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Spectrally starred advertibly complete A-p-normed algebras

Álgebras A-p-normadas espectralmente estrella completas

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Abstract. We show that an advertibly complete A-p normed algebra E is isomorphic to the complex field $\mathbb{C}$, modulo its radical, in any of the following cases: 1) every element of E has a star-shaped spectrum, 2) E is involutive and every normal element of E has a star-shaped spectrum; 3) E is hermitian and every unitary element of E has a star-shaped spectrum.

Key words and phrases. Q-normed algebra, Advertibly complete algebra, Starred spectrum, Spectrally convex algebra, Spectrally starred algebra, Jacobson's radical, Involutive algebra, Normal element, Unitary element.

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Resumen. Probamos que un álgebra A-p-normada E advertible completa es isomorfa al campo complejo $\mathbb{C}$, módulo su radical, si se da una de las siguientes condiciones: 1) todo elemento de E tiene espectro en forma estrella, 2) el álgebra E es involutiva y todo elemento normal de E tiene espectro en forma estrella; 3) el álgebra E es hermitiana and todo elemento unitario de E tiene espectro en forma estrella.

Palabras y frases clave. álgebra Q-normada, espectro estrella, álgebra espectralmente convexa, álgebra espectral estrella, radical de Jacobson, álgebra involutiva, elemento normal, elemento unitario.

1. Introduction

In [2], Bhatt showed that if $(E, \|.\|)$is a Banach algebra each element of which has a convex spectrum, then $E/Rad(E)$ is isomorphic to $\mathbb{C}$. He obtained the same conclusion for involutive Banach algebras assuming the convexity hypothesis only on the spectrum of every normal element. Moreover, in the hermitian case, he showed that it suffices to assume that the spectrum of each unitary element is convex. In [5], the authors studied the previous questions in the general case of locally multiplicatively convex ordinary algebras. In this paper, we discuss similar results for advertibly complete A-p-normed algebra, $0 < p \leq 1$ which are spectrally starred. Our proof does not appeal to the spectral states of the algebra as it is the case of Banach algebras. We show (Theorem 3.3) that if $(E, \|.\|_p), 0 < p \leq 1$, is an advertibly complete A-p-normed algebra (in particular a Q-p-normed algebra) and spectrally starred, then E modulo its Jacobson radical is isomorphic to $\mathbb{C}$. In the involutive case, we obtain the same conclusion for advertibly complete A-p-normed algebra assuming only that each normal element of E has a starred spectrum (Theorem 4.1). If the algebra is additionally hermitian, it suffices to assume that the spectrum of each unitary element is starred (Theorem 4.5) as in the Banach case ([2, Theorem 6, p. 729]). Finally, we obtain the same conclusion for algebras with an involution anti-morphism (Theorem 4.9 and Remark 4.11).

2. Preliminaries

Let E be a complex unital algebra with unit e. If $x \in E$ the symbols $Sp_E(x)$ and $\rho_E(x)$ denote the spectrum of x and its spectral radius, respectively. The Jacobson radical of E will be denoted by $Rad(E)$, that is $Rad(E) = \{a \in E : \rho_E(ab) = 0, \text{ for every } b \in E\}$. A linear functional $f : E \longrightarrow \mathbb{C}$ is called quasimultiplicative if $f(e) = 1$ and $f(a) \in Sp_E(a)$, for all $a \in E$. The algebra E is called ordinary if every quasimultiplicative functional on E is multiplicative. We say that the algebra E is spectrally bounded if, for each $a \in E$, $Sp_E(a)$ is bounded. In the sequel $Sp(E)$ denotes the Gelfand spectrum of E, that is the set of non-zero characters of E. An algebra E is called spectrally convex algebra if $Sp_E(a)$ is a convex set for each $a \in E$, and E is said to be spectrally starred if for each $a \in E$, $Sp_E(a)$ is a star-shaped subset relative to one of its points.

