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Minimum depth of factorization algebra extensions

Profundidad mínima de extensiones de álgebras de factorización

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Abstract. In this paper we study the minimum depth of a subalgebra embedded in a factorization algebra, and outline how subring depth, in this context, is related to module depth of the regular left module representation of the given subalgebra, within the appropriate module ring. As a consequence, we produce specific results for subring depth of a Hopf subalgebra in its Drinfel'd double. Moreover we study a necessary and sufficient condition for normality of a Hopf algebra within a double cross product context, which is equivalent to depth 2, as it is well known by a result of Kadison. Using the sufficient condition, we then prove some results regarding minimum depth 2 for Drinfel'd double Hopf subalgebra pairs, particularly in the case of finite group algebras. Finally, we provide formulas for the centralizer of a normal Hopf subalgebra in a double cross product scenario.

Key words and phrases. Subring depth, Hopf subalgebra, Double cross product Hopf algebras, Drinfel'd double, Normality.

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Resumen. En este artículo estudiamos la profundidad mínima de una subálgebra en el contexto de álgebras de factorización, además describimos como la profundidad de subálgebra se relaciona con la profundidad modular de la representación regular del álgebra en el anillo de representaciones correspondiente. Como consecuencia de esto, obtenemos resultados específicos para extensiones de un álgebra de Hopf en su doble de Drinfel'd. Más aún, estudiamos una condición suficiente y necesaria para la normalidad de un álgebra de Hopf en un producto doble cruzado y utilizando la condición de suficiencia producimos resultados específicos para extensiones normales de un álgebra en su doble de Drinfel'd en el caso de álgebras de grupo finito. Finalmente encontramos fórmulas para el centralizador de una Hopf subalgebra en un producto doble cruzado.

Palabras y frases clave. Profundidad de subanillos, Subalgebras de Hopf, Algebras de Hopf de producto doble cruzado, Normalidad.

1. Introduction and preliminaries

1.1. Motivation

The paper [6], studies in depth the relationship between subgroup depth of a finite-dimensional Hopf algebra extension $R \hookrightarrow H$, and module depth of its quotient module $Q = H/R^+H$ in the finite tensor category of H-modules (respectively R modules), ${}_H\mathcal{M}$, (respectively ${}_R\mathcal{M}$). In particular, [6][Section 5] deals with semisimple group algebra extensions in their Drinfel'd double. Lemma (5.1) and Proposition (5.2) study the structure of Q as a G-module and then establish that its module depth n is implied by the faithfulness of the n-th tensor power $Q^{\otimes n}$, respectively. A converse with stronger conditions is also stated.

Continuing, in [6][Section 6] we study algebra extensions $H \hookrightarrow A\#H$ of a finite-dimensional Hopf algebra H in its smashed product with a given H-module algebra A, Proposition (6) tells us that

$$(A\#H)^{\otimes_H n} \cong A^{\otimes n} \otimes H \tag{1}$$

As a consequence, Theorem (6.2) relates subgroup depth to module depth in this context through the following equation:

$$d(H, A\#H) = 2d(A,{}_H\mathcal{M}) + 1 \tag{2}$$

Motivated by these results, in [4][Chapter 2.3], the author used module depth and Theorem (1.5) to calculate the minimum even depth of the 4-dimensional Sweedler algebra S_4 in its Drinfel'd double $D(S_4)$. This was later generalized in [5] to the minimum even depth of the n^2-dimensional half quantum group H_n in its Drinfel'd double.

Finally, in [10][Example 4.6], it is proved that the minimum even depth of the 8-dimensional small quantum group, $\overline{U}_q(sl_2)$ in its Drinfel'd double, is computed to be less than or equal to 4.

These results, and other examples, set the tone for this study. All of them being instances of factorization algebras, for which at least one of its factors is a finite-dimensional Hopf algebra. For this reason, it is natural to study the subalgebra depth for factorization algebra extensions.

1.2. Outline

Throughout this paper, all rings R, and algebras A, are associative with unit, all algebras are finite-dimensional over a field k of characteristic zero. All modules M are finite-dimensional as well. All subring pairs $S \subseteq R$ satisfy $1_S = 1_R$ and we denote the extension as $S \hookrightarrow R$.

The paper is organized as follows: In Subsection (1.3), we deal with preliminaries on the concept of depth. The main concepts include subring depth, module depth in a tensor category, and we introduce some results that will be of interest further in this study. Other concepts will be introduced when needed.

