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On a Family of Polyhedral Singular Vertices

Sobre una Familia de Vértices Singulares Poliédricos

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Abstract. We provide a local description of the curves with minimal length based at singularities in a family of polyhedral surfaces. These singularities are accumulation points of vertices with conical angles equal to π and 4π (or 3π, in a variation). While a part of the minimizing curves behaves quite like the ones reaching conical vertices, the singularities present features such as being connected to points arbitrarily close to them by exactly two minimizing curves. The spaces containing such singularities are constructed as metric quotients of an euclidean half-disk by certain identification patterns along its edge. These patterns are examples of what is known as paper-folding schemes, and we provide the foundational aspects about them which are necessary for our analysis. The arguments are based on elementary metric geometry and calculus.

Key words and phrases. Polyhedral surfaces, Singularities, Metric quotients.

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Resumen. Proporcionamos una descripción local de las curvas con una longitud mínima basada en singularidades en una familia de superficies poliédricas. Estas singularidades son puntos de acumulación de vértices con ángulos cónicos iguales a π y 4π (o 3π, en una variación). Si bien una parte de las curvas minimizadoras se comporta como las que alcanzan los vértices cónicos, las singularidades tienen características tales como la posibilidad de estar conectadas a puntos arbitrariamente cercanos a ellas mediante exactamente dos curvas minimizadoras. Los espacios que contienen tales singularidades se construyen como cocientes métricos de un semidisco euclidiano por ciertos patrones de identificación de su borde. Estos patrones son ejemplos de lo que se conoce como esquemas de plegado de papel, y proporcionamos los aspectos fundamentales sobre ellos necesarios para nuestro análisis. Las técnicas se basan en geometría métrica elemental y cálculo.

Palabras y frases clave. Superficies poliédricas, Singularidades, Cocientes métricos.

1. Introduction

This paper studies the geometry of polyhedral surfaces with singularities in terms of their length structures. In these spaces, most points have either flat or conical neighborhoods. However, they share the ambient space with points around which the metric is not so well understood, such as accumulation points of conical points. One way of obtaining spaces with this kind of property is an extension of the usual method of constructing surfaces as quotients of polygons by pairing subsegments of its sides. Certain developments in dynamical systems [4] required the pairing of an infinite number of subsegments, leading to the concept of paper-folding schemes [5]. Quotients by these schemes may have singularities as the aforementioned, and also a number of other interesting properties (they may not even be surfaces in the strict sense). We concentrate on the local picture around a kind of singularity produced by a family of paper-folding schemes, providing a depiction of how geodesics reach this special point. This family consists of variations of a pattern in [4], and will be described in the sequence – the dotted lines at the left of Figure 1 indicate the pairings. The technique applies to the original pattern as well, as will be emphasized at the end of the paper.

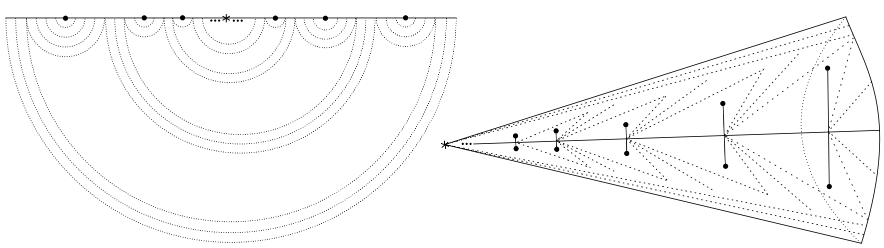

Figure 1. Identification pattern (1), with dotted lines connecting paired points; and its quotient, the dashed lines being the length-minimizing curves of Theorem 1.1. The set V is mostly on the "back side" of the picture.

Let D be a closed half-disk with its euclidean metric and J be the edge of D. Denote by $|J|$ the length of J. Consider three sequences of positive real numbers b_k, $k \geq 0$, and $a_{0,k}$, $a_{1,k}$, $k \geq 1$, such that $b_0 + \sum_{k\geq 1}(a_{0,k} + a_{1,k} + b_k) = |J|/2$. Take compact intervals with lengths given by these sequences: β_k and β'_k, with $|\beta_k| = |\beta'_k| = b_k$, $k \geq 0$; and $\alpha_{j,k}$ and $\alpha'_{j,k}$, with $|\alpha_{j,k}| = |\alpha'_{j,k}| = a_{j,k}$, $k \geq 1$, $j = 0, 1$. Place these intervals side-by-side along J, with disjoint interiors, in the following order:

$$\beta'_0\, \alpha_{0,1}\, \alpha'_{0,1}\, \beta'_1\, \alpha_{0,2}\, \alpha'_{0,2}\, \beta'_2 \cdots * \cdots \beta_2\, \alpha'_{1,2}\, \alpha_{1,2}\, \beta_1\, \alpha'_{1,1}\, \alpha_{1,1}\, \beta_0\,. \tag{1}$$

In this expression, the symbol $*$ represents precisely one point in J, due to the condition over the sequences of lengths. Now, identify each of these subintervals to its prime, isometrically and reversing orientation. The resulting identification pattern is depicted on the left of Figure 1.

Let S be the metric quotient of D associated to this pairing pattern, and let G be the projection in S of J. It looks somewhat like the "singular cone" on the right of Figure 1. In S, the point corresponding to $*$, also denoted as $*$ in this Introduction, is the limit of three sequences of conical vertices, which are the projections of the endpoints of the paired segments. The total angles around them are equal to: π, for two sequences $p_{j,k}$ ($\bullet$ in the Figure); or 4π, for a sequence q_k ($j = 0, 1,\ k \geq 1$) (the crossings between distinct $\bullet$). Our main result describes how each point in S reaches $*$ through length-minimizing curves, assuming a condition over the sequences b_k and $a_{j,k}$.