Let E be a complex algebra and $x \longmapsto x^*$ be an algebra involution on E, that is a linear involution such that $(xy)^* = y^*x^*$ for each x and y in E. A linear involution $*$ on E is said to be anti-morphism if $(xy)^* = x^*y^*$ for each x and y in E. An element $a \in E$ is said to be hermitian (resp., normal) if $a = a^*$ (resp., $aa^* = a^*a$). We denote by $H(E)$ (resp., $N(E)$) the set of hermitian (resp., normal) elements of E. The algebra E is said to be hermitian if the spectrum of every $h \in H(E)$ is real. An element a of E is said to be unitary if $aa^* = a^*a = e$. The set of all unitary elements of E is denoted by $\mathcal{U}(E)$.

Let $\|.\|_p$, $0 < p \leq 1$, be a linear p-norm on E. We say that $\|.\|_p$ is an A-p-norm if left multiplication by elements of the algebra is bounded, that is, for every $x \in E$ there exists $M(x) > 0$ such that $\|xy\|_p \leq M(x) \|y\|_p$, for every $y \in E$. An algebra is said to be A-p-normed if it is equipped with an A-p-norm. A linear p-norm $\|.\|_p$ on E is called an algebra p-norm it is submultiplicative, that is $\|xy\|_p \leq \|x\|_p \|y\|_p$, for all x, $y \in E$. Notice that the p-normed algebras considered are not necessarily complete as is the case in

[9]. Here, a complete p-normed algebra will be called p-Banach. A unital A-p-normed algebra $(E, \|.\|_p),\ 0 < p \leq 1$, is called a Q-algebra if the group $G(E)$ of invertible elements is open. A net $(x_\lambda)_\lambda$ in E is called an approximate inverse of $x \in E$ if $x_\lambda x \longrightarrow e$ and $x x_\lambda \longrightarrow e$. Observe that if $(x_\lambda)_\lambda$ is convergent, then $x_\lambda \longrightarrow x^{-1}$. The algebra E is called advertibly complete if every Cauchy net which is an approximate inverse in E converges in E. This notion was introduced by S. Warner ([8]). In the normed case, it coincides with the Q-property ([8]). Throughout the paper, all considered algebras are complex and unital.

3. Spectrally starred algebras

The following lemma will be useful later

Lemma 3.1. *Let E be an algebra.*

i) If $Sp_E(x)$ is a singleton for all $x \in E$, then $E/Rad\,(E)$ is isomorphic to $\mathbb{C}$.

ii) If $(E, \|.\|_p),\ 0 < p \leq 1$, is a Q-p-normed algebra, then it is ordinary.

Proof. i) Note first that E has only one maximal left ideal. Otherwise let M and M' be two distinct maximal left ideals of E. Then one has $M + M' = E$. Consider $m \in M$ and $m' \in M'$ such that $m + m' = e$. One has $m' = e - m$ is not invertible, so $1 \in Sp_E(m)$. But m is not invertible so $0 \in Sp_E(m)$. It follows that $Sp_E(m)$ contains at least two points, which is impossible. Now let M be the unique maximal left ideal necessarily the unique maximal right ideal of E. Let $x \in E$ and $\lambda \in Sp_E(x)$. Then necessarily $x - \lambda e \in M$ and the latter has codimension 1. This completes the proof.

ii) We will show that $(E, \|.\|_p),\ 0 < p \leq 1$, is spectrally bounded. Then, we conclude with [7], Corollary, p.113. Let $r > 0$ be such that $B(e, r) \subset G(E)$, where $B(e, r)$ is the open ball of E centered at e with radius r. Suppose that there exists $x \in E$ such that $Sp_E(x)$ is unbounded. Let $(\lambda_n)_n$ be a sequence of elements of $Sp_E(x)$ such that $|\lambda_n| \longrightarrow +\infty$. Let $n_0 \in \mathbb{N}$ be such that $-\frac{1}{\lambda_{n_0}}x \in B(0, r)$. Then $-\frac{1}{\lambda_{n_0}}x + e \in G(E)$ and therefore $x - \lambda_{n_0}e \in G(E)$, which is impossible. Thus $Sp_E(x)$ is bounded. ☑

Let $(E, \|.\|_p),\ 0 < p \leq 1$, be an A-p-normed algebra. For each $x \in E$, let

$$\|x\|'_p = \sup\left\{\|xy\|_p : \|y\|_p \leq 1\right\}.$$

Then $\|.\|'_p$ is a submultiplicative linear p-norm on E. Moreover, we have the following lemma, wich is readily seen.