Section (2) introduces the reader to factorization algebras and their subalgebras; it contains our main results in the form of Theorems (2.1), (2.2), and Corollary (2.3). Example (2.4) deals with the case of the minimum depth of a Hopf algebra H in its smash product with an H-module algebra A, in particular the case of the Heisenberg double $\mathcal{H}(H)$ of finite-dimensional Hopf algebras, which motivates the next two sections.

Section (3) deals with the definitions of double cross products as factorization algebras in Propositions (3.2) and (3.3), and explores minimum odd depth for these cases in Theorems (3.4) and (3.6).

Section (4) studies normality (depth ≤ 2) in double cross product Hopf algebras. Theorem (4.1) states a necessary and sufficient condition for normality of a Hopf algebra in its double cross product with another Hopf algebra. This sufficient condition is then used to prove particular cases for Drinfel'd double extensions, in the case of finite group algebras in Corollary (4.2), and to provide formulas for the centralizer of a Hopf subalgebra, in the case of a depth two double cross product extension in Proposition (4.4) and Corollary (4.5).

1.3. Preliminaries on Depth

Let R be a ring and M and N two left (or right) R-modules. We say M is similar to N as an R-module if there are positive integers, p and q, such that $M|pN$ and $N|qM$, where nV means $\oplus^n V$ for every n and every R-module V, and $M|pN$ means that M is a direct summand of pN or equivalently that $M \otimes * \cong pN$. Whenever this is the case, we denote similarity as $M \sim N$. Notice that similarity is compatible with induction and restriction functors on ${}_R\mathcal{M}$, for if $R \hookrightarrow L$ is an extension of R, and K is a right L-module, then $M \sim N$ as R-modules implies $M \otimes_R K \sim N \otimes_R K$ as right L-modules. Moreover, if $S \hookrightarrow R$ is a subring then $M \sim N$ as R-modules implies $M \sim N$ as S-modules.

Consider now a ring extension $B \hookrightarrow A$. Let $n \geq 1$, by $A^{\otimes_B(n)}$ we mean $A \otimes_B A \otimes_B \cdots \otimes_B A$ n times, and define $A^{\otimes_B(0)}$ to be B. Notice that for $n \geq 1$, $A^{\otimes_B(n)}$ has a natural X-Y-bimodule structure where X and Y could be either A or B in all four possible combinations.

Definition 1.1. Let $B \hookrightarrow A$ be a ring extension, we say B has:

(1) **Minimum odd depth 2n +1**, denoted $d(B, A) = 2n + 1$, if :

$$A^{\otimes_B(n+1)} \sim A^{\otimes_B(n)}$$

as B-B modules for $n \geq 0$.

(2) **Minimum even depth 2n**, denoted d(B,A) = 2n, if:

$$A^{\otimes_B (n+1)} \sim A^{\otimes_B (n)}$$

as either B-A or A-B modules for $n \geq 1$.

Notice that by the observation above, one has that for all $n \geq 1$, $d(B,A) = 2n$ implies $d(B,A) = 2n+1$ by module restriction, and that for all $m \geq 0$, $d(A,B) = 2m+1$ implies $d(B,A) = 2m+2$ for all m by module induction. Hence, we are only interested in the minimum values for which any of these relations are satisfied. In case there is no such minimum value we say the extension has **infinite depth**.

A third type of subring depth, called **H-depth**, denoted by $d_h(B,A) = 2n-1$ if $A^{\otimes_B (n+1)} \sim A^{\otimes_B (n)}$ as A-A modules for $n \geq 1$, was introduced by Kadison in [8], as a continuation of the study of H-separable extensions introduced by Hirata, where such extensions are exactly the ones satisfying $d_h(B,A) = 1$. For the purposes of this paper, we will restrict our study to minimum odd and even depth only. In particular the cases $d(B,A) \leq 3$ and $d(A,B) \leq 2$.

Example 1.2. Let $B \hookrightarrow A$ be a ring extension, $R = A^B$ the centralizer and $T = (A \otimes_B A)^B$ the B central tensor square. It is shown in [7, Section 5] that $d(A,B) \leq 2$ implies a Galois A-coring structure in $A \otimes_R T$ in the sense of [2]. Furthermore, it is also shown in [7] that if the extension $B \hookrightarrow A$ is Hopf Galois for a given finite dimensional Hopf algebra H, then $d(B,A) \leq 2$.