Theorem 1.1. *Suppose that, for $k \geq 1$, $b_k/a_{0,k}$ and $b_k/a_{1,k}$ are constant. Then, there exist a closed plane sector in D, based at $*$, with internal angle smaller than π, whose projection V in S has the following properties:*

i) If $x \in V$, there exist an unique curve with minimal length from x to $*$. It is contained in V and meets G only at $*$.

ii) If $x \in S \setminus V$, every curve with minimal length from x to $*$ passes by some q_k and is contained in G from there on. There are precisely either one or two such curves, and the latter happens for points arbitrarily close to $*$.

A more precise version of Theorem 1.1 is given in Theorem 3.3, while Theorem 3.5 is an analogous statement about the original pattern found in [4]. Based on the approach [5, 7] of this kind of quotient by means of metric geometry [1, 3], elementary arguments are employed until the matter reduces to an optimization problem of a real function. The author hopes that this methodology leads to further results on the metrics around singularities similar to the ones treated here. Detailed statements of foundational results about the subject are presented (Theorems 2.7, 2.11 and 2.14), as well as complete proofs of two basic tools for dealing with it (Lemmas 3.1 and 3.2).

In the terminology of [5], a point such as $*$ is a singular 1-vertex, and the identification pattern (1) is a piece of an example of a paper-folding scheme. Considering these schemes along the borders of polygons, the approach introduced in [5] develops topological, metric and conformal theories in order to obtain dynamical applications regarding sequences of quotients. An important result provides sufficient conditions for the complex structure on the regular part of the quotient to extend across a singularity. It is not known if this condition is necessary, while the geometric structure is present in any case. For other applications of polyhedral surfaces in dynamical systems, we refer to [8].

In a similar vein, polyhedral surfaces are central in approximation results on surfaces of bounded curvature [1]. In this context, points as $*$ implies that the space containing it is not of bounded curvature, as every small neighborhood of $*$ contains both thin and fat triangles. An incipient theory showing how to approximate, in the Gromov-Hausdorff sense, such spaces by sequences of polyhedral surfaces whose curvatures explode can be found in [6].

The paper is organized as follows: Section 2 sets basic notations and provides basic statements about the metric quotients of paper-folding schemes, while Section 3 deals with the proof of Theorem 1.1 (re-stated more precisely as Theorem 3.3). The author thanks the organizing comitee of the III Encuentro Matemático del Caribe for the kind invitation to present a talk, and particularly to Jeovanny de Jesus Muentes Acevedo and Raibel de Jesus Arias Cantillo. He also aknowledges: André de Carvalho, for presenting him to paper-folding schemes; Minoru Enrique Akiyama Figueroa, for dicussions on the subject; Jorge Salazar Morales for being helpful during the conclusion of this paper; the Instituto de Ciências Exatas e Naturais of Universidade Federal do Pará for the support during its preparation; and the referees for the considerate comments.

2. Quotients of paper-folding schemes

Throughout the paper, D is a half-disk with its euclidean metric, and J is the edge of D. Points in D will be denoted as z, w, etc., while distances will be denoted by $|z\,w|$. For $z \neq w$, $[z, w]$ is the line segment in D with these endpoints, and (z, w) is $[z, w]$ minus its endpoints. The notations $[z, w)$ and $(z, w]$ will be used accordingly. For a curve γ in D, $|\gamma|$ denotes its length.

In this section, $\mathcal{P}$ is a paper-folding scheme on D, S is the associated metric quotient, and $\pi : D \to S$ is the projection map. These concepts will be defined in the sequence, in accordance with [5] and [3]. The main difference with respect to [5] is that, since we are concerned only with local questions, it is not necessary to define paper-folding schemes along the border of a polygon, but only along the edge of a half-disk. For the sake of brevity, the definitions and statements regarding metric objects were particularized to quotients associated to paper-folding schemes.

***Remark* 2.1.** The assumption that the geometry of D is euclidean is irrelevant for most of the paper. The statements and proofs of basic results are true also for gluings of hyperbolic or, with minor adaptations, spherical pieces. More precisely, the euclidean setting is decisive in a part of the proof of Theorem 3.3, where it will be explicitly mentioned.

Definition 2.2 ([5]). For each isometric compact intervals $[p, q], [q', p'] \subset J$ with disjoint interiors, the associated *pairing* $\langle [p, q], [q', p'] \rangle$ is the family of subsets of $[p, q] \cup [q', p']$ of the form $\{z, z'\}$, with $z \in [p, q]$ and $z' \in [q', p']$ such that $z - p = p' - z$. Each such z and z' are said to be *paired* and, if they belong to the (p, q) and (q', p'), $\{z, z'\}$ is called an *interior pair*. A *folding* is a pairing whose segments have a common endpoint, this point being called a *folding point*. The *length of a pairing* $\langle \alpha, \alpha' \rangle$ is defined by $|\langle \alpha, \alpha' \rangle| = |\alpha| = |\alpha'|$. A *paper-folding scheme on* D is a collection $\mathcal{P} = \{\langle \alpha_j, \alpha'_j \rangle\}_j$ of pairings such that the interiors of all intervals α_j and α'_j are disjoint, and $\sum_j |\langle \alpha_j, \alpha'_j \rangle| = |J|/2$. The collection of subsets of D whose elements are points paired by some element of $\mathcal{P}$ will also be denoted by $\mathcal{P}$.