Lemma 3.2. *Let $(E, \|.\|_p),\ 0 < p \leq 1$, be an A-p-normed algebra. If $(E, \|.\|_p)$ is advertibly complete, then $(E, \|.\|'_p)$ is a Q-p-normed algebra.*

Theorem 3.3. *For $0 < p \leq 1$, let $(E, \|.\|_p)$ be an advertibly complete A-p-normed algebra. If $(E, \|.\|_p)$ is spectrally starred, then $E/Rad\,(E) \simeq \mathbb{C}$.*

Proof. Suppose first that the algebra E is commutative and semi-simple. Let us first show that the spectrum of every element of E is reduced to a singleton. Let $x \in E$. As $(E, \|.\|'_p)$ is a Q-p-normed algebra by Lemma 3.2, one has

$$Sp_E(x) = \{\chi(x) : \chi \in Sp(E)\}\,.$$

Since $Sp_E(x)$ is star-shaped there exists $\alpha \in Sp_E(x)$ such that for every λ in $Sp_E(x)$ the line segment α to λ is in $Sp_E(x)$. It follows that $Sp_E(x) = \{\alpha\}$, otherwise let $\lambda \neq \alpha$ be another element of $Sp_E(x)$. Then, there exist χ_1, $\chi_2 \in Sp(E)$ such that $\alpha = \chi_1(x)$ and $\lambda = \chi_2(x)$. For every $0 < t < 1$, we have

$$t\chi_1(x) + (1-t)\,\chi_2\,(x) \in Sp_E(x).$$

Put $f = t\chi_1 + (1-t)\,\chi_2$. Then f is a linear form on E such that

$$f(a) \in Sp_E(a), \text{ for every } a \in E.$$

As the algebra E is ordinary, it follows that $f \in Sp(E)$, which is impossible. We conclude with Lemma 3.1.

In the general case, given $x \in E$, let C_x be the maximal commutative subalgebra of E containing x and e. Then C_x is Q-p-algebra such that

$$Sp_E(y) = Sp_{C_x}(y), \text{ for every } y \in C_x.$$

This last equality makes C_x and hence $C_x/Rad(C_x)$ spectrally starred convex. By the first part $C_x/Rad(C_x) \simeq \mathbb{C}$. Therefore, for every $y \in C_x$, $Sp_{C_x}(y)$ is a singleton too. In particular, $Sp_E(x)$ is a singleton, for each $x \in E$. We conclude with i) of Lemma 3.1. ☑

Corollary 3.4. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be an advertibly complete A-p-normed algebra. If there exists a neighbourhood V of 0 such that for every $x \in V$, $Sp_E(x)$ is starred, then $E/Rad\,(E) \simeq \mathbb{C}$.*

Theorem 1 in [2] remains true for advertibly complete A-p-normed algebras as it is shown in the following result:

Corollary 3.5. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be an advertibly complete A-p-normed algebra (in particular p-Banach and so Banach) algebra. If $(E, \|.\|_p)$ is spectrally convex, then $E/Rad(E)$ is isomorphic to $\mathbb{C}$.*

***Remark* 3.6.** In Corollary 3.5, the advertibly completeness is not superfluous. Indeed, consider for example the algebra of all complex polynomials on the interval $[0, 1]$, endowed with the pointwise operations and the sup-norm algebra. This space is not of a Q-algebra for the spectrum of any non constant function is equal to $\mathbb{C}$. Then it is not advertibly complete. Let us note that, in the non necessarily unital case, Corollary 3.5 does not hold true even if the algebra is commutative, as any arbitrary radical commutative algebra shows.

4. Involutive case

In [2, Th. 4], Bhatt has shown that if each normal element of a Banach star algebra E has a convex spectrum, then $E/Rad\,(E)$ is isomorphic to $\mathbb{C}$. In fact, this result remains true for involutive advertibly complete A-p-normed algebra, $0 < p \leq 1$, where the spectrum of each normal element of the algebra is starred.