An instance of this result, involving normality of a Hopf algebra extension can be obtained by looking into the quotient module defined for the extension, and the Galois coring structure obtained in this scenario. This result extends a similar result for finite group extensions and the role played by the permutation module of the subgroup pair.

Example 1.3. Let $R \hookrightarrow H$ be a finite dimensional Hopf algebra extension. Define their quotient module Q as H/R^+H, where $R^+ = ker\varepsilon \cap R$ and ε denotes the counit of H. Suppose that R is a normal Hopf subalgebra of H, one can easily show that the extension $R \hookrightarrow H$ is Q-Galois and therefore $d(R,H) \leq 2$. The converse is true as well, and the details can be found in [1, Theorem 2.10].

Hence, the following result holds:

Theorem 1.4. *Let $R \hookrightarrow H$ be a finite dimensional Hopf algebra pair. Then, R is a normal Hopf subalgebra of H if and only if*

$$d(R,H) \leq 2$$

Now, we again consider a k algebra A and an A-module M. Recall that the n-th truncated tensor algebra of M in ${}_A\mathcal{M}$ is defined as

$$T_n(M) = \bigoplus_{i=1}^{n} M^{\otimes(n)} \quad and \quad T_0(M) = k$$

We then define the **module depth** of M in ${}_A\mathcal{M}$ as $d(M,{}_A\mathcal{M}) = n$ if and only if $T_n(M) \sim T_{n+1}(M)$. In case M is an A-module coalgebra (a coalgebra in the category of A-modules), then $d(M,{}_A\mathcal{M}) = n$ if and only if $M^{\otimes(n)} \sim M^{\otimes(n+1)}$, see [9], [6].

We point out that an A-module M has module depth n if and only if it satisfies a polynomial equation $p(M) = q(M)$ in the representation ring of A. Where p and q are polynomials with integer coefficients of degree at most $n+1$. A brief proof of this can be found in [4]. For this reason, we say that a module M has finite module depth in ${}_A\mathcal{M}$ if and only if it is an algebraic element in the representation ring of A.

Finally, we would like to mention that in the case of Hopf subalgebra extensions $R \hookrightarrow H$, there is a way to link subalgebra depth with module depth. The reader will find a proof of the following in [9, Example 5.2]:

Theorem 1.5. *Let $R \hookrightarrow H$ be a Hopf subalgebra pair. Consider their quotient module Q, then the minimum depth of the extension satisfies:*

$$2d(Q,{}_R\mathcal{M}) + 1 \leq d(R,H) \leq 2d(Q,{}_R\mathcal{M}) + 2$$

2. Depth of factorization algebra extensions

The concept of factorization algebras is well-known in algebra. Recall that an algebra S is a factorization if there are two S subalgebras $A, B \subseteq S$ such that multiplication $A \otimes B \longrightarrow S$; $a \otimes b \mapsto ab$ and $B \otimes A \longrightarrow S$; $b \otimes a \mapsto ba$ are isomorphisms of vector spaces.

This can be understood in terms of a map

$$\psi : B \otimes A \longrightarrow A \otimes B \;;\; b \otimes a \longmapsto a_\alpha \otimes b^\alpha \tag{3}$$

satisfying the following octagon for all $a, d \in A$, and all $b, c \in B$:

$$(ad_\alpha)_\beta \otimes b^\beta c^\alpha = a_\beta d_\alpha \otimes (b^\beta c)^\alpha \tag{4}$$

In this case, we denote the factorization as

$$S_\psi := A \otimes_\psi B. \tag{5}$$

Factorization algebras are ubiquitous: setting $\psi(b \otimes a) = a \otimes b$ yields the tensor algebra $A \otimes B$. If H is a Hopf algebra and A is a left H-module algebra, satisfying

$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$, $h \cdot 1_A = \varepsilon(h)1_A$ for all $h \in H$ and $a, b \in A$, define $\psi : H \otimes A; \quad h \otimes a \longmapsto h_1 \cdot a \otimes h_2$, then the product becomes $(a \otimes h)(b \otimes g) = a\psi(h \otimes b)g = a(h_1 \cdot b \otimes h_2)g = ah_1 \cdot b \otimes h_2 g$. It is a routine exercise to verify that $A \otimes_\psi H$ is a factorization algebra and that $A \otimes_\psi H = A\#H$ is the smash product of A and H. Double cross products of Hopf algebras are also examples of factorization algebras, we will study them further in Section (3).

