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On cusps of hyperbolic once-punctured torus bundles over the circle

Acerca de cúspides de haces fibrados hiperbólicos sobre el círculo con fibra el toro con un agujero

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Abstract. The geometry of certain canonical triangulation of once-punctured torus bundles over the circle is applied to the problem of computing their cusp tori. We are also concerned with the problem of finding the limit points of the set formed by such cusp tori, inside the moduli space of the torus. Our discussion generalizes examples which were elaborated by H. Helling (unpublished) and F. Guéritaud.

Key words and phrases. Kleinian group, cusp torus.

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Resumen. Se aplica la geometría de cierta triangulación canónica de haces sobre el círculo con fibra el toro con un agujero al problema de calcular sus toros cuspidales. También se ataca el problema de hallar los puntos límite del conjunto que forman tales toros cuspidales, dentro del espacio moduli de toros. Nuestro método generaliza ejemplos que fueron trabajados por H. Helling (sin publicar) y F. Guéritaud.

Palabras y frases clave. Grupo Kleiniano, toro cuspidal.

1. Introduction

It is well-known that any hyperbolic once-punctured torus bundle over the circle is determined by its monodromy, which is a product $R^{n_1}L^{m_1}\cdots R^{n_k}L^{m_k}$, with $n_j, m_j \in \mathbb{N}$ for $j = 1, \ldots, k$, where L and R represent Dehn twists around two simple closed curves generating the homotopy group of the once-punctured torus. Helling [5] (cf. [2, Section 8]) and Guéritaud [4, Section 10] gave geometric information about the cusp torus when the monodromies are R^nL and R^nL^m, respectively. By letting n tends to ∞, Helling found the point in the moduli space of the torus representing the limit of cusp tori of bundles with monodromy R^nL. Similarly, Guéritaud found the limit point resulting from letting both n and m tend to ∞. In this paper, we are able to find with computer assistance the limit point of cusp tori of bundles with monodromy R^nL^m, for an arbitrary fixed n as m tends to ∞. A question we answer is: how do cusp tori of bundles with monodromy R^nL^m look in the moduli space of the torus? In particular, we prove the following result.

Theorem. *The cusp torus corresponding to monodromy R^nL^m tends to a torus of the form $\mathbb{C}/\langle t_2, t_{i/\cosh(\ell/2)}\rangle$ as m tends to infinity, where ℓ satisfies*

$$\sinh^2\frac{\ell}{2}\left(\frac{\cosh(n\ell/2)-1}{\cosh(n\ell/2)+1}\right) = -1$$

and t_γ denotes the translation $z \mapsto z+\gamma$ for any $\gamma \in \mathbb{C}$.

The paper is organized as follows. In §2 we review a canonical decomposition of the hyperbolic once-punctured torus bundle by hyperbolic ideal tetrahedra, mainly attributed to Jørgensen [6]. In §3 we study the cusp torus in the case of monodromy R^nL^m, following the insight provided by the special cases treated in [5, 4]. The set of cusp tori of bundles with monodromy R^nL^m and their limit points are illustrated in §4. The above Theorem is verified also in §4 (Theorem 4.1). We state in §5 some equations related to the case of arbitrary monodromy.

2. Background: A canonical hyperbolic three-dimensional triangulation

We briefly review in this section some aspects of a canonical tetrahedral decomposition of hyperbolic once-punctured torus bundles over the circle.

Conventions:

- The translation $z \mapsto z+\gamma$ in $\mathbb{C}$ will be denoted by t_γ for any complex number $\gamma \in \mathbb{C}$.
- A Euclidean triangle in $\mathbb{C}$ with vertices z_1, z_2 and z_3 will be denoted by (z_1, z_2, z_3).
- For $n = 1, 2, 3$, an ideal hyperbolic n-simplex in the hyperbolic space $\mathbb{H}^3$ with vertices $z_1, \ldots, z_n$ and z_{n+1} in $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ will be denoted by $(z_1, \ldots, z_{n+1})_\infty$.
- The group generated by the Möbius maps $A_1, \ldots, A_n$ will be denoted by $\langle A_1, \ldots, A_n\rangle$.

- We will fix two positive integers m and n, except in Section 4, where m tends to ∞.

Let M_ϕ be the once punctured torus bundle over the circle with monodromy $\phi = R^n L^m$. By virtue of a theorem of Thurston [15] M_ϕ admits a hyperbolic structure (see also [10, 12]), which is unique by Mostow-Prasad Rigidity (e.g. [8, Section 3.13]). Therefore M_ϕ can be viewed as $\mathbb{H}^3/G$ where G is a Kleinian group generated by two loxodromic maps A and B and a translation t_τ, which is a parabolic isometry leaving invariant the horospheres about ∞.

Following Jørgensen [6], we suppose G normalized so that the subgroup $\langle A, B\rangle$ is a punctured torus group with commutator $B^{-1}A^{-1}BA = t_2$. Then there exists a fiber-preserving hyperelliptic involution h (i.e. $h^{-1} = h$) such that

$$hAh = A^{-1}, \quad hBh = B^{-1} \quad \text{and} \quad B^{-1}A^{-1}h = t_1. \tag{1}$$

It follows from (1) that $ht_\tau h = ABt_\tau B^{-1}A^{-1}$ holds, therefore h belongs to the normalizer of G and induces an isometry of M_ϕ.

