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Sections of the light cone in Minkowski 4-space

Secciones del cono de luz en el espacio de Minkowski 4-dimensional

Antonio de Padua Franco Filho[1],
Anuar Paternina Montalvo[1,✉]

[1]IME, Universidade de São Paulo, São Paulo, Brazil

Abstract. The intersection of an affine hyperplane in $\mathbb{L}^4$ with the light cone $\mathcal{C}$ is called a conic section. In this paper, it is proved that the conic sections in $\mathbb{L}^4$ are either Riemannian spheres, hyperbolic spaces or horospheres, depending on the causal character of the hyperplane. Analogous results for affine sections of de Sitter and hyperbolic spaces in $\mathbb{L}^4$ are also presented at the end.

Key words and phrases. Minkowski 4-space, light cone, conic sections, hyperquadrics.

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Resumen. La intersección de un hiperplano afín en $\mathbb{L}^4$ con el cono de luz $\mathcal{C}$ se llama una sección cónica. En este artículo, probamos que las secciones cónicas de $\mathbb{L}^4$ son esferas de Riemann, espacios hiperbólicos o horoesferas, dependiendo del carácter causal del hiperplano. Al final del artículo presentamos resultados similares para secciones afines de espacios de Sitter y espacios hiperbólicos de $\mathbb{L}^4$.

Palabras y frases clave. Espacio de Minkowski 4-dimensional, cono de luz, secciones cónicas, hipercuádricas.

1. Introduction

If we take an affine hyperplane in $\mathbb{L}^4$ and consider the intersection of this hyperplane with the light cone $\mathcal{C}$, we get a set that is called a conic section. The aim of this paper is to give a formal demonstration that the conic sections in $\mathbb{L}^4$ are either Riemannian spheres, hyperbolic spaces or horospheres, depending on the causal character of the hyperplane. Moreover a corresponding general equation for these affine sections is given.

Conic sections are essential study objects in Lorentzian geometry. There are a wide variety of results in the literature where these objects play an important role. For example, in [4], the author shows that the geometric properties of the intersection of a lightlike hyperplane with a light cone in the Minkowski spacetime are connected with the construction of the trapped surface in [1], and hence provides a better conceptual understanding of the result in [1].

On the other hand, affine sections obtained from the intersection of an affine hyperplane in $\mathbb{L}^4$ with a hyperquadric are also of interest and play essential functions in geometry. For instance, in the study of surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3$, we have that totally umbilical spacelike surfaces in $\mathbb{S}^3_1$ and totally umbilical surfaces in $\mathbb{H}^3$ are contained in these affine sections.

In the next section, we are going to introduce the Minkowski 4-space $\mathbb{L}^4$, the causal character of a vector and a subspace in this space, and the notion of angle between two vectors. Some known facts and properties of space $\mathbb{L}^4$ will also be presented. A more detailed and extensive treatment of the subject, as well as the demonstrations of those results can be found in [3], [2] and [7]. Some references about hyperbolic geometry and horospheres are [5] and [6]. In the last section, we will present our result.

2. The Minkowski 4-space $\mathbb{L}^4$

Let $\mathbb{R}^4$ denote the real vector space with its usual structure. Let $\{e_i : 1 \le i \le 4\}$ the canonical basis of $\mathbb{R}^4$. We denote (x^1, x^2, x^3, x^4) the coordinates of a vector with respect to this basis. We also consider in $\mathbb{R}^4$ its affine structure.

The *Minkowski 4-space* $\mathbb{L}^4$ is the real vector space $\mathbb{R}^4$ endowed with the Lorentz scalar product $\langle \cdot, \cdot \rangle$ defined by the pseudometric

$$ds^2 = -\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2 \tag{2.1}$$

in canonical coordinates and, oriented by $d\mathbb{L}^4 = \left(-dx^1\right) \wedge dx^2 \wedge dx^3 \wedge dx^4$.

A vector $v \in \mathbb{L}^4$ is said to be

(i) spacelike if $\langle v, v \rangle > 0$ or $v = 0$,

(ii) timelike if $\langle v, v \rangle < 0$,

(iii) lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$.

The label spacelike, timelike or lightlike is called the *causal character* of a vector. Moreover, it is said that a timelike or spacelike vector v is *future-directed* if $\langle v, e_1 \rangle < 0$ and *past-directed* if $\langle v, e_1 \rangle > 0$.
The *light cone* of $\mathbb{L}^4$ is the set of all lightlike vectors of $\mathbb{L}^4$:

$$\mathcal{C} = \{(x^1, x^2, x^3, x^4) \in \mathbb{L}^4 : -(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 0\} - \{(0,0,0,0)\}.$$

The set of all timelike vectors in $\mathbb{L}^4$ is

$$\mathcal{T} = \{(x^1, x^2, x^3, x^4) \in \mathbb{L}^4 : -(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 < 0\}.$$

Observe that both $\mathcal{C}$ and $\mathcal{T}$ have two connected components.

Vectors v and w in $\mathbb{L}^4$ are called *orthogonal* if $\langle v, w\rangle = 0$. A set of vectors $\{\epsilon_i : 1 \leq i \leq 4\}$ is an *orthonormal basis* for $\mathbb{L}^4$ if

- $\langle \epsilon_i, \epsilon_j\rangle = 0$ for $i \neq j$,
- $\langle \epsilon_1, \epsilon_1\rangle = -1$,
- $\langle \epsilon_i, \epsilon_i\rangle = 1$ for all $2 \leq i \leq 4$.

Note that the canonical basis of $\mathbb{R}^4$ is an orthonormal basis of $\mathbb{L}^4$ which will also be called canonical. It is said that an orthonormal basis is *positively oriented* or a *Minkowski reference frame* if it is compatible with the orientation $d\mathbb{L}^4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) < 0$ and, in addition, the timelike vector ϵ_1 is future-directed.