Now let S_ψ be a factorization algebra via $\psi : B \otimes A \longmapsto A \otimes B$. We point out that due to multiplication in S_ψ and the fact that both A and B are subalgebras of S_ψ, we get that for every $n \geq 1$, $S_\psi^{\otimes_B (n)} \in {}_{S_\psi}\mathcal{M}_{S_\psi}$ in the following way:

$$\begin{aligned} &(a \otimes_\psi b)(a_1 \otimes b_1 \otimes_B \cdots \otimes_B a_n \otimes b_n)(c \otimes_\psi d) \\ &\quad = a\psi(b \otimes a_1)b_1 \otimes_B \cdots \otimes_B a_n\psi(b_n \otimes c)d \\ &\quad = aa_{1\alpha} \otimes b^\alpha b_1 \otimes_B \cdots \otimes_B a_n c_\alpha \otimes b_n^\alpha d. \end{aligned} \tag{6}$$

The same condition holds for S_ψ as either a left or right B module via subalgebra restriction. In this case, we can assume $n \geq 0$ and define $S_\psi^{\otimes_B (0)} = B$. This allows us to consider the following isomorphism:

Theorem 2.1. *Let A and B be algebras, $\psi : B \otimes A \longmapsto A \otimes B$ a factorization and S_ψ the corresponding factorization algebra. Then*

$$S_\psi^{\otimes_B (n)} \cong A^{\otimes(n)} \otimes B \tag{7}$$

as S_ψ-B or B-S_ψ-bimodules for $n \geq 1$ and as B-B-bimodules for $n \geq 0$.

Proof. First notice that for $n = 1$ the result follows since $A, B \subset S_\psi$ is a factorization.

Now, for every $n > 1$, $(A \otimes_\psi B)^{\otimes_B (n)} \cong (A \otimes_\psi B)^{\otimes_B (n-1)} \otimes_B (A \otimes_\psi B)$. By induction on n and noting that $B \otimes_B A \cong A$ one gets

$$\begin{aligned} &(A \otimes_\psi B)^{\otimes_B (n-1)} \otimes_B A \otimes_\psi B \cong A^{\otimes_B (n-1)} \otimes B \otimes_B A \otimes B \\ &\quad \cong A^{\otimes(n-1)} \otimes A \otimes B \cong A^{\otimes(n)} \otimes B. \end{aligned} \tag{8}$$

Finally for $n = 0$ we get $S_\psi^{\otimes_B (0)} = B \cong k \otimes B \cong A^{\otimes(0)} \otimes B$ as B-B bimodules. ☑

Recall that a Krull-Schmidt category is a generalization of categories where the Krull-Schmidt Theorem holds. They are additive categories such that each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings, also this decompositions are unique in a categorical sense. For example categories of modules having finite composition length are Krull-Schmidt.

Theorem (2.1), in the context of a Krull-Schmidt category, allows us to relate subalgebra depth in a factorization algebra with module depth in the finite tensor category of finite dimensional left B-modules. In turn, this will allow us to compute minimum odd depth values in the case of Smash Product algebras and Drinfel'd Double Hopf algebras at the end of this section, as well as in Section (3). The following theorem and its corollary provide this connection and generalize [9, Equation 19] and [6, Equation 23].

Theorem 2.2. *Let $A \otimes_\psi B$ be a factorization algebra with ${}_B\mathcal{M}_B$ a Krull-Schmidt category, and $A \in {}_B\mathcal{M}$. Then, the minimum odd depth of the extension satisfies:*

$$d(B, S_\psi) \leq 2d(A, {}_B\mathcal{M}) + 1. \tag{9}$$

Proof. Let $d(A, {}_B\mathcal{M}_B) = n$. Since ${}_B\mathcal{M}_B$ is a Krull-Schmidt category, standard face and degeneracy functors imply $A^{\otimes_B(m)} | A^{\otimes_B(m+1)}$ for $m \geq 0$. Then $T_n(A) \sim T_{n+1}(A)$ implies $A^{\otimes(n+1)} \sim A^{\otimes(n)}$. Tensoring on the right by $(- \otimes B)$ one gets $A^{\otimes(n+1)} \otimes B \sim A^{\otimes(n)} \otimes B$. By Theorem (2.1), this is equivalent to $(A \otimes_\psi B)^{\otimes_B(n+1)} \sim (A \otimes_\psi B)^{\otimes_B(n)}$. This by definition is $d(B, S_\psi) \leq 2n+1$. ☑