There is a tetrahedral decomposition of the geometric structure of M_ϕ into ideal hyperbolic tetrahedra, which is unique if it is required to be invariant under the involution h [7]. We refer to it as the canonical tetrahedral decomposition.

The canonical tetrahedral decomposition is also dual to the Ford domain of G (e.g. [4, 13, 14, 1]). In other words, each face of the Ford domain in $\mathbb{H}^3$ is a portion of a geodesic plane which is a Euclidean hemisphere with center in $\mathbb{C}$, considered as the boundary at infinity of $\mathbb{H}^3$, and the geodesics running from infinity down to the centers of these hemispheres are edges of the canonical tetrahedral decomposition. Recall that the centers of these hemispheres involved in the Ford domain are poles of generators of the punctured-torus group $\langle A, B\rangle$.

Since the difference between the pole of a generator and the pole of its inverse is equal to ± 1, (see the sentence which follows Equation 2.1 of [6]), we can suppose, after conjugation by a translation and interchanging A with A^{-1} or B with B^{-1} if necessary, that

$$A(\infty) = 1/2, \quad A^{-1}(\infty) = -1/2, \quad B(\infty) = Q_n \text{ and } B^{-1}(\infty) = Q_n + 1 \tag{2}$$

for some $Q_n \in \mathbb{C}$. Define the points $Q_j = A^{j-n}B(\infty)$ for $j = 0, \dots, n+1$. Then the $2n$ ideal tetrahedra $(\infty, -1/2, Q_{j-1}, Q_j)_\infty$ and $(\infty, 1/2, Q_j, Q_{j+1})_\infty$, with $j = 1, \dots, n$, belong to the lifting of the canonical decomposition of M_ϕ under the quotient projection $\mathbb{H}^3 \to M_\phi$ (see e.g. [13, Section 4]). Since A maps the tetrahedron $(\infty, -1/2, Q_{j-1}, Q_j)_\infty$ to $(1/2, \infty, Q_j, Q_{j+1})_\infty$, both tetrahedra are projected onto the same ideal tetrahedron in the quotient M_ϕ for each $j \in \{1, \dots, n\}$. This and the well-known fact that opposite edges of an ideal hyperbolic tetrahedron have the same dihedral angle imply the following.

Fact 1. $(-1/2, Q_{j-1}, Q_j)$ and $(1/2, Q_{j+1}, Q_j)$ are similar Euclidean triangles for each $j \in \{1, \dots, n\}$.

Likewise, define the points $P_j^* = B^j A(\infty)$ for $j = 0, \dots, m+1$, then the $2m$ ideal tetrahedra $(\infty, Q_n+1, P_{j-1}^*, P_j^*)_\infty$ and $(\infty, Q_n, P_j^*, P_{j+1}^*)_\infty$, with $j = 1, \dots, m$, belong to the lifting of the canonical decomposition of M_ϕ under the quotient projection $\mathbb{H}^3 \to M_\phi$. Since B maps $(\infty, Q_n+1, P_{j-1}^*, P_j^*)_\infty$ to $(\infty, Q_n, P_j^*, P_{j+1}^*)_\infty$, the same arguments for the points Q_j imply the following.

Fact 2. $(Q_n+1, P_{j-1}^*, P_j^*)$ and (Q_n, P_{j+1}^*, P_j^*) are similar Euclidean triangles for each $j \in \{1, \dots, m\}$.

An immediate consequence of the Jørgensen's normalization $BA = ABt_2$ and the definition of P_j^* is the following.

Fact 3. $P_0^* = 1/2$ and $P_1^* = Q_{n+1}$ (hence the tetrahedra $(\infty, 1/2, Q_n, Q_{n+1})_\infty$ and $(\infty, P_0^*, Q_n, P_1^*)_\infty$ are the same).

See Fig. 1.

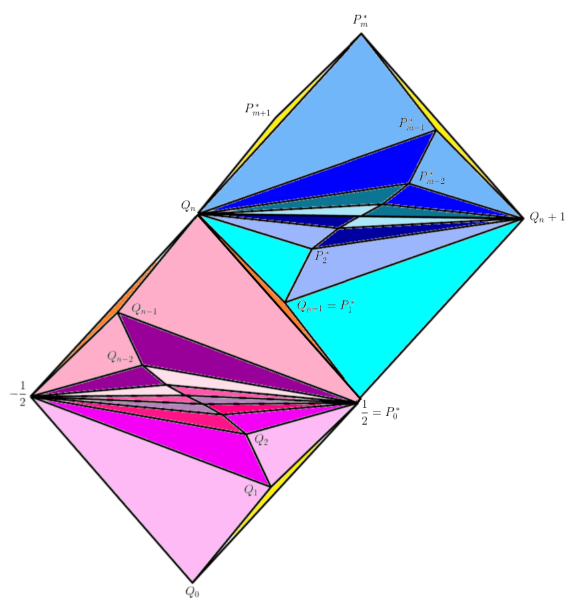

FIGURE 1. Points $(Q_j)_{j=0}^{n+1}$ and $(P_j^*)_{j=0}^{m+1}$ satisfying Facts 1, 2 and 3. Similar triangles are shown with the same color.