Definition 2.3. For each $z, w \in D$, write $z\,\mathcal{P}\,w$ if, either $z = w$, or z and w are paired by some element of $\mathcal{P}$. The associated *quotient semi-metric* is defined on D by:

$$|z\,w|_{\mathcal{P}} = \inf \sum_{j=1}^{N} |z_j\,w_j|\,, \tag{2}$$

where the infimum is taken over every $\{z_j, w_j\}_{j=1}^{N}$ such that $z_1 = z$, $w_N = w$ and $w_j\,\mathcal{P}\,z_{j+1}$ for every $j = 1, \ldots, N-1$. Define the equivalence relation $\sim_{\mathcal{P}}$ on D by $z \sim_{\mathcal{P}} w$ if, and only if, $|z\,w|_{\mathcal{P}} = 0$. Denote the equivalence class of each $z \in D$ with respect to $\sim_{\mathcal{P}}$ by $\bar{z}$, and let $S = D/{\sim_{\mathcal{P}}}$ be the set of all these equivalence classes. Then $|\quad|_{\mathcal{P}}$ induces a metric in S, called the *quotient metric associated to* $\mathcal{P}$. The associated *projection map* $\pi : D \to S$ is defined by $\pi(z) = \bar{z}$. The set $G = \pi(J)$ is the *scar* of S.

Mostly, we are going to employ an abuse of notation, denoting the metrics of distinct spaces as $|\quad|$ and letting the points indicate the space in which distances are being taken. For instance, while $|z\,w|$ denotes a distance in D, $|\bar{z}\,\bar{w}|$ denotes the distance between the projections of these points in S.

Definition 2.4. The *length* of a curve $\gamma : [0,1] \to S$ is defined by $|\gamma| = \sup \sum_{i=0}^{N-1} |\gamma(t_i)\,\gamma(t_{i+1})|$, where the supremum is taken over every partition $0 = t_0 < \cdots < t_N = 1$.

We are going to call γ *minimizing* if $|\gamma| \le |\gamma'|$ for every curve γ' joining $\gamma(0)$ and $\gamma(1)$.

We now list a number of basic results on the subjects.

Theorem 2.5 ([3])**.** *The projection map $\pi : D \to S$ does not increase distances. In particular, π is a continuous surjection, and S is compact. It follows that the metric of S is strictly intrinsic, in the sense that, for every $\bar{z}, \bar{w} \in S$, $|\bar{z}\,\bar{w}| = \inf |\gamma|$, where the infimum is taken over every curve γ in S joining $\bar{z}$ and $\bar{w}$, and is attained by some curve of this form.*

A simple property of minimizing curves in S that will be used repeatedly is that any restriction of a minimizing curve is minimizing. The following terminology on extensions is not standard, but is helpful in our context. The meaning of expressions as "a geodesic can be further extended from a point on" will hopefully be made clear by the context.

Definition 2.6. A curve $\gamma : I \to S$, where I is any interval, is a *geodesic of S* if every $t_0 \in I$ has a neighborhood I' in I such that the restriction of γ to I' is minimizing. A geodesic can be *extended* if its image is properly contained in a geodesic, and can be *uniquely extended* if this geodesic is unique.

In the following results, the map π is employed to show that the metric of S is either flat or conical around most of its points. The conclusions of Theorems 2.7, 2.11 and 2.14 were stated, without proof and in a simplified form, as the "Metric Structure Theorem" in [5]. The proofs are straightforward, but technical (and, to the the author's knowledge, a bit long, at least when done by elementary means). They can be found in its entirety in [7] and, in a similar vein, in [2]. These statements could also be strengthened to guarantee convexity of the regions in the quotients, but this will not be relevant in our results.

The interior and the boundary of D as subset of the euclidean plane are denoted by $\operatorname{Int} D$ and ∂D. For any interval I, $\operatorname{Int} I$ denotes I minus its endpoints.

Theorem 2.7 (Interior Points). *For each $z \in \operatorname{Int} D$, and $\bar{z} = \pi(z)$, the radius $r = d(z, \partial D)/2$ is such that:*

i) $\pi^{-1}(B(\bar{z}, r)) = B(z, r)$ *and* $\pi(B(z, r)) = B(\bar{z}, r)$.

ii) $\pi : B(z, r) \to B(\bar{z}, r)$ *is an isometry between these subspaces of D and S.*

iii) The restriction of π to $\operatorname{Int} D$ is a length-preserving homeomorphism onto its image.

iv) Every non-constant geodesic contained in $\pi(\operatorname{Int} D)$ can be uniquely extended to a geodesic of the form $\pi((p, q))$, where $p, q \in \partial D$.

We give a precise meaning to the concept of "conical vertex" using metric cones over circles.

Definition 2.8. For each $\eta > 0$, let C_η be the circle with length equal to η, with its intrinsic metric. The set $\mathbb{B}_\eta$ is defined as the (set theoretic) quotient $[0, +\infty) \times C_\eta$ by the equivalence relation that identifies any two elements of the form $(0, \theta)$, and identify any other point only to itself. Denote by $[r, \theta] \in \mathbb{B}_\eta$ the equivalence class of each $(r, \theta) \in [0, +\infty) \times C_\eta$. The element $O = [0, \theta] \in \mathbb{B}_\eta$, with arbitrary θ, is the *vertex* of $\mathbb{B}_\eta$. The *conical metric* in $\mathbb{B}_\eta$ is defined by:

$$|[r, \theta]\, [s, \theta']| = \begin{cases} \sqrt{r^2 + s^2 - 2rs\cos|\theta\,\theta'|} & |\theta\,\theta'| \leq \pi \\ r + s & |\theta\,\theta'| \geq \pi. \end{cases}$$

With this metric, $\mathbb{B}_\eta$ is the *metric cone over C_η*. To each $r > 0$, the ball in $\mathbb{B}_\eta$ with radius r centered at O, with its subspace metric, will be denoted by $\mathbb{B}_\eta(r)$ and called a *conical neighborhood*. The number η is the *conical angle* of the vertex, or the *total angle around it*.