Recall that B is a bialgebra if it is both an algebra and a coalgebra, such that the coalgebra morphisms are algebra maps, i.e., B is a coalgebra in the category of k algebras. This means that the counit $\varepsilon : B \longrightarrow k$ is an algebra map that splits the coproduct: $(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id$. Via the counit, the ground field k becomes a trivial right B module: $k \cdot b = k\varepsilon(b)$. Hence, a k vector space V becomes a right B-module: $V \cong V \otimes k$.

Corollary 2.3. *Let B be a bialgebra. Then, the inequality in Theorem* (2.2) *becomes an equality.*

Proof. Let B be a bialgebra, since k becomes a B-module via the counit of B, tensoring by $- \otimes_B k$ or $k \otimes_B -$ is a morphism of B-modules. Let $d(B, S_\psi) = 2n + 1$, then by definition $S_\psi^{\otimes_B(n)} \sim S_\psi^{\otimes_B(n+1)}$ as B-B-bimodules, and by the isomorphism in Theorem (2.1), this implies $A^{\otimes(n)} \otimes B \sim A^{\otimes(n+1)} \otimes B$, then it suffices to tensor on the right by $(- \otimes_B k)$ on both sides of the similarity to get $A^{\otimes n+1} \sim A^{\otimes n}$, which in turn implies $d(A, {}_B\mathcal{M}) \leq n$. ☑

Notice that assuming that $A \in_B \mathcal{M}$ makes sense, since the factorization algebras we are considering all depend on this fact to be well defined. On the other hand this result says nothing about *even* depth since by no means one should expect A to be a right or left S_ψ-module.

The reader may have already noticed that the three previous results are sufficient to recover the main results stated in the Motivation subsection at the beginning of this manuscript, in particular we point out the following:

Example 2.4. [6, Theorem 6.2] Let H be a Hopf algebra and A an H-module algebra, consider their smash product algebra $A\#H$, and the algebra extension $H \hookrightarrow A\#H$. The extension satisfies.

$$d(H, A\#H) = d(A,_H \mathcal{M}) + 1.$$

Moreover, one can show the following: let H be a cocommutative Hopf algebra such that $dim_k(H) \geq 2$ and consider H^* as a H-module algebra via $h \rightharpoonup f$ and their smash product $H^*\#H$, also known as their Heisenberg double, then the extension $H \hookrightarrow H^*\#H$ satisfies

$$d(H, H^*\#H) = 3.$$

This follows since H is a factor $H^*\#H$ subalgebra and the fact that ${}_HH^* \cong H_H$ by self duality and Frobenius reciprocity. Finally minimum module depth satisfies $d(H, \mathcal{M}_H) = 1$.

This example motivates the question of whether this result (or an equivalent one) can be attained for a more general class of extensions of Hopf algebras into factorization algebras. The next two sections deal with this question in the context of the Drinfel'd double $D(H)$ of a Hopf algebra and more generally in the case of the double cross product $A \bowtie B$ of a matched pair of Hopf algebras A and B.

3. Double cross products and minimum odd depth

The study of double cross products was started in the early seventies by W. Singer with the introduction of matched pairs of Hopf algebras satisfying certain module-comodule factorization conditions in the setting of connected module categories, [14]. Later M. Takeuchi [15] furthered the study of matched pairs in the ungraded case, in particular, he aimed at describing natural properties of braided groups. Later S. Majid [11] studied bicrossed products as a means to construct self dual objects in the category of Hopf algebras, primarily in the non cocommutative cases, in some sense motivated by the possibility to construct models for quantum gravity. We follow Majid's definition of double cross products as in [12].