The fibers of the punctured torus bundle are easy to visualize in the canonical tetrahedral decomposition. For instance, let $\mathfrak{P}_0$ be the pleated quadrilateral formed by the union of the ideal triangles $(\infty, Q_n, Q_{n+1})_\infty$ and $(\infty, 1/2, Q_{n+1})_\infty$. The edges $(Q_n, \infty)_\infty$ and $(Q_{n+1}, 1/2)_\infty$ of $\mathfrak{P}_0$ are glued by A, and the edges $(1/2, \infty)_\infty$ and $(Q_{n+1}, Q_n)_\infty$ are glued by B. This is verified by using Eq. (2) and Fact 3. Thus $\mathfrak{P}_0$ is projected onto an embedded punctured torus in M_ϕ which can be viewed as fiber of the punctured torus bundle. It is useful to consider other two pleated quadrilaterals $\mathfrak{P}_- = (\infty, Q_0, Q_1)_\infty \cup (\infty, 1/2, Q_1)_\infty$ and $\mathfrak{P}_+ = (\infty, Q_n, P^*_{m+1})_\infty \cup (\infty, P^*_m, P^*_{m+1})_\infty$. Identify each pair of their opposite edges by the following isometries:

$$\begin{aligned}
&\text{edges } (Q_0, \infty)_\infty \text{ and } (Q_1, 1/2)_\infty \text{ of } \mathfrak{P}_- \text{ are glued by } A,\\
&\text{edges } (1/2, \infty)_\infty \text{ and } (Q_1, Q_0)_\infty \text{ of } \mathfrak{P}_- \text{ are glued by } A^{-n}B,\\
&\text{edges } (Q_n, \infty)_\infty \text{ and } (P^*_{m+1}, P^*_m)_\infty \text{ of } \mathfrak{P}_+ \text{ are glued by } B^mA, \text{ and}\\
&\text{edges } (P^*_m, \infty)_\infty \text{ and } (P^*_{m+1}, Q_n)_\infty \text{ of } \mathfrak{P}_+ \text{ are glued by } B.
\end{aligned} \tag{3}$$

The theory says the following about the translation t_τ in the uniformizing Kleinian group of M_ϕ.

Fact 4. $t_\tau(Q_0) = Q_n$, $t_\tau(Q_1) = P^*_{m+1}$ and $t_\tau(1/2) = P^*_m$ (in other words, $t_\tau(\mathfrak{P}_-) = \mathfrak{P}_+$).

In fact, $\mathfrak{P}_-$ and $\mathfrak{P}_+$ are projected onto the same embedded punctured torus T^* in M_ϕ, which also can be viewed as fiber of the punctured torus bundle, and t_τ corresponds to the monodromy of the bundle. Let us give an interpretation of t_τ as a monodromy of the form R^nL^m. Since t_τ takes $(Q_0, \infty)_\infty$ to $(Q_n, \infty)_\infty$, Eq. (3) shows that t_τ takes the curve corresponding to A to the curve corresponding to B^mA. Similarly, t_τ takes $(1/2, \infty)_\infty$ to $(P^*_m, \infty)_\infty$, which implies that t_τ takes the curve corresponding to $A^{-n}B$ to the curve corresponding to B. Then we can think t_τ as inducing the transformation $(A, A^{-n}B) \mapsto (B^mA, B)$ in the homotopy classes of curves in T^*. This corresponds to the n-th iterate of a Dehn twist about the curve corresponding to A and the m-th iterate of a Dehn twist about the curve corresponding to B.

We can get a view of the canonical tetrahedral decomposition of M_ϕ in a neighborhood around ∞. If we move by the translation t_1 the $2(n+m)$ tetrahedra $(\infty, -1/2, Q_{j-1}, Q_j)_\infty$, $(\infty, 1/2, Q_j, Q_{j+1})_\infty$, $(\infty, Q_n+1, P^*_{i-1}, P^*_i)_\infty$ and $(\infty, Q_n, P^*_i, P^*_{i+1})_\infty$, with $j = 1, \ldots, n$ and $i = 1, \ldots, m$, we obtain $4(n+m)$ tetrahedra which are projected onto $n+m$ ideal tetrahedra in M_ϕ, because the tetrahedral decomposition is h-invariant and $t_1 = (B)^{-1}A^{-1}h$. This is the canonical decomposition of M_ϕ by $n+m$ ideal tetrahedra. The action of $\langle t_2, t_\tau \rangle$ on the above $4(n+m)$ tetrahedra gives all the tetrahedra adjacent to ∞ in the lifting of the canonical decomposition.

2.1. The cusp torus

The cusp torus of M_ϕ is the quotient $\mathbb{C}/\langle t_2, t_\tau \rangle$. Since t_2 and t_τ are parabolic isometries leaving invariant the horospheres about ∞, the cusp torus has a natural flat geometry, up to scaling factor. The cusp torus inherits a triangulation by

$4(n+m)$ triangles, namely, $(-1/2, Q_{j-1}, Q_j)$, $(1/2, Q_{j+1}, Q_j)$, $(Q_n+1, P^*_{i-1}, P^*_i)$ and (Q_n, P^*_{i+1}, P^*_i), with $j = 1, \dots, n$, $i = 1, \dots, m$, and their translations by t_1 (see Fig. 2).