The *Lorentzian norm* $\|v\|$ of a vector v of $\mathbb{L}^4$ is defined by $\|v\| = \sqrt{|\langle v, v\rangle|}$. The vector v is called unitary if its norm is 1.

***Remark* 2.1.** An orthonormal basis of $\mathbb{L}^4$ cannot contain lightlike vectors, since $\langle\cdot,\cdot\rangle$ is non-degenerate.

Let V be a vector subspace of $\mathbb{L}^4$, it is said to be

(i) spacelike if $\langle\cdot,\cdot\rangle|_V$ is positive definite,

(ii) timelike if $\langle\cdot,\cdot\rangle|_V$ is non-degenerate and has index 1 (the maximum possible dimension of a negative definite subspace is 1),

(iii) lightlike if $\langle\cdot,\cdot\rangle|_V$ is degenerate.

The *causal character* of a subspace is the property to be spacelike, timelike or lightlike. Note that this definition is consistent with the definition of causal character of a vector in the sense that the causal character of an individual vector v is the same as the causal character of the subspace *span* $\{v\}$ it generates.

***Remark* 2.2.** An unidimensional subspace is usually called a *light ray*. Every subspace of $\mathbb{L}^4$ necessarily has one of the three causal types above. For simplicity, when a subspace is not lightlike, it is said only that it is non-degenerate.

It is said that a set $S \subseteq \mathbb{L}^4$ is *orthogonal* if any two distinct vectors in S are orthogonal. Moreover, if all vectors in S are unitary vectors, then S is called *orthonormal*. We see that an orthonormal basis is an example of an orthonormal set.

For any subset $S \subseteq \mathbb{L}^4$, the *orthogonal space* of S in $\mathbb{L}^4$ is defined by

$$S^\perp = \{w \in \mathbb{L}^4 : \langle v, w\rangle = 0\ \forall v \in S\}.$$

***Remark* 2.3.** The orthogonal space of a subset in $\mathbb{L}^4$ is always a subspace of $\mathbb{L}^4$, even if the subset itself is not. We use the term "orthogonal space" instead of "orthogonal complement" because, in this case, we do not necessarily have that, if $S \subseteq \mathbb{L}^4$ is a subspace, then $\mathbb{L}^4 = S \oplus S^\perp$. A counterexample can be obtained by taking the subspace $V = \{(x^1, x^2, x^3, x^4) \in \mathbb{L}^4 : x^1 = x^3\}$. The orthogonal space to this subspace is $V^\perp = \{(x^1, x^2, x^3, x^4) \in \mathbb{L}^4 : x^1 = x^3\ ,\ x^2 = x^4 = 0\}$.

The notion of causal character for subspaces is naturally related to the notion of orthogonality, as it can be concluded from the following results.

Proposition 2.4. *If V is a subspace of $\mathbb{L}^4$, then*

(i) $\dim V + \dim V^\perp = 4$,

(ii) $(V^\perp)^\perp = V$.

Corollary 2.5. *Let $V \subseteq \mathbb{L}^4$ be a subspace. Then V is non-degenerate if and only if $\mathbb{L}^4 = V \oplus V^\perp$. In particular, V is non-degenerate if and only if $V^\perp$ is also non-degenerate.*

Lemma 2.6. *Let $V \subseteq \mathbb{L}^4$ be a non-degenerate subspace. Then V is timelike (spacelike) if and only if $V^\perp$ is spacelike (timelike). In particular, if v is a timelike (spacelike) vector of $\mathbb{L}^4$, then the subspace span $\{v\}^\perp$ is spacelike (timelike) and $\mathbb{L}^4 =$ span $\{v\} \oplus$ span $\{v\}^\perp$.*

Comparing with Euclidean space $\mathbb{R}^4$, the existence of timelike and lightlike vectors in $\mathbb{L}^4$ gives rise to some "strange" properties, as the following:

Proposition 2.7. *In $\mathbb{L}^4$, we have that:*

(i) Two lightlike vectors are orthogonal if and only if they are proportional.

(ii) A timelike vector cannot be orthogonal to a lightlike vector or to another timelike vector.

(iii) If V is a lightlike subspace, then $\dim(V \cap V^\perp) = 1$.

Now, we consider some criteria for a subspace of dimension ≥ 2 to be timelike.

Proposition 2.8. *Let $V \subseteq \mathbb{L}^4$ be a subspace of dimension ≥ 2. Then, the following conditions are equivalent:*

(i) V is a timelike subspace.

(ii) *V contains two linearly independent lightlike vectors.*

(iii) *V contains a timelike vector.*

Now, we characterize lightlike subspaces.

Proposition 2.9. *Let $V \subseteq \mathbb{L}^4$ be a subspace. The following statements are equivalent:*

(i) *V is a lightlike subspace.*

(ii) *V contains a lightlike vector but not a timelike vector.*

(iii) *$V \cap \mathcal{C} = L - \{0\}$, where L is a one-dimensional subspace.*

2.1. Angles between two vectors

For $u \in \mathcal{T}$, the open set:

$$C(u) = \{v \in \mathcal{T} : \langle u, v \rangle < 0\}$$

is the *timecone* of $\mathbb{L}^4$ containing u. The *opposite* timecone is

$$C(-u) = -C(u) = \{v \in \mathcal{T} : \langle u, v \rangle > 0\}.$$

The set $C(u)$ is non-empty since $u \in C(u)$. Moreover, if v is another timelike vector, by Proposition 2.7 we have that $\langle u, v \rangle < 0$ or $\langle u, v \rangle > 0$. This means that $\mathcal{T}$ is the disjoint union of these two timecones.