Theorem 2.9 ([3]). *For each $\theta \in C_\eta$, consider the geodesic $\gamma = \{[r, \theta] \in \mathbb{B}_\eta \,|\, r \geq 0\}$. Then γ can be extended if, and only if, $\eta \geq 2\pi$; and can be uniquely extended if, and only if, $\eta = 2\pi$.*

Theorems 2.11 and 2.14 locally factors π as a composition of an isometry and a "gluing map" to obtain flat and conical neighborhoods around images of points interior to paired segments, and images of basepoints of cyclic chains of pairings, respectively.

Definition 2.10. Let $\langle \alpha, \alpha' \rangle \in \mathcal{P}$, $z \in \operatorname{Int} \alpha$ be paired to $z' \in \operatorname{Int} \alpha'$, and $r > 0$ be such that $B(z, r) \cap \alpha \subset \operatorname{Int} \alpha$ and $B(z', r) \cap \alpha' \subset \operatorname{Int} \alpha'$. In particular, $B(z, r) \cap B(z', r) = \varnothing$. Denote $B(r) = B(z, r) \cup B(z', r)$. An associated *gluing map* $i : B(r) \to \mathbb{B}_{2\pi}(r)$ takes each of $B(z, r)$ and $B(z', r)$ to a half-disk of $\mathbb{B}_{2\pi}(r)$ isometrically, preserving orientation, and with the property that $i(z_1) = i(z_2)$ if, and only if, $z_1 = z_2$ or z_1 and z_2 are paired by $\langle \alpha, \alpha' \rangle$.

Theorem 2.11 (Interior Pairs). *Let $\langle \alpha, \alpha' \rangle \in \mathcal{P}$. For each $z \in \operatorname{Int} \alpha$ paired to $z' \in \operatorname{Int} \alpha'$, there exist $r > 0$ such that, with the notations of Definition 2.10:*

i) $\pi^{-1}(B(\bar{z}, r)) = B(r)$ *and* $\pi(B(r)) = B(\bar{z}, r)$.

ii) The equation $\pi = \phi \circ i$, where i is a gluing map, defines an isometry $\phi : \mathbb{B}_{2\pi}(r) \to B(\bar{z}, r)$, the latter being considered with the subspace metric.

iii) The restrictions of π to $\operatorname{Int} \alpha$ and $\operatorname{Int} \alpha'$ define length-preserving homeomorphisms onto its images.

iv) $\pi(\operatorname{Int} \alpha) = \pi(\operatorname{Int} \alpha')$ *is the unique geodesic extension of any non-constant geodesic contained in it.*

Definition 2.12. Let $\{p_j\}_{j=1}^m$ be a sequence of pairwise distinct points in J. These points are *basepoints of a cyclic chain of pairings of* $\mathcal{P}$ if: either $m = 1$ and p_1 is the common endpoint of two paired segments, a so called *folding point*; or $m > 1$, p_j is paired to p_{j+1}, for $j = 1, \ldots, m-1$, and p_m is paired to p_1. The *pairings of the chain* are the pairings whose set of endpoints contain $\{p_j\}_{j=1}^m$.

Definition 2.13. Let $\{p_j\}_{j=1}^m \subset \operatorname{Int} J$ be the basepoints of a cyclic chain of pairings of $\mathcal{P}$, and let $r > 0$ be such that, for each $j = 1, \ldots, m$, the only point of $B(p_j, r) \cap J$ that is not interior to some pairing of the chain is p_j. In particular, these balls are pairwise disjoint. Denote $B(r) = \cup_{j=1}^m B(p_j, r)$. An associated *gluing map* $i : B(r) \to \mathbb{B}_{m\pi}(r)$ takes each $B(p_j, r)$ to a sector with internal angle π of $\mathbb{B}_{m\pi}(r)$ isometrically, preserving orientation, and with the property that $i(z_1) = i(z_2)$ if, and only if, $z_1 = z_2$ or z_1 and z_2 are paired by some pairing of the chain.

Theorem 2.14 (Conical Vertices). *Let $\{p_i\}_{i=1}^m \subset \operatorname{Int} J$ be the basepoints of a cyclic chain of pairings of $\mathcal{P}$. Then there exist $r > 0$ such that, with the notations of Definition 2.13:*

i) $\pi^{-1}(B(\bar{p}, r)) = B(r)$ *and* $\pi(B(r)) = B(\bar{p}, r)$.

ii) The equation $\pi = \phi \circ i$, where i is a gluing map associated to $B(r)$, defines an isometry $\phi : \mathbb{B}_{m\pi}(r) \to B(\bar{z}, r)$, where the latter is considered with its subspace metric.

iii) Every geodesic in S of the form $\pi((z,p))$ extends to $\pi((z,p])$. It can be further extended if, and only if, $m \geq 2$; and can be further extended uniquelly if, and only if, $m = 2$.

Notice that the last item in Theorem 2.14 is just Theorem 2.9. As will be made clear in the sequence, there may be geodesics of the form $\pi((z,w))$ in S such that $\pi((z,w])$ is not a geodesic. This property may be called *singular*, as it implies that $\bar{w}$ doesn't have flat or conical neighborhoods.

3. Proof of Theorem 1.1 and Variations

Keeping the notations of Section 2, we first prove a general Lemma on quotients associated to paper-folding schemes on D. Let C be the closed half-circle in ∂D, and $\operatorname{Int} C$ be C minus its endpoints.

Lemma 3.1. There are no extensions of geodesics in S through points in $\pi(\operatorname{Int} C)$.