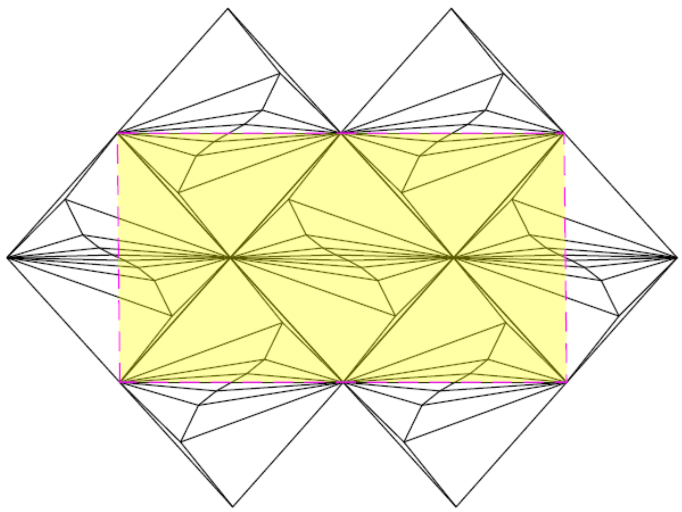

FIGURE 2. Cusp torus for monodromy R^9L^8. The yellow region is a fundamental domain for the action of $\langle t_2, t_\tau \rangle$. We have $\tau \approx -0.01084 + 1.11663i$ in this case.

3. Recovering the cusp torus for monodromy R^nL^m

Let ϕ a monodromy of the form R^nL^m. The main goal of this section is to prove Theorem 3.7, in order to recover the cusp torus (as well as the hyperbolic structure of M_ϕ and its uniformizing Kleinian group) from the construction of two sequences of complex numbers $(Q_j)_{j=0}^{n+1}$ and $(P_j)_{j=0}^{m+1}$ satisfying the following four conditions.

Condition 1. $(Q_j)_{j=0}^{n+1} \subset \mathbb{C} \setminus \{\pm 1/2\}$ is a sequence of pairwise distinct points such that

$$Q_{j+1} - 1/2 = \frac{(Q_{j-1} + 1/2)(Q_j - 1/2)}{Q_j + 1/2}, \quad \text{for} \quad j = 1, \dots, n. \tag{4}$$

(Condition 1 is precisely Fact 1 of Section 2.)

Condition 2. $(P_j)_{j=0}^{m+1} \subset \mathbb{C} \setminus \{\pm 1/2\}$ is a sequence of pairwise distinct points such that

$$P_{j+1} + 1/2 = \frac{(P_{j-1} - 1/2)(P_j + 1/2)}{P_j - 1/2}, \quad \text{for} \quad j = 1, \dots, m. \tag{5}$$

(Replacing each P_j by P_j^*, $-1/2$ by Q_n and $1/2$ by $Q_n + 1$ in the Condition 2, we obtain the Fact 2.)

Condition 3. $t_{Q_n+1/2}$ satisfies $t_{Q_n+1/2}(P_0) = 1/2$ and $t_{Q_n+1/2}(P_1) = Q_{n+1}$.

(Letting $t_{Q_n+1/2}(P_j) = P_j^*$ for each j, the Condition 3 and the Fact 3 are equivalent. See Fig. 3).

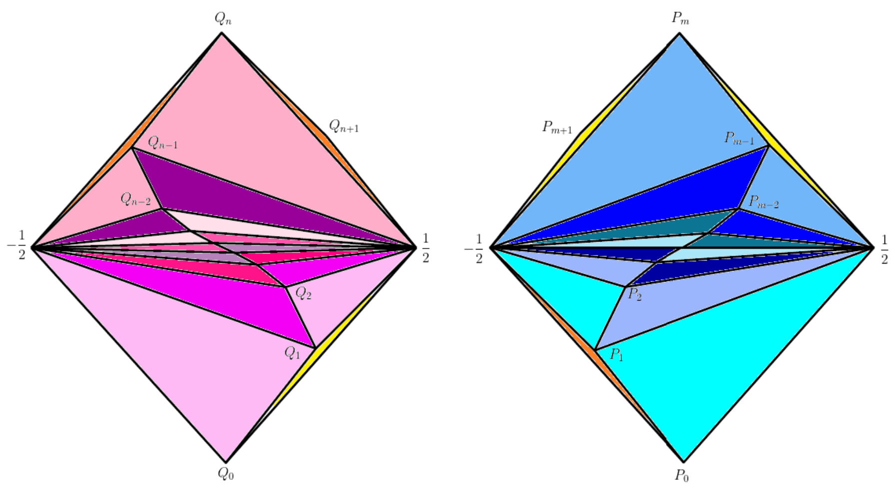

FIGURE 3. Sequences $(Q_j)_{j=0}^{n+1}$ and $(P_j)_{j=0}^{m+1}$. After translating the right quadrilateral by $t_{Q_n+1/2}$, we obtain Fig. 1.

Condition 4. There is a translation taking $(1/2, Q_0, Q_1)$ to $(P_m, -1/2, P_{m+1})$.

(Condition 4 and the Fact 4 are equivalent by defining $t_\tau = t_{Q_n+1/2} \circ t$, where t is the translation mentioned in the Condition 4.)

Convention: By "mod(λ)", $\lambda \in \mathbb{R}$, we will mean "up to an additive constant in $\lambda\mathbb{Z}$".

3.1. The sequence $(Q_j)_{j=0}^{n+1}$

We will show that a sequence $(Q_j)_{j=0}^{n+1}$ satisfying Condition 1 results from the iteration of Q_0 by powers of a Möbius map (Lemma 3.1).