Proposition 2.10. *Let $u, v \in \mathbb{L}^4$ be timelike vectors. Then,*

(i) *They are in the same timecone if and only if $\langle u, v \rangle < 0$.*

(ii) *$u \in C(v)$ if and only if $v \in C(u)$.*

A difference that we find between $\mathbb{R}^4$ and $\mathbb{L}^4$ refers to the Cauchy–Schwarz inequality. Recall that if $u, v \in \mathbb{R}^4$, the Cauchy–Schwarz inequality asserts $|\langle u, v \rangle| \leq \|u\|\|v\|$ and the equality holds if and only if u and v are proportional. This inequality permits the definition of angle between two vectors.

In Minkowski space, and for timelike vectors, there exists a “reverse” inequality called *backwards Cauchy–Schwarz inequality.*

Theorem 2.11. *Let u and v be timelike vectors in $\mathbb{L}^4$. Then*

$$|\langle u, v \rangle| \geq \|u\|\|v\|$$

and the equality holds if and only if u and v are proportional. In the case that both vectors lie in the same timelike cone, there exists a unique number $\varphi \geq 0$ such that

$$\langle u, v \rangle = -\|u\|\|v\| \cosh \varphi. \tag{2.2}$$

Since the Cauchy–Schwarz inequality runs backwards in this context, so does the triangle inequality.

Corollary 2.12. *If u and v are timelike vectors in the same timecone, then $\|u+v\| \geq \|u\| + \|v\|$, with equality if and only if u and v are proportional.*

The number $\varphi \geq 0$ of the Theorem 2.11 is called the *hyperbolic angle* between u and v. The angle φ is also called rapidity in more physical literature.

After the definition of angle between two timelike vectors that lie in the same timecone, we ask how to define the angle between any two vectors $u, v \in \mathbb{L}^4$. Assume that u, v are linearly independent and that they are not lightlike. The angle is defined depending on the causal character of the plane (2-dimensional subspace) P spanned by u and v. The induced metric on P can be Riemannian, Lorentzian or degenerate.

Let u and v be spacelike vectors in $\mathbb{L}^4$ that span a spacelike subspace. It is obvious that we have

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

with equality if and only if u and v are proportional. Hence, there exists a unique number $0 \leq \theta \leq \pi$ such that

$$\langle u, v\rangle = \|u\|\|v\| \cos\theta. \tag{2.3}$$

The number θ is called the *Lorentzian spacelike angle* [5, p. 68] between u and v.

Let u and v be spacelike vectors in $\mathbb{L}^4$ that span a timelike subspace. In [5], it was proved that

$$|\langle u, v\rangle| > \|u\|\|v\|$$

and hence, that there exists a unique real number $\theta > 0$ such that

$$\langle u, v\rangle = \|u\|\|v\| \cosh\theta. \tag{2.4}$$

The *Lorentzian timelike angle* [5, p. 69] between spacelike vectors u and v is defined to be θ.

Let u be a spacelike vector and v be a timelike vector in $\mathbb{L}^4$. In [5], it was proved that there exists a unique real number $\theta \geq 0$ such that

$$\langle u, v\rangle = \|u\|\|v\| \sinh\theta. \tag{2.5}$$

The *Lorentzian timelike angle* [5, p. 71] between u and v is defined to be θ.

2.2. Hyperquadrics

Let's consider $q(u) = \langle u, u\rangle$ the associated quadratic form of $\langle\cdot,\cdot\rangle$. Relative to standard coordinates,

$$q(u) = -(u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2.$$

The sets, for $r > 0$ and $\epsilon = \pm 1$, $Q = q^{-1}(\epsilon r^2)$ are called the *(central) hyperquadrics* or *pseudo-spheres* of $\mathbb{L}^4$. The two families $\epsilon = 1$ and $\epsilon = -1$ fill all of $\mathbb{L}^4$ except the set $q^{-1}(0)$, which consists of the light cone $\mathcal{C}$ and the origin 0.

The hyperquadric of $\mathbb{L}^4$ given by

$$\mathbb{S}^3_1(r) = q^{-1}(r^2) = \{x \in \mathbb{L}^4 : \langle x, x\rangle = r^2\}$$

is called *de Sitter space* (with center at the origin and radius r).

The hyperquadric of $\mathbb{L}^4$ given by

$$\mathbb{H}^3_1(r) = q^{-1}(-r^2) = \{x \in \mathbb{L}^4 : \langle x, x\rangle = -r^2\}$$

is called *Anti-de Sitter space* or *hyperbolic space* (with center at the origin and radius r).

For the purposes of this work it is sufficient to consider $r = 1$ in both cases and hence denote $\mathbb{S}^3_1(1)$ by $\mathbb{S}^3_1$ and $\mathbb{H}^3_1(1)$ by $\mathbb{H}^3$.

The translations of theses spaces are also considered:

$$\mathbb{S}^3_1(x_0, r) = \{x \in \mathbb{L}^4 : \langle x - x_0, x - x_0\rangle = r^2\}$$

and

$$\mathbb{H}^3_1(x_0, r) = \{x \in \mathbb{L}^4 : \langle x - x_0, x - x_0\rangle = -r^2\},$$

which are called de Sitter space and hyperbolic space with center at x_0 and radius r, respectively.

***Remark* 2.13.** A linear transformation $A : \mathbb{L}^4 \to \mathbb{L}^4$ is called a Lorentz transformation if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{L}^4$. The fact that a change of reference frame is determined by a Lorentz transformation shows us that the central hyperquadrics remain fixed when taking any reference frame.

In the next section we will study the following algebraic system of equations:

$$\begin{cases} -a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b \\ \quad -x_1^2 + x_2^2 + x_3^2 + x_4^2 = \epsilon r^2, \end{cases} \tag{2.6}$$

for $a_i, b, r \in \mathbb{R}$ with $r > 0$, and $\epsilon \in \{-1, 0, 1\}$, and we will interpret the solutions of this system as a surface in $\mathbb{L}^4$.

The first equation of the system above, can be seen as an equation of an affine hyperplane and the second equation, can be seen as a leaf of the foliation of $\mathbb{L}^4$ given by its central hyperquadrics or pseudo-spheres.