Definition 3.1. Let A and B be two Hopf algebras such that A is a left B-module coalgebra and B a right A-module coalgebra. We say B and A are a matched pair [12, Definition 7.2.1] if there are coalgebra maps

$$\alpha : B \otimes A \longrightarrow B; \quad h \otimes k \longmapsto h \triangleleft k \quad \textit{and} \quad \beta : B \otimes A \longrightarrow A; \quad h \otimes k \longmapsto h \triangleright k$$

such that the following compatibility conditions hold:

$$(hg) \triangleleft k = \sum (h \triangleleft (g_1 \triangleright k_1))(g_2 \triangleleft k_2); \quad 1_B \triangleleft k = \varepsilon_A(k) 1_B \tag{10}$$

and

$$h \triangleright (kl) = \sum (h_1 \triangleright k_1)((h_2 \triangleleft k_2) \triangleright l); \quad h \triangleright 1_A = \varepsilon_B(h)1_A. \tag{11}$$

Define a product by

$$(k \bowtie h)(l \bowtie g) = \sum k(h_1 \triangleright l_1) \bowtie (h_2 \triangleleft l_2)g; \tag{12}$$

the resulting algebra $B \bowtie A$ is called the **double crossed product** of A and B [12, Theorem 7.2.2], and is a Hopf algebra with coproduct, counit and antipode given by

$$\Delta(k \bowtie h) = k_1 \bowtie h_1 \otimes k_2 \bowtie h_2 \tag{13}$$

$$\varepsilon(k \otimes h) = \varepsilon_K(k)\varepsilon_H(h), \tag{14}$$

$$S(k \bowtie h) = (1_K \bowtie S_H(h))(S_K(k) \bowtie k) \tag{15}$$

$$= S_H(h_1) \triangleright S_K(k_1) \bowtie S_H(h_2) \triangleleft S_K(k_2)$$

respectively.

The following are well known results and are cited here for the sake of completeness, they summarize the fact that Double Cross Products of Hopf algebras are exactly the Hopf algebras that factorize as the product of two Hopf subalgebras. More information about them can be found in [12] and [11] as well as in [3].

Proposition 3.2. *Double cross products are factorization algebras.*

This is evident and a consequence of the definition, more importantly the converse is also true:

Proposition 3.3. [12, Theorem 2.7.3] *Suppose H is a Hopf algebra and L and A two sub-Hopf algebras, such that $H \cong A \otimes_\psi L$ is a factorisation, then H is a double crossed product.*

Proof. The multiplication $m : L \otimes A \longrightarrow H$ defined by $a \otimes l \longmapsto al$ is a bijection. This implies $A \bigcap L = k$. Then consider the map:

$$\mu : L \otimes A \longrightarrow A \otimes L; \quad l \otimes a \longmapsto m^{-1}(la)$$

then define

$$\triangleright : L \otimes A \longrightarrow A; \quad l \triangleright a = ((\varepsilon_L \otimes Id) \circ \mu)(l \otimes a),$$

$$\triangleleft : L \otimes A \longrightarrow L; \quad l \triangleleft a = ((Id \otimes \varepsilon_A) \circ \mu)(l \otimes a).$$

☑

We wrote the proof of this last Proposition since it allows us to construct examples such as Example (3.5).

Now, let H be any Hopf algebra with bijective antipode S with composition inverse $\overline{S}$. Let S^* be the bijective antipode of H^* and $\overline{S^*}$ its composition inverse, then H is a right H^{*cop}-module coalgebra via

$$h \leftharpoonup\!\!\leftharpoonup f = \sum \overline{S^*}(f_2) \rightharpoonup h \leftharpoonup f_1$$

and H^* is a left H-module coalgebra via

$$h \rightharpoonup\!\!\rightharpoonup f = \sum h_1 \rightharpoonup f \leftharpoonup \overline{S}(h_2);$$

see [13, Chapter 10] for details on this actions. Define the Drinfel'd double of H, $D(H)$ as the double cross product $H^{*cop} \bowtie H$ with product

$$(f \bowtie h)(g \bowtie k) = \sum f(h_1 \rightharpoonup\!\!\rightharpoonup g_2) \bowtie (h_2 \leftharpoonup\!\!\leftharpoonup g_1)k.$$

The coproduct, counit and antipode are given by

$$\Delta(f \bowtie h) = \sum (f_2 \bowtie h_1) \otimes (f_1 \bowtie h_2),$$

$$\varepsilon_{D(H)}(f \bowtie h) = \varepsilon_{H^*}(f)\varepsilon_H(h)$$

and

$$S_{D(H)}(f \bowtie h) = \sum (S(h_2) \rightharpoonup S(f_1)) \bowtie (f_2 \leftharpoonup S(h_1))$$

respectively.