Proof. First we show that, in fact, the restriction of π to $\operatorname{Int} P \cup \operatorname{Int} C$ is a length-preserving homeomorphism onto its image. This property holds on $\operatorname{Int} P$ due to Theorem 2.7, so the first step is to prove that π is injective on $\operatorname{Int} C$. Let $z \in \operatorname{Int} C$ and $w \in D$, $w \neq z$. Consider $\{z_j, w_j\}_{j=1}^{N}$ as in Definition 2.3, and call $\ell = \sum_{j=1}^{N} |z_j\, w_j|$ its length. If $w_1 = w$, then $\ell = |z\, w| > 0$. Otherwise, suppose that $w_1 \in \operatorname{Int} P$. Then, $z_2\, \mathcal{P}\, w_1$ implies that $z_2 = w_1$, and the triangle inequality gives $|z_1\, w_1| + |z_2\, w_2| > |z_1\, w_2|$. Repeating this argument, a new sequence $\{z'_j, w'_j\}_{j=1}^{N'}$ can be obtained from $\{z_j, w_j\}_{j=1}^{N}$, satisfying the conditions of Definition 2.3, with length $\ell' < \ell$, and such that, either $w'_1 = w$, or there exist $z_2 \neq w'_1$ such that $z_2\, \mathcal{P}\, w'_1$. This last property implies that $w'_1 \in J$, so $|z_1\, w_1| \geq d(z_1, J) > 0$. This shows that the length of any sequence as in Definition 2.3 is bounded below by $d(z, J)$ and, therefore, $|\bar{z}\, \bar{w}| > 0$. Notice that, in fact, this argument is also the first step in a proof of Theorem 2.7.

So, the restriction of π to $\operatorname{Int} D \cup \operatorname{Int} C$ is a continuous injective map onto its image. We leave to the reader the verification that its inverse is continuous. Let us see that this restriction preserves lengths. For a curve $\gamma : [0,1] \to D$, if $\gamma(0) \in \operatorname{Int} C$ and $\gamma(t) \in \operatorname{Int} D$ for $t > 0$, the length of $\bar{\gamma} = \pi \circ \gamma$ can be computed as follows:

$$|\bar{\gamma}| = \lim_{s \to 0} |\bar{\gamma}|_{[s,1]}| = \lim_{s \to 0} |\gamma|_{[s,1]}| = |\gamma| \,. \tag{3}$$

For this, we used the continuity of length (see [3]) and Theorem 2.7. In particular, if $p, q \in \operatorname{Int} C$, then $|\pi([p,q])| = |[p,q]|$. Now, if γ is contained in C and

$\{t_j\}_{j=1}^N$ is a sufficiently fine partition of $[0,1]$,

$$\sum_{j=1}^{N-1} |\bar\gamma(t_i)\,\bar\gamma(t_{i+1})| = \sum_{j=1}^{N-1} |\pi([\gamma(t_i)\,\gamma(t_{i+1})])| = \sum_{j=1}^{N-1} |\gamma(t_i)\,\gamma(t_{i+1})|. \tag{4}$$

This suffices to prove that $|\bar\gamma| = |\gamma|$.

Finally, suppose that $\bar\gamma : (-\varepsilon, 0] \to S$ is a geodesic with $\gamma(0) \in \pi(\operatorname{Int} C)$, where $\varepsilon > 0$. The homeomorphism obtained above defines a geodesic γ in D with $\gamma(0) \in \operatorname{Int} C$. As γ clearly cannot be extended through $\gamma(0)$, the result follows. ☑

For each $w \in J$, denote by $w^\perp$ the line segment of D that contains w and is perpendicular to J.

Lemma 3.2. *For every $w \in J$ and $z \in w^\perp$, $\pi([z,w])$ is the unique minimizing curve in S from $\bar z$ to $\bar w$. It follows that $|\bar z\,\bar w| = |z\,w|$.*

Proof. Take $z \in w^\perp$, $z \neq w$, and consider a minimizing curve $\bar\gamma$ from z to w, whose existence is guaranteed by Theorem 2.5. First assume that $z \in \operatorname{Int} D$, so $\bar z \in \pi(\operatorname{Int} D)$. Then, Theorem 2.7 says that an initial portion of $\bar\gamma$ is contained in $\pi((p,q))$ for some $p, q \in \partial P$. At least one among p and q does not lie in C, otherwise $\bar\gamma$ wouldn't reach $\bar w$, due to Lemma 3.1. Assume that $q \in J$ and suppose, for a contradiction, that $q \neq w$. So $\bar\gamma \supset \pi([z,q])$, and $|\bar z\,\bar w| = |\bar\gamma| \geq |\pi([z,q])|$. Now, by continuity (see [3]), this length can be obtained as the limit when $q' \to q$ along $[z,q]$. Since $q' \in \operatorname{Int} D$, Theorem 2.7 applies. This, with the continuity of distance, gives:

$$|\pi([z,q])| = \lim_{q' \to q} |\pi([z,q'])| = \lim_{q' \to q} |z\,q'| = |z\,q|\,. \tag{5}$$

However, w is the closest point to z in J, and $|z\,q| > |z\,w|$. The conclusion is that $|\bar z\,\bar w| > |z\,w|$, contradicting Theorem 2.5. Therefore, $q = w$, and $\bar\gamma = \pi([z,w])$. ☑