In view of $\tanh^2(\ell/2) = \alpha^2 \Leftrightarrow e^\ell = (1-\alpha)^{\epsilon_1}(1+\alpha)^{\epsilon_2}$, where $\{\epsilon_1, \epsilon_2\} = \{-1, 1\}$, the equation $\tanh^2(\ell/2) = \alpha^2 \in \mathbb{C} \setminus \{1\}$ has exactly two solutions ℓ and $-\ell$ mod($2\pi i$). It follows that there is a Möbius map A of the form

$$A(z) = \frac{z + \frac{1}{2}\tanh^2 \frac{\ell}{2}}{2z + 1}$$

so that $A(Q_0) = Q_1$ (unless $4Q_1Q_0 + 2Q_1 - 2Q_0 = 1$, i.e. $Q_0 = -1/2$ or $Q_1 = 1/2$, which is impossible since $Q_0, Q_1 \in \mathbb{C} \setminus \{\pm 1/2\}$).

This Möbius map A cannot be parabolic since $\tanh(\ell/2) \neq 0$. Note that A is a loxodromic map (i.e. it has exactly two fixed points, one attracting and one repelling) if and only if $\Re(\ell) \neq 0$. When $\Re(\ell) > 0$ there is a well-known interpretation for ℓ as the complex length of A (e.g. [8, Section 7.4]). In particular, the hyperbolic geodesic joining the fixed points of A is projected to a closed geodesic in $\mathbb{H}^3/\langle A \rangle$ whose length is the real part $\Re(\ell)$.

Lemma 3.1. *$(Q_j)_{j=0}^{n+1}$ satisfies $Q_j = A(Q_{j-1})$, for $j = 1, \ldots, n+1$.*

Proof. It is an induction starting with $Q_1 = A(Q_0)$. Assume that $Q_j = A(Q_{j-1})$. Hence $Q_{j-1} = A^{-1}(Q_j) = \left[Q_j - (1/2)\tanh^2(\ell/2)\right]/(-2Q_j + 1)$. Substituting Q_{j-1} into Eq. (4) we obtain

$$Q_{j+1} = \left[Q_j + (1/2)\tanh^2(\ell/2)\right]/(2Q_j + 1) = A(Q_j). \qquad ☑$$

We will now show that $(Q_j)_{j=0}^{n+1}$ can be written in terms of hyperbolic functions (Lemma 3.2). In view of $\tanh(\nu\ell/2) = \beta \Leftrightarrow e^{\nu\ell} = (1+\beta)/(1-\beta)$, the equation $\tanh(\nu\ell/2) = \beta \in \mathbb{C} \setminus \{\pm 1\}$ has exactly one solution $\nu \bmod (2\pi i/\ell)$. It follows that if ℓ is chosen, then there exists a complex number ν so that

$$(1/2)\tanh(\ell/2)\tanh(\nu\ell/2) = Q_0 \tag{6}$$

(unless $2Q_0/\tanh(\ell/2) = \pm 1$, i.e. $Q_1 = A(Q_0) = Q_0$, which is impossible since Q_0 and Q_1 are distinct).

Lemma 3.2.

$$Q_j = \frac{1}{2}\tanh\frac{\ell}{2}\tanh\frac{(\nu+j)\ell}{2}, \quad \text{for } j = 0, \ldots, n+1. \tag{7}$$

Proof. This is an induction starting with $Q_0 = (1/2)\tanh(\ell/2)\tanh(\nu\ell/2)$. Assume that Q_j can be expressed in the form of Eq. (7). Hence

$$Q_{j+1} = A\left(\frac{1}{2}\tanh\frac{\ell}{2}\tanh\frac{(\nu+j)\ell}{2}\right)$$

also satisfies Eq. (7). ☑

3.2. The sequence $(P_j)_{j=0}^{m+1}$

By way of analogy to Section 3.1, there is a Möbius map of the form

$$B^*(z) = \frac{z - \frac{1}{2}\tanh^2\frac{l}{2}}{-2z+1}$$

so that $B^*(P_0) = P_1$ (unless $4P_1P_0+2P_0-2P_1 = 1$, i.e. $P_0 = 1/2$ or $P_1 = -1/2$, but these situations are excluded by Condition 2).

With l chosen, there is a complex number μ satisfying

$$(1/2)\tanh(l/2)\tanh(\mu l/2) = P_0 \tag{8}$$

(unless $2P_0/\tanh(l/2) = \pm 1$, i.e. $P_1 = B^*(P_0) = P_0$ which is impossible since P_0 and P_1 are distinct).

Lemma 3.3. *$P_j = (1/2)\tanh(l/2)\tanh[(\mu - j)l/2]$ for $j = 0, \dots, m+1$ and $P_j = B^*(P_{j-1})$ for $j = 1, \dots, m+1$.*

Proof. Carry out the same process as in Lemma 3.1 and Lemma 3.2, except that we now apply Eq. (5) instead of Eq. (4) to prove that $P_j = B^*(P_{j-1})$. ☑

3.3. Equations for ℓ and l

The following result was motivated by [4, Section 10].

Proposition 3.4. *Let $(Q_j)_{j=0}^{n+1} \subset \mathbb{C} \setminus \{\pm 1/2\}$ and $(P_j)_{j=0}^{m+1} \subset \mathbb{C} \setminus \{\pm 1/2\}$ be sequences of points that satisfy the Conditions 1, 2, 3 and 4. Then*

(1) $Q_0 = P_0$ and $Q_n = P_m$.

(2) $Q_j = -Q_{n-j}$, for all $j = 0, \dots, n$.

(3) $P_j = -P_{m-j}$, for all $j = 0, \dots, m$.