The foliation given by the second equation induces a foliation of the affine hyperplane given by the quadrics of this 3-dimensional subspace.

3. Sections of the light cone

A hyperplane of $\mathbb{L}^4$ is a vector subspace of codimension one in $\mathbb{L}^4$. The translation of a hyperplane is known as an affine hyperplane. We know that there are three mutually exclusive cases for a hyperplane H in $\mathbb{L}^4$: H is a spacelike hyperplane, H is a timelike hyperplane or H is a lightlike hyperplane.

If H is a spacelike (timelike) affine hyperplane in $\mathbb{L}^4$, by Lemma 2.6 there exists a timelike (spacelike) vector τ and a real number c (that can be either positive, negative or zero) such that

$$H = \{x \in \mathbb{L}^4 : \langle x, \tau \rangle = c\}.$$

3.1. Spacelike case

Let us consider a spacelike affine hyperplane that intersects the x^1-axis at the point $p = (a, 0, 0, 0)$, and denote it by $H(p)$. We can assume without loss of generality that $a > 0$. Thus, there exists an unique future-directed unitary timelike vector τ such that the affine hyperplane $H(p)$ has the following equation:

$$x \in H(p) \quad \text{if and only if} \quad \langle x - p, \tau \rangle = 0. \tag{3.1}$$

The solution of the following system of equations:

$$\begin{cases} \langle x, x \rangle = 0 \\ \langle x - p, \tau \rangle = 0 \end{cases} \tag{3.2}$$

is the intersection of the affine hyperplane $H(p)$ with the light cone $\mathcal{C}$.

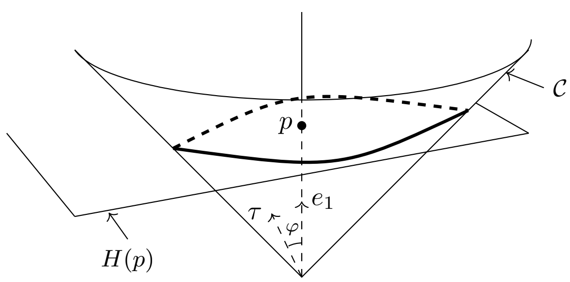

Figure 1. Affine section obtained from the intersection of a spacelike affine hyperplane in $\mathbb{L}^4$ with the like cone $\mathcal{C}$.

To solve the previous system (3.2), we are going to find an appropriate Minkowski reference frame adapted to the timelike vector τ. Let $\cosh \varphi = -\langle e_1, \tau \rangle$ be the hyperbolic angle between the future-directed timelike vectors

$e_1 = (1, 0, 0, 0)$ and τ (see Fig. 1). Firstly, assume that $e_1 \neq \tau$. Take an unitary vector $\overline{e}_4$ of the subspace $span\,\{e_1\}^{\perp}$, and an orthonormal basis $\{\tau, \nu\}$ for the vector subspace $span\,\{e_1, \tau\}$ such that

$$\tau = \cosh\varphi e_1 + \sinh\varphi \overline{e}_4 \quad \text{and} \quad \nu = \sinh\varphi e_1 + \cosh\varphi \overline{e}_4. \tag{3.3}$$

Observe that ν is an unitary vector of $H(p)$.

Now, consider a Minkowski reference frame $\{f_1, f_2, f_3, f_4\}$ of $\mathbb{L}^4$, where $f_1 = \tau$ and $f_4 = \nu$. Thus, we have the following vectorial equation for the hyperplane $H(p)$

$$x(r, s, t) = p + rf_2 + sf_3 + tf_4, \tag{3.4}$$

r,s and t any real numbers.

Hence, if x satisfies the system (3.2), we have that

$$x = p + \overline{r}f_2 + \overline{s}f_3 + \overline{t}f_4 \quad \text{and} \quad \langle x, x\rangle = 0, \tag{3.5}$$

for certain real numbers $\overline{r}, \overline{s}$ and $\overline{t}$.

Therefore, from (3.5), (3.3) and the above considerations, we get

$$\overline{r}^2 + \overline{s}^2 + (\overline{t} - a\sinh\varphi)^2 = a^2\cosh^2\varphi. \tag{3.6}$$

Secondly, assume that $e_1 = \tau$. Then, the following equation is a vectorial equation for the hyperplane $H(p)$

$$x(r, s, t) = p + re_2 + se_3 + te_4, \tag{3.7}$$

r, s, and t any real numbers.

Thus, if x satisfies the system (3.2), we have that

$$x = p + \overline{r}e_2 + \overline{s}e_3 + \overline{t}e_4 \quad \text{and} \quad \langle x, x\rangle = 0, \tag{3.8}$$

for certain real numbers $\overline{r}, \overline{s}$ and $\overline{t}$.
Therefore, from (3.8), we obtain

$$\overline{r}^2 + \overline{s}^2 + \overline{t}^2 = a^2. \tag{3.9}$$

It follows from the above, the next result:

Proposition 3.1. *The section $H(p) \cap \mathcal{C}$ in $\mathbb{L}^4$ is a 2-dimensional Riemannian sphere. More precisely, with the above notations, we have that*

$$H(p) \cap \mathcal{C} = S^2(x_0, \rho), \tag{3.10}$$

where $x_0 = a\cosh\varphi f_1$, $\rho = a\cosh\varphi$ and $S^2(x_0, \rho)$ is the sphere in $H(p)$ with center at x_0 and radius ρ.

3.2. Timelike case

Now, let us consider a timelike affine hyperplane. This hyperplane intersects some spatial axis, let us suppose that it cuts the x^4-axis. Let $p = (0, 0, 0, a)$ be the point of intersection between the hyperplane and the x^4-axis, and denote the hyperplane by $H(p)$. We can assume without loss of generality that $a > 0$. Thus, there exists an unitary spacelike vector ν such that the affine hyperplane $H(p)$ has the following equation:

$$x \in H(p) \quad \text{if and only if} \quad \langle x - p, \nu\rangle = 0. \tag{3.11}$$

We want to solve the following system of equations:

$$\begin{cases} \langle x, x\rangle = 0, \\ \langle x - p, \nu\rangle = 0, \end{cases} \tag{3.12}$$

i.e., to find the intersection of the timelike affine hyperplane $H(p)$ with the light cone $\mathcal{C}$.

