{19}$$

Corollary 2.11. *Combining the inequalities* (18) *and* (19) *follows*

$$\left|\frac{-ma+w}{b-ma}\left[k(1)f(w) - k(0)f\left(\frac{ma+w}{2}\right)\right] - \frac{2}{(b-ma)}\,{}^{ma}J^{k}_{w^-}f\left(\frac{ma+w}{2}\right)\right.$$
$$\left. + \frac{w-b}{b-ma}\left[k(-1)f(w) - k(0)f\left(\frac{w+b}{2}\right)\right] - \frac{2}{b-ma}\,{}^{b}J^{k}_{w^+}f\left(\frac{w+b}{2}\right)\right|.$$

Theorem 2.12. *Let $0 < m \leq 1$; $f : \left[ma, \frac{b}{m}\right] \to \mathbb{R}$ a differentiable function, $0 \leq a < b$ such that $f \in L^1\left(\left[ma, \frac{b}{m}\right]\right)$. If $|f'|$ is a $(\alpha, m)-$convex function, with $(\alpha, m) \in (0,1]^2$ for $1 < p$ with $\frac{1}{p} + \frac{1}{q} = 1$, so for $w \in [ma, b]$ we have*

$$|S| \leq \frac{(-ma+w)^2}{2(b-ma)}\left(\int_0^1 |k(t)|^p \, dt\right)^{\frac{1}{p}}$$
$$\left[\left[\frac{1}{2^{q.\alpha}(q.\alpha+1)}\right]^{\frac{1}{q}} m\left|f'(a)\right| + \left[\frac{2^{q.\alpha+1}-1}{2^{q.\alpha}(q.\alpha+1)}\right]^{\frac{1}{q}}\left|f'(w)\right|\right]$$
$$+ \frac{(w-b)^2}{2(b-ma)}\left(\int_0^1 |k(-t)|^p \, dt\right)^{\frac{1}{p}}$$
$$\left[\left[\frac{1}{2^{q.\alpha}(q.\alpha+1)}\right]^{\frac{1}{q}} m\left|f'\left(\frac{b}{m}\right)\right| + \left[\frac{2^{q.\alpha+1}-1}{2^{q.\alpha}(q.\alpha+1)}\right]^{\frac{1}{q}}\left|f'(w)\right|\right]$$

Proof. Using the Lemma 2.1 and that $|f'|$ is (α, m)-convex, we have

$$|S| \leq \frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t).\left|f'\left(\frac{1-t}{2}ma + \frac{1+t}{2}w\right)\right| dt \tag{20}$$
$$+ \frac{(w-b)^2}{2(b-ma)}\int_0^1 k(-t).\left|f'\left(\frac{1+t}{2}w + \frac{1-t}{2}b\right)\right| dt$$
$$\leq \frac{(-ma+w)^2}{2(b-ma)}\int_0^1 k(t)\left[\left(\frac{1-t}{2}\right)^{\alpha} m\left|f'(a)\right| + \left(\frac{1+t}{2}\right)^{\alpha}\left|f'(w)\right|\right] dt$$
$$+ \frac{(w-b)^2}{2(b-ma)}\int_0^1 k(-t)\left[\left(\frac{1-t}{2}\right)^{\alpha} m\left|f'\left(\frac{b}{m}\right)\right| + \left(\frac{1+t}{2}\right)^{\alpha}\left|f'(w)\right|\right] dt.$$

Then

$$|S| \leq \frac{(-ma+w)^2}{2(b-ma)} \left(\int_0^1 |k(t)|^p \, dt \right)^{\frac{1}{p}} \left[\left[\int_0^1 \left(\frac{1-t}{2} \right)^{q\alpha} dt \right]^{\frac{1}{q}} m \left| f'(a) \right| + \left[\int_0^1 \left(\frac{1+t}{2} \right)^{q\alpha} dt \right]^{\frac{1}{q}} \left| f'(w) \right| \right]$$

$$+ \frac{(w-b)^2}{2(b-ma)} \left(\int_0^1 |k(-t)|^p \, dt \right)^{\frac{1}{p}} \left[\left[\int_0^1 \left(\frac{1-t}{2} \right)^{q\alpha} dt \right]^{\frac{1}{q}} m \left| f'\left(\frac{b}{m} \right) \right| + \left[\int_0^1 \left(\frac{1+t}{2} \right)^{q\alpha} dt \right]^{\frac{1}{q}} \left| f'(w) \right| \right].$$

Let's notice that

$$\int_0^1 \left(\frac{1+t}{2} \right)^{q\alpha} dt = \frac{2^{q.\alpha+1}-1}{2^{q.\alpha}(q\alpha+1)} \tag{21}$$

and

$$\int_0^1 \left(\frac{1-t}{2} \right)^{q\alpha} dt = \frac{1}{2^{q.\alpha}(q\alpha+1)}. \tag{22}$$

In this way, we obtain the desired inequality. ☑

***Remark* 2.13.** If we take $k(t) = \frac{t^\alpha}{2} - \frac{1}{5}$, we have the Theorem 3.7 of [14].

Corollary 2.14. *If we take $w = \frac{ma+b}{2}$ in Theorem 2.12 we have the following inequality*

$$|S| \leq \frac{b-ma}{8} \left(\int_0^1 |k(t)|^p \, dt \right)^{\frac{1}{p}} \left(\frac{1}{2^{q\alpha}(q\alpha+1)} \right)^{\frac{1}{q}} m \left| f'(a) \right| + 2 \left(\frac{2^{q\alpha+1}-1}{2^{q\alpha}(q\alpha+1)} \right)^{\frac{1}{q}} \left| f'\left(\frac{ma+b}{2} \right) \right| + \left(\frac{1}{2^{q\alpha}(q\alpha+1)} \right)^{\frac{1}{q}} m \left| f'\left(\frac{b}{m} \right) \right|.$$

3. Conclusions

In this note we obtain new inequalities of the Simpson type, with different notions of convexity, using weighted integrals. The results obtained contain several known ones reported in the literature.

On the other hand, the results obtained can be generalized using the recently defined convex (h,m)-modified functions (see [4] and [17]).

Taking into account the Remark 1.4 the results obtained opens up new possibilities for future work, to which several fractional integrals can be used to establish new specific fractional integral inequalities.

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(Recibido en septiembre de 2023. Aceptado en noviembre de 2023)

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