Moreover, if ℓ and l satisfy

$$Q_1 = \frac{Q_0 + \frac{1}{2}\tanh^2\frac{\ell}{2}}{2Q_0 + 1} \quad \text{and} \quad P_1 = \frac{P_0 - \frac{1}{2}\tanh^2\frac{l}{2}}{-2P_0 + 1},$$

then the classes $\pm\ell \bmod (2\pi i)$ and $\pm l \bmod (2\pi i)$ have representatives ℓ and l, respectively, such that

$$\begin{aligned} \tanh(\ell/2)\cosh(l/2) &= \cosh(n\ell/4), \\ \tanh(l/2)\cosh(\ell/2) &= \cosh(ml/4). \end{aligned} \tag{9}$$

Note that the existence of ℓ and l is guaranteed by the arguments in the Sections 3.1 and 3.2. In the proof, we also make use of the Lemmas 3.1, 3.2 and 3.3.

Proof. (1) Condition 4 says that $P_m = -Q_0$ and $P_{m+1} + 1/2 = Q_1 - Q_0$. By putting $P_{m+1} = B^*(P_m)$ and $Q_1 = A(Q_0)$ we obtain

$$4Q_0^2 = \tanh^2\frac{\ell}{2} + \tanh^2\frac{l}{2} - 1. \tag{10}$$

Condition 3 says that $P_0 = -Q_n$ and $P_1 - P_0 = Q_{n+1} - 1/2$. By putting $P_1 = B^*(P_0)$ and $Q_{n+1} = A(Q_n)$ we obtain

$$4P_0^2 = \tanh^2 \frac{\ell}{2} + \tanh^2 \frac{l}{2} - 1. \tag{11}$$

From this we deduce $P_0 = Q_0$, since $Q_0 \neq Q_n$. In addition, $P_m = Q_n$.

(2) It is an induction starting with $j = 0$. Assume that $Q_j = -Q_{n-j}$. Then

$$Q_{j+1} = A(Q_j) = \frac{Q_j + \frac{1}{2}\tanh^2\frac{\ell}{2}}{2Q_j + 1} = \\ -\frac{Q_{n-j} - \frac{1}{2}\tanh^2\frac{\ell}{2}}{-2Q_{n-j} + 1} = -A^{-1}(Q_{n-j}) = -Q_{n-j-1}.$$

(3) It is a similar induction to the previous one.

Substituting Eq. (6) and Eq. (8) respectively into Eq. (10) and Eq. (11) result

$$\tanh^2(\ell/2)\cosh^2(l/2) = \cosh^2(\nu\ell/2), \\ \tanh^2(l/2)\cosh^2(\ell/2) = \cosh^2(\mu l/2).$$

From Eq. (7), $Q_n = -Q_0 \Leftrightarrow \tanh[(\nu + n)\ell/2] = -\tanh(\nu\ell/2) \Leftrightarrow e^{(2\nu+n)\ell} = 1$. Similarly, $P_m = -P_0 \Leftrightarrow e^{(2\mu-m)l} = 1$. This proves the existence of integers k_1 and k_2 such that

$$\nu\ell/2 = -n\ell/4 + \pi i k_1/2, \\ \mu l/2 = ml/4 + \pi i k_2/2.$$

Now we prove that k_1 can be taken to be even. There are two cases:

- If n is even then k_1 is automatically even. Otherwise, odd k_1 implies

$$Q_j = \frac{1}{2}\tanh\frac{\ell}{2}\tanh\left(-\frac{n\ell}{4} + \frac{\pi i k_1}{2} + \frac{j\ell}{2}\right) = \frac{1}{2}\tanh\frac{\ell}{2}\coth\left(-\frac{n\ell}{4} + \frac{j\ell}{2}\right)$$

  and the vertex $Q_{n/2}$ is not defined.

- If n is odd then k_1 can be taken to be even. To prove this, notice that $\ell' = \ell + 2\pi i$ satisfies

$$\tanh^2(\ell'/2)\cosh^2(l/2) = \tanh^2(\ell/2)\cosh^2(l/2) = \\ \cosh^2\left(-\frac{n\ell}{4} + \frac{\pi i k_1}{2}\right) = \cosh^2\left(-\frac{n\ell'}{4} + \frac{\pi i(k_1 + n)}{2}\right).$$

  Therefore, if ℓ is a solution with odd k_1, then ℓ' is a solution with even $k_1 + n$.

Similarly, k_2 can be taken to be even.

We conclude that the classes $\ell \bmod(2\pi i)$ and $l \bmod(2\pi i)$ have representatives ℓ and l, respectively, satisfying

$$\tanh^2(\ell/2)\cosh^2(l/2) = \cosh^2(n\ell/4),$$
$$\tanh^2(l/2)\cosh^2(\ell/2) = \cosh^2(ml/4).$$

By removing the squares from the above equations, we obtain four systems

$$\begin{array}{rcl} \pm\tanh(\ell/2)\cosh(l/2) & = & \cosh(n\ell/4) \\ \pm\tanh(l/2)\cosh(\ell/2) & = & \cosh(ml/4) \end{array},$$

which we will denote by $S(+,+)$, $S(-,-)$, $S(+,-)$ and $S(-,+)$, according to the choice of signs. If (ℓ, l) is a solution of $S(+,+)$ then $(\pm\ell, \pm l)$ is a solution of $S(\pm,\pm)$, respectively.