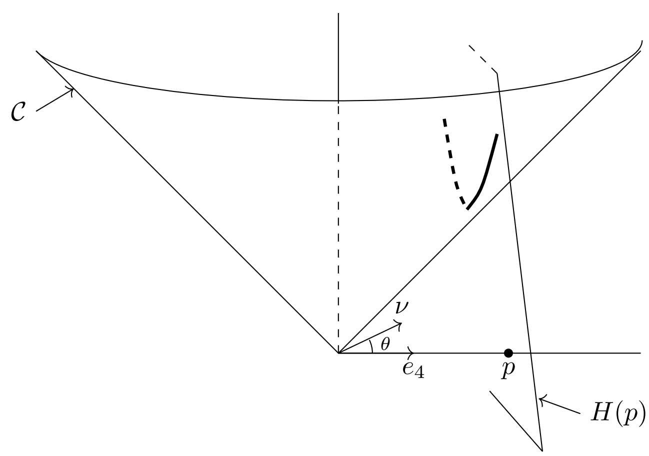

FIGURE 2. Affine section obtained from the intersection of a timelike affine hyperplane in $\mathbb{L}^4$ with the like cone $\mathcal{C}$.

To solve the system (3.12), we are going to find a very good Minkowski reference frame adapted to the spacelike vector ν. Firstly, assume that $e_4 \neq \nu$. Consider the plane *span* $\{e_4, \nu\}$, this plane can be spacelike or timelike.

If *span* $\{e_4, \nu\}$ is a spacelike plane, then take $\cos\theta = \langle e_4, \nu\rangle$ the Lorentzian spacelike angle between e_4 and the spacelike vector ν (see Fig. 2). Take an unitary spacelike vector $\overline{e}_2$ of the subspace *span* $\{e_4\}^{\perp}$, and an orthonormal basis $\{v, \nu\}$ for the vectorial subspace *span* $\{e_4, \nu\}$ such that

$$v = \cos\theta \overline{e}_2 - \sin\theta e_4 \quad \text{and} \quad \nu = \sin\theta \overline{e}_2 + \cos\theta e_4. \tag{3.13}$$

Now, consider a Minkowski reference frame $\{f_1, f_2, f_3, f_4\}$ of $\mathbb{L}^4$, where f_1 is a future-directed unitary timelike vector, $f_2 = v$ and $f_4 = \nu$. Then, we obtain the following vectorial equation for the hyperplane $H(p)$

$$x(r, s, t) = p + rf_1 + sf_2 + tf_3, \tag{3.14}$$

r,s and t any real numbers.

Therefore, if x satisfies the system (3.12), we have that

$$x = p + \overline{r}f_1 + \overline{s}f_2 + \overline{t}f_3 \quad \text{and} \quad \langle x, x\rangle = 0, \tag{3.15}$$

for certain real numbers $\overline{r}, \overline{s}$ and $\overline{t}$.

Thus, from (3.15), (3.13) and the above considerations, we get

$$-\overline{r}^2 + (\overline{s} - a\sin\theta)^2 + \overline{t}^2 = -a^2\cos^2\theta. \tag{3.16}$$

On the other hand, if *span* $\{e_4, \nu\}$ is a timelike plane, then take $\cosh\psi = \langle e_4, \nu\rangle$ the Lorentzian timelike angle between e_4 and the spacelike vector ν. Take an unitary timelike vector $\overline{e}_1$ of the subspace *span* $\{e_4\}^{\perp}$, and an orthonormal basis $\{\tau, \nu\}$ for the vectorial subspace *span* $\{e_4, \nu\}$ such that

$$\tau = \cosh\psi \overline{e}_1 + \sinh\psi e_4 \quad \text{and} \quad \nu = \sinh\psi \overline{e}_1 + \cosh\psi e_4. \tag{3.17}$$

Consider a Minkowski reference frame $\{f_1, f_2, f_3, f_4\}$ of $\mathbb{L}^4$, where $f_1 = \tau$ and $f_4 = \nu$. Thus, a vectorial equation for the hyperplane $H(p)$ is

$$x(r, s, t) = p + rf_1 + sf_2 + tf_3, \tag{3.18}$$

r,s and t any real numbers.

Hence, if x satisfies the system (3.12), we have that

$$x = p + \overline{r}f_1 + \overline{s}f_2 + \overline{t}f_3 \quad \text{and} \quad \langle x, x\rangle = 0, \tag{3.19}$$

for certain real numbers $\overline{r}, \overline{s}$ and $\overline{t}$.

Therefore, from (3.19), (3.17) and the above considerations, we get

$$-(\overline{r} - a\sinh\psi)^2 + \overline{s}^2 + \overline{t}^2 = -a^2\cosh^2\psi. \tag{3.20}$$

Secondly, assume that $e_4 = \nu$. Then, the following equation is a vectorial equation for the hyperplane $H(p)$:

$$x(r, s, t) = p + re_1 + se_2 + te_3, \tag{3.21}$$

r, s, and t any real numbers.

Thus, if x satisfies the system (3.12), we have that

$$x = p + \overline{r}e_1 + \overline{s}e_2 + \overline{t}e_3 \quad \text{and} \quad \langle x, x\rangle = 0, \tag{3.22}$$

for certain real numbers $\overline{r}, \overline{s}$ and $\overline{t}$.