Theorem 4.1. *Let* $(E, \|.\|_p)$, $0 < p \leq 1$*, be an advertibly complete* A-p-*normed algebra with algebra involution* $x \longmapsto x^*$. *If each normal element of* E *has a starred spectrum, then* $E/Rad\,(E) \simeq \mathbb{C}$.

Proof. Let x be a normal element of E. Then x is contained in a closed maximal commutative $*$-subalgebra M_x of E and

$$Sp_{M_x}(y) = Sp_E(y)\,, \text{ for every } y \in M_x.$$

Since each element of M_x is obviously normal, it follows from the hypothesis of the theorem that M_x is spectrally starred. Hence, by Theorem 3.3, $M_x/Rad\,(M_x) \simeq \mathbb{C}$. It follows that $Sp_E(x)$ is a singleton, for every normal element x of E. In particular, $Sp_E(h)$ is a singleton, for every $h \in H(E)$. This implies that E is a hermitian algebra. Consider the semi-simple algebra $B = E/Rad\,(E)$ provided with the canonical involution resulting from that of E. Since E is a hermitian, B is also hermitian. As in [6], one can prove that the Ptàk function p_B, given by

$$p_B(x) = \rho_B\,(x^*x)^{\frac{1}{2}}\,, \text{ for every } x \in B,$$

is an algebra norm such that $Rad(B) = \ker(p_B)$. Now, for $h \in H(B)$, let M_h be the closed maximal commutative $*$-subalgebra of B generated by e and h. One has that $Sp_{M_h}(x)$ is a singleton, for every $x \in M_h$. Furthermore, if $a \in Rad(M_h)$, then $a = 0$. Indeed $a^*a \in Rad(M_h)$ and so

$$0{=}\rho_{M_h}(a^*a) = \rho_B(a^*a) = p_B(a).$$

It follows that M_h is semi-simple for every $h \in H(B)$. By Lemma 3.1, M_h is isomorphic to $\mathbb{C}$. Hence $B = E/Rad\,(E)$ is also isomorphic to $\mathbb{C}$. ☑

As an application one has:

Corollary 4.2. *For* $0 < p \leq 1$, *let* $(E, \|.\|_p)$ *be an involutive advertibly complete* A-p-*normed algebra (in particular involutive* p-*Banach and so involutive Banach) algebra. If each normal element of* E *has a convex spectrum, then* $E/Rad\,(E) \simeq \mathbb{C}$.

***Remark* 4.3.** As in Corollary 3.5, the reader will notice that the second result of Bhatt ([2, Theorem 4, p.728]) remains true in an involutive advertibly complete A-p-normed algebra under the assumption that each normal element has a starred spectrum.

The following result may be proved in much the same way as Proposition 14 of Bonsall and Duncan ([3, p. 66]):

Lemma 4.4. *For $0 < p \leq 1$, let $(E, \|.\|_p)$, $0 < p \leq 1$, be an involutive p-Banach algebra. Let $h \in H(E)$ and $\rho_A(h) < 1$. Then there exists $u \in \mathcal{U}(E)$ such that $h = (u + u^*)/2$. In particular, $H(E)$ is a linear span of $\mathcal{U}(E)$.*

Theorem 4.5. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be a hermitian p-Banach algebra. If each unitary element of E has a starred spectrum, then $E/Rad\,(E) \simeq \mathbb{C}$.*

***Proof*.** Let $u \in \mathcal{U}(E)$. Then, one has $\rho_E(u) \leq p_E(u) = 1$, for E is hermitian. This implies that $|\lambda| \leq 1$, for every $\lambda \in Sp_E(u)$. Moreover,

$$\overline{Sp_E(u)} = Sp_E(u^*) = Sp_E(u^{-1}) = \left\{\lambda^{-1} : \lambda \in Sp_E(u)\right\},$$

this yields $|\lambda| = 1$, for every $\lambda \in Sp_E(u)$. Hence the spectrum of a unitary element of E is contained in the unit circle. Since each unitary element of E has a starred spectrum, it follows that $Sp_E(u)$ is a singleton for every $u \in \mathcal{U}(E)$. Now, by Lemma 4.4, the spectrum of each hermitian element of E is a singleton. The rest of the proof runs as in Theorem 3.3. ☑