We now restrict the discussion to the paper-folding scheme $\mathcal{P}$ on D defined by (1) in the Introduction. Recall that it depends on sequences of positive real numbers $b_k = |\beta_k| = |\beta'_k|$, $k \geq 0$, and $a_{j,k} = |\alpha_{j,k}| = |\alpha'_{j,k}|$, $j = 0,1$, $k \geq 1$ satisfying $b_0 + \sum_{k\geq 1}(a_{0,k} + a_{1,k} + b_k) = |J|/2$. Denoting $\bar\beta_k = \pi(\beta_k) = \pi(\beta'_k)$ and $\bar\alpha_{j,k} = \pi(\alpha_{j,k}) = \pi(\alpha_{j,k})$, Theorem 2.11 implies that $|\bar\beta_k| = b_k$ and $|\bar\alpha_{j,k}| = a_{j,k}$. Let's fix notations for the endpoints of paired segments: $\beta'_k = [q^-_{k,0}, q^-_{k+1,0}]$, $\alpha_{0,k} = [q^-_{k,0}, p_{0,k}]$, $\alpha'_{0,k} = [p_{0,k}, q^-_{k,1}]$, $\ldots$, $\alpha'_{1,k} = [q^+_{1,k}, p_{1,k}]$, $\alpha_{1,k} = [p_{1,k}, q^+_{k,0}]$, $\beta_k = [q^+_{k+1,0}, q^+_{k,1}]$, where $k \geq 0$ when indexing the betas, and $k \geq 1$ when indexing the alphas. Notice that each $p_{j,k}$, $j = 0,1$ and $k \geq 1$, is the folding point of the fold $\langle \alpha_{j,k}, \alpha'_{j,k} \rangle$, while $q^+_{k,0}$, $q^+_{k,1}$, $q^-_{k,1}$ and $q^-_{k,0}$, for $k \geq 1$ are the basepoints of a chain of pairings. Therefore, Theorem 2.14 guarantee that their projections $\bar{p}_{j,k}$ and $\bar{q}_k$ indeed are conical vertices with total angles equal to π and 4π around it.

We denote by $z_0 \in D$ the point $*$ of the Introduction. Parametrize the points in D as $z = z(r,\theta)$, where $r = |z\, z_0|$ and, for $z \neq z_0$, θ is the angle formed at z_0, in the clockwise direction, by $z_0^{\perp}$ and $[z_0, z]$ (see Figure 2). Thus, $\theta \in [-\pi/2, \pi/2]$ and r is a non-negative number bounded above by a function of θ. We now rephrase Theorem 1.1 in a form that is more precise and convenient for its proof. The set V of the previous statement is equal to $\pi(\{-\theta_0 \leq \theta \leq \theta_1\})$. If $c_j = b_k/a_{j,k}$ are the constants in Theorem 1.1, its relation to the s_j of Theorem 3.3 is given by $s_j = c_j/(2 + c_j)$.

Theorem 3.3. *Suppose that there exists constants* $0 < s_j < 1$*,* $j = 0, 1$*, such that, for every* $k \geq 1$*:*

$$b_k = s_j(2a_{j,k} + b_k) \tag{6}$$

Let $\theta_j \in \arcsin(s_j) \in (0, \pi/2)$*. For each* $z = z(r,\theta) \in D$ *the following holds:*

i) If $\theta \in [-\theta_0, \theta_1]$*, then* $\pi([z, z_0])$ *is the unique minimizing curve of* S *from* $\bar{z}$ *to* $\bar{z}_0$*.*

ii) If $\theta \in [-\pi/2, -\theta_0) \cup (\theta_1, \pi/2]$*, then every minimizing curve of* S *from* $\bar{z}$ *to* $\bar{z}_0$ *is the concatenation of a curve from* $\bar{z}$ *to some* $\bar{q}_k$ *and* $\cup_{j\geq k}\bar{\beta}_j$*. There are points* $\bar{z}$ *of this form arbitrarily close to* $\bar{z}_0$ *that can be joined to* $\bar{z}_0$ *by exactly two minimizing curves.*

Proof. Let $z = z(r,\theta) \in D_0$, $z \neq z_0$, and consider a minimizing curve $\bar{\gamma}$ from $\bar{z}$ to $\bar{z}_0$. We are going to show that $\bar{\gamma}$ has the stated form by eliminating every other possibility. First, in case $\theta = 0$, then $z \in z_0^{\perp}$, and the uniqueness in Lemma 3.2 guarantee that $\bar{\gamma} = \pi([z, z_0])$. In this case, denote this curve by $[\bar{z}, \bar{z}_0]$.

Consider the cases $\theta = \pm\pi/2$, so $\bar{z} \in G = \pi(J)$. Suppose, by contradiction, that $\bar{\gamma}$ leaves G.

Then, due to Theorem 2.7, $\bar{\gamma}$ can be uniquely prolonged until it reaches $\pi(\operatorname{Int} C)$, and cannot be further prolonged by Lemma 3.1. Therefore, $\bar{\gamma}$ does not reach $\bar{z}_0$. We conclude that $\bar{\gamma}$ does not leave G. By a similar reason, $\bar{\gamma}$ does not enter the the $\bar{\alpha}_{j,k}$, otherwise it ends at $\bar{p}_{j,k}$, by Theorems 2.11 and 2.14. Finally, if $\bar{\gamma}$ passes by some $\bar{q}_k$, then $\bar{q}_j \notin \bar{\gamma}$ for every $0 \leq j < k$, on the contrary $\bar{\gamma}$ ends at $\bar{q}_0$. The only remaining possibility is that $\bar{\gamma}$ goes from $\bar{z}$ to some $\bar{q}_k$, and from there on it coincides with $\cup_{j\geq k}\bar{\beta}_j$, as stated. In this case, we also denote this curve by $[\bar{z}, \bar{z}_0]$. As a consequence, $|\bar{q}_k\, \bar{z}_0| = B_k$ for every $k \geq 0$, and $|\bar{p}_k\, \bar{z}_0| = a_k + B_k$, where B_k is defined by:

$$B_k = \sum_{i\geq k} b_i\,. \tag{7}$$

From here on we treat the remaining cases: $-\pi/2 < \theta < \pi/2$, $\theta \neq 0$. Notice that $\bar{\gamma}$ does not meet $\pi(z_0^{\perp})$ except at $\bar{z}_0$. Otherwise, from the meeting point $\bar{w}$ on, the first case in this proof says that $\bar{\gamma}$ coincides with $[\bar{w}, \bar{z}_0]$. Then, the triangle inequality, applied to a flat neighborhood of $\bar{w}$ (Theorem 2.7), implies that $\bar{\gamma}$ is not minimizing around $\bar{w}$, contradicting the minimizing property of restrictions of minimizing curves. By a reasoning similar to the one on the second case, $\bar{\gamma}$ doesn't meet the interior of any $\bar{\alpha}_{j,k}$. It is also clear that no $\bar{p}_{j,k}$ belongs to $\bar{\gamma}$, as there are no extensions of geodesics through it. And, if $\bar{\gamma}$ passes by some $\bar{q}_k$, it coincides from there on with $[\bar{q}_k, \bar{z}_0]$.