Therefore, from (3.22), we obtain

$$-\overline{r}^2 + \overline{s}^2 + \overline{t}^2 = -a^2. \tag{3.23}$$

It follows from the above, the next result:

Proposition 3.2. *The section $H(p) \cap \mathcal{C}$ in $\mathbb{L}^4$ is a 2-dimensional hyperbolic space. More precisely, with the above notations, we have that*

$$H(p) \cap \mathcal{C} = \mathbb{H}^2_1(x_0, \rho), \tag{3.24}$$

where $x_0 = a\cos\theta f_4$ and $\rho = a\cos\theta$, or $x_0 = a\cosh\psi f_4$ and $\rho = a\cosh\psi$, and $\mathbb{H}^2_1(x_0, \rho)$ is the hyperbolic space in $H(p)$ with center at x_0 and radius ρ.

3.3. Lightlike case

At last, let us consider a lightlike affine hyperplane. We can assume without loss of generality that this hyperplane intersects the upper part of the light cone. Let $p = (a, p^2, p^3, p^4)$ be the point of intersection between this hyperplane and the light cone $\mathcal{C}$ such that

$$a = \min\{x^1 > 0 : (x^1, x^2, x^3, x^4) \text{ is at the intersection of the hyperplane with the light cone } \mathcal{C}\}.$$

Denote the hyperplane by $H(p)$. We have that there exists a lightlike vector υ in the upper part of the light cone such that the affine hyperplane $H(p)$ has the following equation:

$$x \in H(p) \quad \text{if and only if} \quad \langle x - p, \upsilon\rangle = 0. \tag{3.25}$$

We want to solve the following system of equations:

$$\begin{cases} \langle x, x\rangle = 0, \\ \langle x - p, \upsilon\rangle = 0, \end{cases} \tag{3.26}$$

and so, to find the intersection of the lightlike affine hyperplane $H(p)$ with the light cone $\mathcal{C}$.

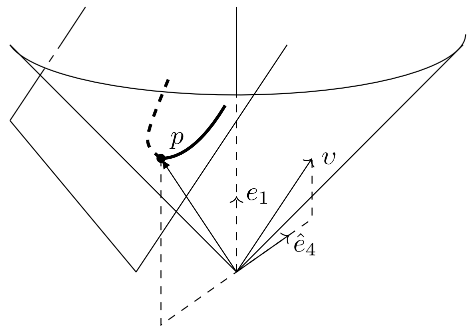

FIGURE 3. Affine section obtained from the intersection of a lightlike affine hyperplane in $\mathbb{L}^4$ with the like cone $\mathcal{C}$.

Consider $T_\upsilon\mathcal{C}$ the lightlike tangent hyperplane to $\mathcal{C}$ at the point υ. Now, take a basis $\{\overline{e}_2, \overline{e}_3, \upsilon\}$ of $T_\upsilon\mathcal{C}$ in such a way that $\{\overline{e}_2, \overline{e}_3\}$ is an orthonormal basis of the plane $T_\upsilon\mathcal{C} \cap span\{e_1\}^\perp$, and for the positive real number a

$$\upsilon = ae_1 + a\hat{e}_4 \quad \text{and} \quad p = ae_1 - a\hat{e}_4, \tag{3.27}$$

where $\hat{e}_4 = \overline{e}_2 \wedge \overline{e}_3$ and "$\wedge$" is the vector product in $span\{e_1\}^\perp$ (see Fig. 3). Observe that $\{e_1, \overline{e}_2, \overline{e}_3, \hat{e}_4\}$ is a Minkowski reference frame of $\mathbb{L}^4$.

We have that, a vectorial equation for the hyperplane $H(p)$ is

$$x(r, s, t) = p + r\overline{e}_2 + s\overline{e}_3 + t\upsilon, \tag{3.28}$$

for any real numbers r,s and t.

Hence, if x satisfies the system (3.26), we have that

$$x = p + \tilde{r}\overline{e}_2 + \tilde{s}\overline{e}_3 + \tilde{t}\upsilon \quad \text{and} \quad \langle x, x\rangle = 0, \tag{3.29}$$

for certain real numbers $\tilde{r}$, $\tilde{s}$ and $\tilde{t}$.

Therefore, from (3.29), (3.27) and the above considerations, we get

$$\tilde{r}^2 + \tilde{s}^2 = 4a^2\tilde{t}. \tag{3.30}$$

It follows from the above, the next result:

Proposition 3.3. *The section $H(p) \cap \mathcal{C}$ in $\mathbb{L}^4$ is a horosphere. More precisely, with the above notations, we have that $H(p) \cap \mathcal{C}$ is the horosphere*

$$\tilde{r}^2 + \tilde{s}^2 = 4a^2\tilde{t}. \tag{3.31}$$

(See Corollary 4.5 and Corollary 4.6).

4. An application to hyperbolic geometry

Let $\{\epsilon_i : 1 \leq i \leq 4\}$ be a Minkowski reference frame of $\mathbb{L}^4$. We denote the coordinates of a vector with respect to this basis by (t, x, y, z).

Consider $\mathbb{H}^3_+ = \left\{(t, x, y, z) : -t^2 + x^2 + y^2 + z^2 = -1\ ,\ t > 0\right\}$ the upper part of the hyperbolic space, and $\mathbb{B}^3 = \left\{(0, u, v, w) : u^2 + v^2 + w^2 < 1\right\}$ the Euclidean open ball contained in the slice $\{0\} \times \mathbb{R}^3$ of the space $\mathbb{L}^4$.