Since double crossed products of Hopf algebras are both factorization algebras and Hopf algebras Corollary (2.3) becomes:

Proposition 3.4. *Let H and K be a matched pair of Hopf algebras and consider their double crossed product $H \bowtie K$, then the Hopf algebra extension $H \hookrightarrow H \bowtie K$ satisfies*

$$d(H, H \bowtie K) = 2d(K,_H \mathcal{M}) + 1$$

Example 3.5. Recall that two Hopf algebras A and B are said to be paired [11, 1.4.3] if there is a bilinear map

$$A \otimes B \longrightarrow k; a \otimes b \longmapsto \langle a, b\rangle$$

satisfying $\langle ac, b\rangle = \langle a \otimes c, \Delta b\rangle$, $\langle a, 1\rangle = \varepsilon(a)$, $\langle 1, b\rangle = \varepsilon(b)$ and $\langle Sa, b\rangle = \langle a, Sb\rangle$. We also say it is nondegenerate if and only if $\langle a, b\rangle = 0$ for all $b \in B$ implies $a = 0$ and $\langle a, b\rangle = 0$ for all $a \in A$ implies $b = 0$. Assume now that A and B are paired and that $\langle, \rangle$ is convolution invertible, define

$$a \triangleleft b = \sum a_2 \langle a_1, b_1\rangle^{-1} \langle a_3, b_2\rangle.$$

$$a \triangleright b = \sum b_2 \langle a_1, b_1 \rangle^{-1} \langle a_2, b_3 \rangle$$

With this action we can endow $A^{op} \bowtie B$ with a double cross product structure. Consider then H to be a finite dimensional Hopf algebra and

$$\langle , \rangle : H \otimes H \longrightarrow k; h \otimes g \longmapsto \varepsilon(a)\varepsilon(b)$$

then $\langle , \rangle$ satisfies the conditions above, is nondegenerate if and only if H is semisimple via Maschke's theorem and is convolution invertible via $\langle , \rangle \langle , \rangle = \varepsilon$ Then $H^{op} \bowtie H$ is a double cross product isomorphic to the tensor Hopf algebra $H^{op} \otimes H$, Proposition (3.3), and the minimum odd depth satisfies

$$d(H, H^{op} \bowtie H) = 3$$

since $d(H, {}_{H^{op}}\mathcal{M}) = 1$.

Proposition 3.6. *Let H be a finite-dimensional Hopf algebra of dimension $m \geq 2$ and consider $D(H) = H^{*cop} \bowtie H$ its Drinfel'd double. Then the minimum odd depth satisfies:*

$$d(H, D(H)) \geq 3.$$

Proof. The proof is analogous to the one in Example (2.4). ☑

4. Depth two

In this last section we will focus on depth two, for double cross product Hopf subalgebra extensions. Results are motivated by the following example:

Consider a finite group algebra kG and its dual $(kG)^* = k\langle p_x | x \in G \rangle$ where the $\{p_x\}$ form the dual basis of G satisfying $p_x(y) = \delta_{x,y}$ for all $x, y \in G$. This is an algebra via convolution product, and the identity element is $\varepsilon = \sum_{y \in G} p_y$. It is easy to check that $(kG)^*$ has a Hopf algebra structure given by

$$\Delta^* p_x = \sum_{lk=x} p_l \otimes p_k,$$

$$\varepsilon^*(p_x) = \delta_{x,1},$$

and antipode S^*.

Consider then $R = kG$ a finite group algebra and $H = D(kG) = (kG)^{*cop} \bowtie kG$ its Drinfel'd double. Multiplication is given by

$$(p_x \bowtie g)(p_y \bowtie k) = p_x p_{gyg^{-1}} \bowtie gk$$

and the antipode is

$$S(p_x \bowtie g) = (\varepsilon \bowtie g^{-1})(S^* p_x \bowtie e) = S^* p_{g^{-1}xg} \bowtie g^{-1}.$$

Let $p_x = p_x \bowtie e \in (kG)^*$, and $p_y \bowtie g \in H$. The right adjoint action of H on $(kG)^*$ is given by

$$S(p_y \bowtie g)_1(p_x \bowtie e)(p_y \bowtie g)_2 = \sum_{lk=y} S(p_l \bowtie g)(p_x \bowtie e)(p_k \bowtie g).$$