Assume that $\theta > 0$, the case $\theta < 0$ being analogous. The conclusion of the discussion above is that either $\bar{\gamma} = \pi([z, z_0])$; or, for some $k \geq 0$ and $n = 0, 1$, $\bar{\gamma} = \bar{\gamma}_{k,n}$, defined as the concatenation of $\pi([z, q_{k,n}^{+}])$ with $[\bar{q}_k, \bar{z}_0]$. Moreover, the possibility $n = 1$ is excluded, as it violates the triangle inequality in a conical neighborhood of $\bar{q}_k$. Simplify the notation to $q_{k,0}^{+} = q_k$, $\bar{\gamma}_{k,0} = \bar{\gamma}_k$, and $s_1 = s$. The lengths of our possible $\bar{\gamma}$, namely $\pi([z, z_0])$ and $\bar{\gamma}_k$, are equal to r and $|z\, q_k| + B_k$, respectively. In order to compare these quantities, we solve an elementary calculus optimization problem. Due to the hypothesis (6),

$$|z\, q_k| + B_k = |z\, q_k| + s \sum_{j \geq k} (2a_j + b_j) = |z\, q_k| + s|q_k\, z_0| \,. \tag{8}$$

The values assumed by right-hand side, as well as the value $r = |z\, z_0|$, are contained in the values assumed by $|z\, w| + s|w\, z_0|$ when w varies in J. For our purposes, we found convenient to parametrize such w and to express (8) as follows. Here, for the first time in this proof, we used the relation between b_k and a_k. And now the assumption that the geometry of D is euclidean will finally be employed (see the remark at the start of Section 2).

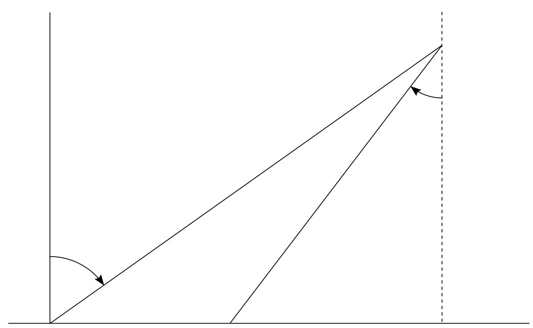

FIGURE 2. Notations in the proof of Theorem 3.3

For each $\varphi \in (-\pi/2, \pi/2)$, consider the geodesic ray that leaves z forming an angle equal to φ, in the clockwise direction, with the perpendicular to J

through z (see Figure 2). This geodesic ray reaches either J, or a prolongation of it, in a point $w = w(\varphi)$ satisfying

$$|z\,w| = r\cos\theta\sec\varphi\,. \tag{9}$$

Let $z' \in J$ be the orthogonal projection of z. Then $|w\,z_0| = |z'\,z_0| \pm |w\,z'|$, depending on whether $\varphi < 0$ or $\varphi \geq 0$. In any case, summing (9) with $s|w\,z_0|$, and expressing in terms of φ gives:

$$\begin{aligned}
|z\,w| + s|w\,z_0| &= |z\,w| + s(r\sin\theta - |z\,w|\sin\varphi) && (10)\\
&= r\cos\theta\sec\varphi + s(r\sin\theta - r\cos\theta\tan\varphi) && (11)\\
&= f(\varphi), && (12)
\end{aligned}$$

where $f : (-\pi/2, \pi/2) \to \mathbb{R}$ is defined by:

$$f(\varphi) = r\cos\theta(\sec\varphi - s\tan\varphi) + sr\sin\theta\,. \tag{13}$$

Therefore, the lengths of the candidates to minimizing curves are given by values of f: in fact, $f(\theta) = r$; and if $w(\varphi) = q_k$, then the left-hand side of (10) coincides with (8).

Looking for critical points of f,

$$f'(\varphi) = r\cos\theta(\tan\varphi\sec\varphi - s\sec^2\varphi) = 0, \tag{14}$$

we find out that $\varphi^* = \arcsin(s)$ is the unique critical point. Since $0 < s < 1$, $0 < \varphi^* < \pi/2$. It is a global minimum, as

$$f''(\varphi) = r\cos\theta\sec\varphi(\sec^2\varphi + \tan^2\varphi - 2s\sec\varphi\tan\varphi) > 0 \tag{15}$$

for every $\varphi \in (-\pi/2, \pi/2)$. This is clear if $\tan\varphi \leq 0$, and it follows from $0 < s < 1$ if $\tan\varphi > 0$, as the last factor is greater than $(\sec\varphi - \tan\varphi)^2$ in this case.

The location of φ^* with respect to θ provides the two cases of the statement. If $\theta \leq \varphi^*$, then $\pi([z, z_0])$ is a minimizing curve from $\bar{z}$ to $\bar{z}_0$. It is unique, as the lengths of $\bar{\gamma}_k$ are values of f for $\varphi < \theta$. And, if $\theta > \varphi^*$, then $\bar{\gamma}_k$ is minimizing for some $k \geq 0$ such that q_k is nearby $w(\varphi^*)$. More precisely, the behaviour of f implies that, if $w(\varphi^*) \in [q_{k+1}, q_k]$, then $\bar{\gamma}_k$ or $\bar{\gamma}_{k+1}$ is minimizing. It is possible that both have this property, as will be shown in the sequence by a continuity argument.