The hyperbolic stereographic projection. We fix the point $S = (-1, 0, 0, 0)$ (south pole of $\mathbb{H}^3$), and for each point $A = (0, u, v, w) \in \mathbb{B}^3$ we consider the equation

$$P(\lambda) = S + \lambda(A - S) = (-1, 0, 0, 0) + \lambda(1, u, v, w) \quad \text{for each} \quad \lambda > 0.$$

If $\langle P, P\rangle = -(\lambda - 1)^2 + \lambda^2(u^2 + v^2 + w^2) = -1$ and $\lambda > 0$, then

$$\lambda = \frac{2}{1 - u^2 - v^2 - w^2}, \tag{4.1}$$

which implies that $P(\lambda) = F(u, v, w)$ is given by

$$F(u, v, w) = \frac{1}{1 - u^2 - v^2 - w^2}(1 + u^2 + v^2 + w^2, 2u, 2v, 2w). \tag{4.2}$$

The inverse of the function $F : \mathbb{B}^3 \to \mathbb{H}^3_+$ it will denote by $\mathrm{St}(t, x, y, z) = (0, u, v, w)$, and can be obtained from the equation

$$Q(\psi) = S + \psi(Y - S) \quad \text{for each } \psi > 0, \text{ where } Y = (t, x, y, z).$$

Thus, we have that $\mathrm{St}(t, x, y, z) = A = (0, u, v, w)$ for $\psi = \dfrac{1}{1+t}$, hence:

$$\mathrm{St}(t, x, y, z) = \left(0, \frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right). \tag{4.3}$$

St is called the stereographic projection from $\mathbb{H}^3_+$ onto the Euclidean ball $\mathbb{B}^3$.

It can be observed that the straight line segment given by $0 \leq \psi \leq 1$ cuts the future directed light cone $\mathcal{C}^+$ at a unique point given by $\langle Q(\psi), Q(\psi)\rangle = 0$, since the light cone is the asymptotic hyperquadric in the set of all central hyperquadrics of $\mathbb{L}^4$.

Proposition 4.1. *For each geometric figure $\mathcal{F}$ contained in the open ball $\mathbb{B}^3$ there exists an unique figure $\mathcal{F}_{\mathcal{C}^+}$ in the light cone $\mathcal{C}$ and an unique figure $\mathcal{F}_{\mathbb{H}^3_+}$ in the hyperbolic space $\mathbb{H}^3$ determined by the stereographic projection.*

4.1. Horospheres

To define the horospheres and their parametrization in this context we start by introducing the *conformal ball model* of the hyperbolic 3-space as follows.

Definition 4.2. The line element given by the parametrization in equation (4.1) of the upper part of the hyperbolic space $\mathbb{H}^3_+$ is

$$ds^2(F) = \frac{4}{(1-u^2-v^2-w^2)^2}(du^2+dv^2+dw^2). \tag{4.4}$$

The Poincaré ball model is the ball $\mathbb{B}^3$ equipped with the line element determined by equation (4.4).

Since this metric is conformal it preserves Euclidean angles. The Euclidean spheres are also hyperbolic spheres.

Let A be a point on a hyperbolic sphere S of $\mathbb{B}^3$, and let R be the geodesic ray of $\mathbb{B}^3$ starting at A and passing through the center C of S. If we expand S by moving C away from A on R at a constant rate while keeping A on S, the sphere tends to a limiting hypersurface Σ in $\mathbb{B}^3$ containing A. By moving A to 0, we see that Σ is a Euclidean sphere minus the ideal endpoint B of R and that the Euclidean sphere $\overline{\Sigma}$ is tangent to $\partial\mathbb{B}^3$ at B.

Definition 4.3. A horosphere Σ of $\mathbb{B}^3$, based at a point B of $\partial\mathbb{B}^3$, is the intersection with $\mathbb{B}^3$ of a Euclidean sphere in $\overline{\mathbb{B}^3}$ tangent to $\partial\mathbb{B}^3$ at B.

Parametrization of a horosphere. We take the following family of parametric surfaces in $\mathbb{L}^4$:

$$X(x,y) = (\alpha,0,0,\beta) + \left(\frac{x^2+y^2}{2m}, x, y, \frac{x^2+y^2}{2m}\right),$$

defined for each $(x,y)\in\mathbb{R}^2$ and constants $m,\alpha,\beta\in\mathbb{R}$.

Since the tangent vectors $\dfrac{\partial X}{\partial x}$ and $\dfrac{\partial X}{\partial y}$ are unitary spacelike and orthogonal to each other, these surfaces are conformal to the Euclidean plane $\mathbb{R}^2$. The first quadratic form is given by

$$ds^2(X) = dx^2 + dy^2.$$

As a consequence, the Gauss curvature $K(X)(x,y)=0$ for each $(x,y)\in\mathbb{R}^2$. Furthermore, the mean curvature vectors of each of these surfaces are lightlike vectors

$$H(X)(x,y) = \frac{1}{m}(\epsilon_1 + \boldsymbol{n}).$$

Now, we have that

$$\langle X(x,y), X(x,y)\rangle = -\alpha^2+\beta^2+x^2+y^2+\frac{x^2+y^2}{m}(-\alpha+\beta) = -1$$

if and only if $\alpha^2-\beta^2=1$ and $\beta-\alpha=-m$. Solving these equations, we obtain

$$\alpha = \frac{1+m^2}{2m} \quad \text{and} \quad \beta = \frac{1-m^2}{2m}.$$

We have shown the following result:

Proposition 4.4. *The parametric surface given by*

$$X(x,y) = \left(\frac{x^2+y^2+m^2+1}{2m}, x, y, \frac{x^2+y^2-m^2+1}{2m} \right),$$

is a surface contained in the hyperbolic space $\mathbb{H}^3$ and in the lightlike affine hyperplane with lightlike direction $\epsilon_1 + \boldsymbol{n}$, passing through the point $V = X(0,0) = \big((m^2+1)/2m, 0, 0, (1-m^2)/2m\big)$.

Corollary 4.5. *Using the stereographic projection, we obtain the following parametric surface in the Euclidean ball $\mathbb{B}^3$ of $\{0\} \times \mathbb{R}^3$:*

$$Y(x,y) = \left(0, \frac{2mx}{x^2+y^2+(m+1)^2}, \frac{2my}{x^2+y^2+(m+1)^2}, \frac{x^2+y^2-m^2+1}{x^2+y^2+(m+1)^2} \right), \tag{4.5}$$

which is a horosphere passing through the point $A = (0,0,0,(1-m)/(1+m))$, with center at $C = (0,0,0,1/(1+m))$, radius $m/(1+m)$ and point of tangency $P = (0,0,0,1) \in \partial\mathbb{B}^3$.