Corollary 4.6. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be a hermitian p-Banach algebra. If each unitary element of E has a convex spectrum, then $E/Rad\,(E) \simeq \mathbb{C}$.*

For the proof of the analog of Theorem 4.1 in the case of an involution anti-morphism, we will need two results. The first one is the analog of the result of Kaplansky ([4, Theorem 4.8]) in p-Banach algabras. Using theorem 3.10 of [9] and the fact that the quotient of a p-Banach algebra by a primitive ideal is also primitive, one can prove that the previous result of Kaplansky extends as follows:

Theorem 4.7. *A semisimple p-Banach algebra, $0 < p \leq 1$, in which every square has a quasi-inverse is necessarily commutative.*

The second result we will need concerns the existence of a closed and commutative subalgebra containing a normal element in the case of an involutive anti-morphism. For this, the proof of Lemma 6.1.3. of [1, p. 117-118], applies, mutatis mutandis, to this setting as well and one has:

Lemma 4.8. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be a p-Banach algebra with an involutive anti-morphism and x be normal in E. Then there exists a closed and commutative subalgebra F containing x, stable by involution, such that $Sp_F\,(x) = Sp_E\,(x)$.*

Theorem 4.1 remains true in p-Banach algebas with an involution anti-morphisman as showing by the following result.

Theorem 4.9. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be a p-Banach with an involution anti-morphism $x \longmapsto x^*$. If each normal element of E has a starred spectrum, then $E/Rad\,(E) \simeq \mathbb{C}$.*

Proof. As in the first step of the proof of Theorem 4.1 and taking into account Lemma 6.1.3, the algebra $B = E/Rad\,(E)$, provided with the canonical involution resulting from that of E, is hermitian. We will show that B is commutative and we conclude as in the first step of Theorem 3.3. As B is semi-simple, the anti-morphism $x \longmapsto x^*$ is continuous. Therefore, the algebra $H(B)$ is p-Banach such that

$$Sp_{H(B)}(h) = Sp_B(h) \subset \mathbb{R}, \text{ for every } h \in H(B).$$

Now by Theorem 3.7, $H(B)/Rad\,(H(B))$ is commutative. Finally, let us show that $H(B)$ is semi-simple. Let $h \in Rad\,(H(B))$ and $a = u + iv \in B$, with $u, v \in H(B)$. Then $\rho\,(hu) = 0$ and $\rho\,(hv) = 0$. Therefore $\rho\,(ha) = 0$. Indeed let $\lambda = \alpha + i\beta \in Sp_B\,(ha)$, with $\alpha, \beta \in \mathbb{R}$. If $hu - \alpha$ is invertible, then

$$ha - \lambda = (hu - \alpha)\left[e + i(hu - \alpha)^{-1}(hv - \beta)\right].$$

Since B is hermitian and $(hu-\alpha)^{-1}(hv-\beta) \in H(B)$, it follows that $e+i(hu-\alpha)^{-1}(hv-\beta)$ is invertible, which is impossible. Therefore $hu-\alpha$ is not invertible i.e., $\alpha \in Sp_B\,(hu)$. Hence $\alpha = 0$. Likewise, we show that $\beta = 0$ and therefore $h \in Rad(B) = \{0\}$. This completes the proof. ☑

As a consequence, one has

Corollary 4.10. *Let $(E, \|.\|_p)$, $0 < p \leq 1$, be a p-Banach with an involution anti-morphism $x \longmapsto x^*$. If each normal element of E has a convex spectrum, then $E/Rad\,(E) \simeq \mathbb{C}$.*

***Remark* 4.11.** Let $(E, \|.\|_p)$, $0 < p \leq 1$, be a p-Banach algebra with an involution anti-morphism $x \longmapsto x^*$. Suppose that E is hermitian and each unitary element of which has a starred spectrum. Using the same argument as in Theorem 4.5, the reader can prove that $E/Rad\,(E) \simeq \mathbb{C}$.

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