The conclusion is the existence of representatives in the classes $\pm\ell \bmod(2\pi i)$ and $\pm l \bmod(2\pi i)$ which satisfy Eq. (9). ☑

3.4. Triangulations and numerical experiments

Eq. (9) has many distinct pairs of solutions (ℓ, l). A generic solution (ℓ, l) determines two sequences $(Q_j)_{j=0}^n$ and $(P_j)_{j=0}^m$ where the triangles do not fit together consistently (see Fig. 4).

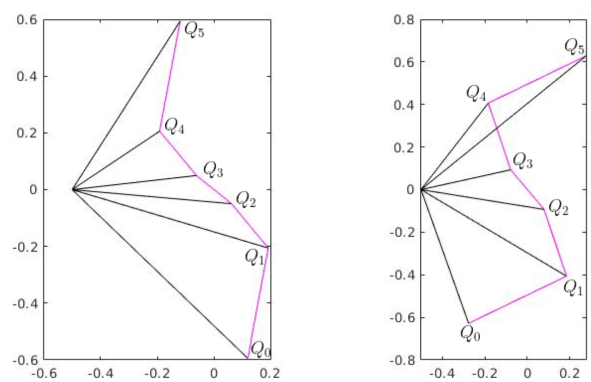

FIGURE 4. Two sequences $(Q_j)_{j=0}^5$ determined by distinct solutions (ℓ, l) for the case R^5L^3, one giving a triangulation (left) and the other not (right).

Let $\mathcal{Q}$ be the quadrilateral with vertices at $-1/2$, Q_0, $1/2$ and Q_n. Let $\mathcal{P}$ be the quadrilateral with vertices at $-1/2$, P_0, $1/2$ and P_m. Since $Q_n = -Q_0 = -P_0 = P_n$, the quadrilaterals $\mathcal{Q}$ and $\mathcal{P}$ are congruent parallelograms.

***Remark* 3.5.** The existence of the canonical decomposition of M_ϕ (see Section 2) implies that Eq. (9) has a particular solution (ℓ, l) which gives triangulations of both $\mathcal{Q}$ and $\mathcal{P}$.

The situation of Remark 5 will lead in a triangulation of $\mathbb{C}$ by the translations of $\mathcal{Q} \cup t_{Q_n+1/2}(\mathcal{P})$ under the elements of the group $\langle t_1, t_{2Q_n}\rangle$ (see Fig. 2 and Fig. 5). Moreover, $\mathcal{Q} \cup t_{Q_n+1/2}(\mathcal{P}) \cup t_1(\mathcal{Q} \cup t_{Q_n+1/2}(\mathcal{P}))$ will be a fundamental domain for the torus $\mathbb{C}/\langle t_2, t_{2Q_n}\rangle$.

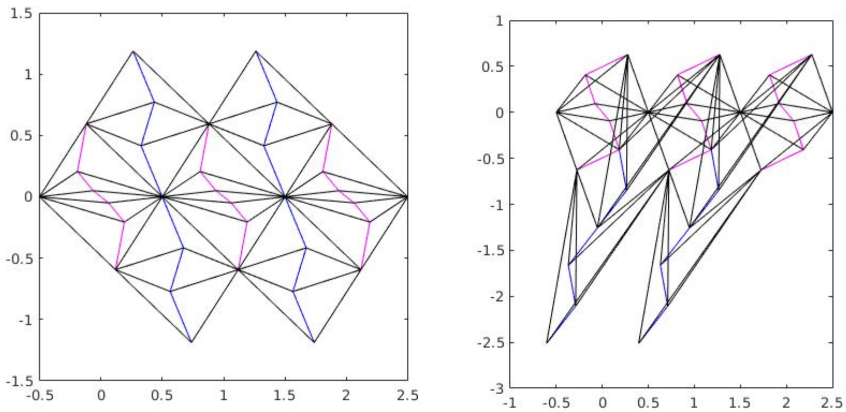

FIGURE 5. Fig. 4 can be extended by translations of $\mathcal{Q} \cup t_{Q_n+1/2}(\mathcal{P})$ under the elements of the group $\langle t_1, t_{2Q_n}\rangle$.

By using *vpasolve*$(Eq.(9), (\ell, l), (c_1, c_2))$, Matlab numerically solves the system of equations Eq. (7) for the variables ℓ and l, using respectively the initial guess c_1 and c_2. (The figures in this paper were created by this method.)

From our experimental work we propose the following.

Conjecture 1. Eq. (9) has a solution (ℓ, l) such that

$$\begin{array}{lrl} 0 < \Re(\ell), & 0 & < \Im(\ell) < \ 2\pi/n, \\ 0 < \Re(l), & -2\pi/m & < \Im(l) < \ 0. \end{array}$$

Proposition 3.6. *Suppose that (ℓ, l) is a witness for Conjecture 1. Then the sequences $(Q)_{j=1}^n$ and $(P)_{j=1}^m$ corresponding to (ℓ, l) determine triangulations of the parallelograms $\mathcal{Q}$ and $\mathcal{P}$, respectively.*

The following proof was motivated by the proof of [5, Lemma 3].

Proof. In this proof, let us consider $-\pi < \arg z \leq \pi$ for any $z \in \mathbb{C}$.