Proposition 4.6. *The surfaces given by*

$$X(x,y) = (\alpha, 0, 0, \beta) + \left(\frac{x^2+y^2}{2m}, x, y, \frac{x^2+y^2}{2m} \right),$$

defined for each $(x,y) \in \mathbb{R}^2$ and constants $m, \alpha, \beta \in \mathbb{R}$, are congruent by a translation to the horosphere $F(Y(\mathbb{R}^2)) \subset \mathbb{H}^3$ where $Y(x,y)$ is given by equation (4.5).

5. Sections of $\mathbb{S}^3_1$ and $\mathbb{H}^3$

In like manner as considering a conic section we can also consider an affine section obtained from the intersection of an affine hyperplane in $\mathbb{L}^4$ with the de Sitter space $\mathbb{S}^3_1$, as well as one obtained by cutting the hyperbolic space $\mathbb{H}^3$ with an affine hyperplane in $\mathbb{L}^4$.

We have the following results:

Proposition 5.1. *(i) If $H(p)$ is a spacelike affine hyperplane, the section $H(p) \cap \mathbb{S}^3_1$ in $\mathbb{L}^4$ is a 2-dimensional Riemannian sphere. More precisely, with the above notations, we have that*

$$H(p) \cap \mathbb{S}^3_1 = S^2(x_0, \rho), \tag{5.1}$$

where $x_0 = a \cosh\varphi f_1$, $\rho = \sqrt{a^2\cosh^2\varphi + 1}$ and $S^2(x_0,\rho)$ is the sphere in $H(p)$ with center at x_0 and radius ρ.

(ii) If $H(p)$ is a timelike affine hyperplane, with the above notations, we have that:

If $1 < a\cos\theta$ *or* $1 < a\cosh\psi$*, then* $H(p)\cap\mathbb{S}^3_1$ *is the two-sheeted hyperbolic space with center at* $x_0 = a\cos\theta f_4$ *and radius* $\rho = \sqrt{a^2\cos^2\theta - 1}$*, or with center at* $x_0 = a\cosh\psi f_4$ *and radius* $\rho = \sqrt{a^2\cosh^2\psi - 1}$*.*

If $1 = a\cos\theta$ *or* $1 = a\cosh\psi$*, then* $H(p)\cap\mathbb{S}^3_1$ *is the two-sheeted light cone with center at* $x_0 = a\cos\theta f_4$ *or* $x_0 = a\cosh\psi f_4$*.*

If $1 > a\cos\theta$ *or* $1 > a\cosh\psi$*, then* $H(p)\cap\mathbb{S}^3_1$ *is the 2-dimensional de Sitter space with center at* $x_0 = a\cos\theta f_4$ *and radius* $\rho = \sqrt{1 - a^2\cos^2\theta}$*, or with center at* $x_0 = a\cosh\psi f_4$ *and radius* $\rho = \sqrt{1 - a^2\cosh^2\psi}$*.*

(iii) If $H(p)$ *is a lightlike affine hyperplane, the section* $H(p)\cap\mathbb{S}^3_1$ *in* $\mathbb{L}^4$ *is a horosphere. More precisely, with the above notations,* $H(p)\cap\mathbb{S}^3_1$ *is the horosphere*

$$\tilde{r}^2 + \tilde{s}^2 = 4a^2\tilde{t} + 1. \tag{5.2}$$

(See Corollary 4.5 and Corollary 4.6).

Proposition 5.2. *(i) If* $H(p)$ *is a spacelike affine hyperplane, the section* $H(p)\cap\mathbb{H}^3$ *in* $\mathbb{L}^4$ *is a 2-dimensional Riemannian sphere. More precisely, with the above notations, if* $a\cosh\varphi > 1$ *then we have that*

$$H(p)\cap\mathbb{H}^3 = S^2(x_0,\rho), \tag{5.3}$$

where $x_0 = a\cosh\varphi f_1$*,* $\rho = \sqrt{a^2\cosh^2\varphi - 1}$ *and* $S^2(x_0,\rho)$ *is the sphere in* $H(p)$ *with center at* x_0 *and radius* ρ*.*

(ii) If $H(p)$ *is a timelike affine hyperplane, the section* $H(p)\cap\mathbb{H}^3$ *in* $\mathbb{L}^4$ *is a two-sheeted hyperbolic plane. More precisely, with the above notations, we have that* $H(p)\cap\mathbb{H}^3$ *is a two-sheeted hyperbolic plane with center at* $x_0 = a\cos\theta f_4$ *and radius* $\rho = \sqrt{a^2\cos^2\theta + 1}$*, or with center at* $x_0 = a\cosh\psi f_4$ *and radius* $\rho = \sqrt{a^2\cosh^2\psi + 1}$*.*

(iii) If $H(p)$ *is a lightlike affine hyperplane, the section* $H(p)\cap\mathbb{H}^3$ *in* $\mathbb{L}^4$ *is a horosphere. More precisely, with the above notations,* $H(p)\cap\mathbb{H}^3$ *is the horosphere*

$$\tilde{r}^2 + \tilde{s}^2 = 4a^2\tilde{t} - 1, \tag{5.4}$$

with center at $\dfrac{1}{1+2a}\hat{e}_4$*, radius* $\dfrac{2a}{1+2a}$ *and point of tangency* $\hat{e}_4$ *(see Corollary 4.5).*

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Departamento de Matemática
IME, Universidade de São Paulo
Caixa Postal 66281, CEP 05311-970, São Paulo, Brazil
e-mail: apadua@ime.usp.br
e-mail: anuarep@ime.usp.br