Since f attains each of its values at most two times, there are at most two minimizing curves from $\bar{z}$ to $\bar{z}_0$. One condition that ensures uniqueness is $w(\varphi^*) = q_k$. For a fixed $\theta_0 > \varphi^*$, there exist $z_k = z(r_k, \theta_0)$ such that this happens provided that k is big enough. More precisely, let $k_0 \geq 0$ be such that, for every $k \geq k_0$, the geodesic ray leaving q_k with angle equal to φ^* formed in the counter-clockwise direction with $q_k^{\perp}$, meets $\{\theta = \theta_0\}$ in the point z_k.

Of course, $r_k \to 0$ as $k \to +\infty$. For each $k \geq k_0$, let $[\bar{z}_k, \bar{z}_0]$ be the unique minimizing curve from $\bar{z}_k$ to $\bar{z}_0$. Since the point $w(\varphi^*)$ associated to each $z \in [z_k, z_{k+1}]$ lies in $[q_{k+1}, q_k]$, the possible minimizing curves from such z to $\bar{z}_0$ are the concatenations $\pi([z, q_{k+1}]) \cup [\bar{q}_{k+1}, \bar{z}_0]$ and $\pi([z, q_k]) \cup [\bar{q}_k, \bar{z}_0]$. Let $g : [z_k, z_{k+1}] \to \mathbb{R}$ be the function defined by the difference of the lengths of these curves. It is continuous, and the signs of its values at z_k and z_{k+1} are opposite, due to the uniqueness of $[\bar{z}_k, \bar{z}_0]$ and $[\bar{z}_{k+1}, \bar{z}_0]$. Therefore, the lengths are equal for some z_1 in the domain of g, and $\bar{z}_1$ can be joined to $\bar{z}_0$ by exactly two minimizing curves. For a further refinement on the location of z_1, one can check that it lies between z_k and the intersection of $\{\theta = \theta_0\}$ with $p_{0,k}^{\perp}$. This concludes the proof of the theorem. ☑

***Remark* 3.4.** While $V = \pi(\{-\theta_0 \leq \theta \leq \theta_1\})$ is a plane sector in S when considered with its intrinsic metric induced by the metric of S, this is not true if the metric considered in V is the subspace one. For instance, by choosing big $s_0, s_1 \in (0, 1)$, it is easy to see that there are points among the projections of the intersections of $(q_{j,k}^{\pm})^{\perp}$ with $\{\theta = -\theta_0\}$ and $\{\theta = \theta_1\}$ that go to each other minimizing length passing by the q_k. The question of determining $|\bar{z}\,\bar{w}|$ for points $\bar{w} \neq \bar{z}_0$ will be left for future investigations.

To conclude the paper, we briefly describe an identification pattern in [4], depicted in Figure 3 with its quotient. The pairings are:

$$\beta_0'\,\beta_1'\,\beta_2' \cdots * \cdots \beta_2\,\alpha_2'\,\alpha_2\,\beta_1\,\alpha_1'\,\alpha_1\,\beta_1\,\beta_0\,. \tag{16}$$

It is similar to (1) and would coincide with it if $a_{0,k} = 0$ for every $k \geq 1$, corresponding to $s_0 = 1$ and $\theta_0 = \pi/2$. Following step-by-step the above argument, one can prove:

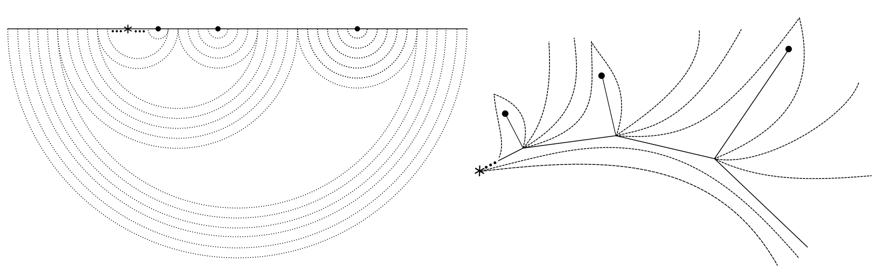

FIGURE 3. Paper-folding scheme (16) and its quotient, depicted in a more loose way when compared to Figure 1. The points • and $\perp$ are conical vertices with angles π and 3π, respectively. Dashed lines are the minimizing curves as in Theorem 3.5.

Theorem 3.5. *Let S be the quotient of D associated to a paper-folding scheme of the form (16). Suppose that $|\beta_k|/|\alpha_k|$ is constant. Let $z_0 \in D$ correspond to*

$*$, and employ the notations introduced before Theorem 3.3. Then, there exist $\theta_1 \in (0, \pi/2)$ such that, for each $z = z(r, \theta) \in D$:

i) If $\theta \in [0, \theta_0]$, then $\pi([z, z_0])$ is the unique minimizing curve from $\bar{z}$ to $\bar{z}_0$.

ii) If $\theta \in (\theta_0, \pi/2]$, then every minimizing curve of S from $\bar{z}$ to $\bar{z}_0$ is the concatenation of a curve from $\bar{z}$ to some $\bar{q}_k$ and $\cup_{j \geq k} \bar{\beta}_j$. There are points $\bar{z}$ of this form arbitrarily close to $\bar{z}_0$ that can be joined to $\bar{z}_0$ by exactly two minimizing curves.

Finally, notice that, in case both sequences $a_{j,k}$ are equal to zero, then $*$ is just a folding point. So, a fold can be seem as a degeneration of the families of singularities considered in this paper.

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