A quick calculation and using the formulas above shows that the latter equals

$$\sum_{lk=y} S^*(p_{g^{-1}lg})p_{g_{-1}xg}p_{g^{-1}kg} \bowtie e.$$

A similar calculation shows that the left adjoint action of H on $(kG)^*$ yields

$$(p_y \bowtie g)_1(p_x \bowtie e)S(p_y \bowtie g)_2 = \sum_{lk=y} p_l p_{gxg^{-1}} p_k \bowtie e$$

and hence, $(kg)^*$ is H left and right ad stable and hence normal. As it is shown in Theorem (1.4) this implies that

$$d((kG)^*, H) \leq 2.$$

We point out that this is true since the left coadjoint action of $(kG)^*$ on kG given by $\leftharpoonup$ is trivial on the generators:

$$g \leftharpoonup p_x = g$$

The following proposition tells us that this is in fact a necessary and sufficient condition for depth 2 in the more general case of double cross products:

Proposition 4.1. *Let A, B be a matched pair of Hopf algebras, and let $H = A \bowtie B$ be their double cross product. Then $d(A,H) \leq 2$ (Equivalently $d(B,H) \leq 2$) if and only if $B \triangleleft A$ (Equivalently $B \triangleright A$) is trivial.*

To prove that $d(A,H) \leq 2$ implies $B \triangleleft A$ is trivial is a standard calculation involving the left and right adjoint action of $A \bowtie B$ on A and assuming A is ad-stable, or in simpler terms, a normal $A \bowtie B$ subalgebra. The converse is standard as well. It is in fact easy to check that if $B \triangleleft A$ is trivial, then A is ad-stable in $A \bowtie B$. The equivalent condition, that $d(B,H) \leq 2$ if and only if $B \triangleright A$ is trivial follows by the symmetry of the argument.

The following are special cases and consequences of this proposition. In particular for the case of $D(kG)$, the Drinfel'd double of a finite group algebra.

Corollary 4.2. *Let G be a finite group and consider $D(kG) = (kG)^{*cop} \bowtie kG$, then*

$$d(kG, D(kG)) \leq 2$$

if and only if G is abelian.

Proof. Let $g, x \in G$. Recall that the left coadjoint action of kG on $(kG)^*$ is given by $g \rightharpoonup p_x = p_{gxg^{-1}}$ which is trivial (i.e $p_{gxg^{-1}} = p_x$ for all $g, x \in G$) if and only if G is abelian. ☑

Example 4.3. Consider $H^{op} \bowtie H$ as in Example (3.5), then the minimum depth satisfies

$$d(H, H^{op} \bowtie H) \geq 2$$

since $h \triangleright g = \sum g_2 \langle h_1, g_1 \rangle^{-1} \langle h_2, g_3 \rangle = g_2 \varepsilon(h_1)\varepsilon(g_1)\varepsilon(h_2)\varepsilon(g_3) = g\varepsilon(h)$ for all $h, g \in H$ and hence $H \triangleright H^{op}$ is trivial.

Now consider the double cross product $H = A \bowtie B$, $Z(A)$, $C_H(A)$ and $N_H(B)$ the center of A, the centralizer of A in H and the normal core of B in H respectively. Then $C_H(A)$ satisfies the following:

Proposition 4.4. *Let $H = A \bowtie B$ be a double cross product such that $d(A, H) \leq 2$. Then*

$$C_H(A) = Z(A) \bowtie N_H(B)$$

as algebras

Proof. Let $f \bowtie k \in C_H(A)$ and $a \bowtie 1_B \in A$. Then $(f \bowtie k)(a \bowtie 1_B) = (a \bowtie 1_B)(f \bowtie k)$. On one hand we have

$$(a \bowtie 1_B)(f \bowtie k) = af_1 \bowtie (1_B \triangleleft f_2)k = af \bowtie k$$

Since depth two implies $A \triangleleft B$ is trivial. On the other hand

$$(f \bowtie k)(a \bowtie 1_B) = f(k_1 \triangleright a) \bowtie k_2$$

Now

$$f(k_1 \triangleright a) \bowtie k_2 = af \bowtie k$$

if and only if $k \triangleright a = \varepsilon(k)a$ and $fa = af$ for all $a \in A$ if and only if $k \in N_H(B)$ and $f \in Z(A)$. ☑

Corollary 4.5. *Let kG be a finite group algebra and consider $H = D(kG)$ its Drinfel'd double. Then*

$$C_H((kG)^*) = Z((kG)^*) \bowtie Z(kG)$$

as algebras.

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