If we substitute the expressions of Q_j and P_j involving ℓ and l, we find that

$$\begin{aligned} \tfrac{Q_{j+1}-1/2}{Q_{j+1}-Q_j} &= \tfrac{Q_{j-1}+1/2}{Q_{j-1}-Q_j} = -\frac{\cosh^2\left(\frac{j\ell}{2}-\frac{n\ell}{4}\right)}{\sinh^2\frac{\ell}{2}}, \qquad j=1,\dots,n,\\ \tfrac{P_{j+1}+1/2}{P_{j+1}-P_j} &= \tfrac{P_{j-1}-1/2}{P_{j-1}-P_j} = -\frac{\cosh^2\left(\frac{ml}{4}-\frac{jl}{2}\right)}{\sinh^2\frac{l}{2}}, \qquad j=1,\dots,m. \end{aligned}$$

In fact these quotients are invariants of the similarity classes of the triangles $(-1/2, Q_{j-1}, Q_j)$, $(1/2, Q_{j+1}, Q_j)$, $(Q_n+1, P^*_{i-1}, P^*_i)$ and (Q_n, P^*_{i+1}, P^*_i). In order to understand the orientation of these triangles, we consider the functions of a real variable given by the arguments of these invariants

$$\begin{aligned} \vartheta(s) &= \arg\left(-\frac{\cosh^2\left(\frac{s\ell}{2}-\frac{n\ell}{4}\right)}{\sinh^2\frac{\ell}{2}}\right), \qquad 0\le s\le n,\\ \theta(s) &= \arg\left(-\frac{\cosh^2\left(\frac{ml}{4}-\frac{sl}{2}\right)}{\sinh^2\frac{l}{2}}\right), \qquad 0\le s\le m. \end{aligned} \tag{12}$$

A consequence of Eq. (9) is that

$$\vartheta(0) = \arg\left(-\frac{\cosh^2(l/2)}{\cosh^2(\ell/2)}\right) \quad \text{and} \quad \theta(0) = \arg\left(-\frac{\cosh^2(\ell/2)}{\cosh^2(l/2)}\right),$$

then $\vartheta(0)$ and $\theta(0)$ have opposite signs.

Since $\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$, we see that

$$\tan\left(\arg\cosh\left(\frac{s\ell}{2}-\frac{n\ell}{4}\right)\right) = \tanh\left(\frac{2s-n}{4}\Re(\ell)\right)\tan\left(\frac{2s-n}{4}\Im(\ell)\right) \tag{13}$$

is the product of two nonnegative, continuous and increasing functions for all $n/2 \le s \le n$ (under the hypotheses $0 < \Re(\ell)$ and $0 < \Im(\ell) < 2\pi/n$). It follows that

$$0 \le \arg\cosh\left(\frac{s\ell}{2}-\frac{n\ell}{4}\right) < \frac{\pi}{2}$$

for all $0 \le s \le n$, and $\vartheta(s)$ is a continuous function. In contrast,

$$\tan\left(\arg\cosh\left(\frac{ml}{4}-\frac{sl}{2}\right)\right) = \tanh\left(\frac{m-2s}{4}\Re(l)\right)\tan\left(\frac{m-2s}{4}\Im(l)\right) < 0$$

(under the hypotheses $0 < \Re(l)$, and $-2\pi/m < \Im(l) < 0$) implies

$$-\frac{\pi}{2} \le \arg\cosh\left(\frac{ml}{4}-\frac{sl}{2}\right) < 0$$

for all $0 \le s \le m$, and also $\theta(s)$ is a continuous function.

We claim that $\vartheta(s) \neq 0$ for all $0 \le s \le n$ and $\theta(s) \neq 0$ for all $0 \le s \le m$. Assuming this is true, our job is finished: the triangles of $\mathcal{Q}$ have the same orientation since $\vartheta(0), \vartheta(1), \dots, \vartheta(n)$ have the same sign, and the triangles of $\mathcal{P}$ are well-oriented too since $\theta(0), \theta(1), \dots, \theta(m)$ have the same sign, with $\vartheta(0) = -\theta(0)$.

To confirm the claim, we split $i\sinh(x+iy) = -\cosh x \sin y + i \sinh x \cos y$ to get

$$\tan(\arg i \sinh(x+iy))) = -\tanh x \cot y.$$

Then $\vartheta(s) = 0$ leads to

$$\tanh\left(\frac{2s-n}{4}\Re\ell\right)\tan\left(\frac{2s-n}{4}\Im\ell\right) = -\tanh\frac{\Re\ell}{2}\cot\frac{\Im\ell}{2}$$

which is impossible by checking signs on either side. If instead $\theta(s) = 0$, we likewise get a contradiction. ☑

3.5. Recovering the hyperbolic structure of M_ϕ

The hyperbolic structure of M_ϕ can be recovered from $(Q_j)_{j=0}^n$, $(P_j)_{j=0}^m$ and Remark 3.5 by using the Poincare's Polyhedron Theorem. For this, we set $P_k^* = t_{Q_n+1/2}(P_k)$ for $k = 0, \dots, m$ and $B = t_{Q_n+1/2} B^* t_{Q_n+1/2}^{-1}$.

Let $\Omega \subset \mathbb{H}^3$ be the ideal polyhedron

$$\Omega = \left(\bigcup_{j=1}^{n} (\infty, 1/2, Q_{j-1}, Q_j)_\infty\right) \bigcup \left(\bigcup_{j=1}^{n} (\infty, Q_n, P_{j-1}^*, P_j^*)_\infty\right)$$

composed from $m+n$ ideal tetrahedra (recall Fig. 1).

The following result was motivated by [5, Theorem 1].

Theorem 3.7. *Suppose that $(Q_j)_{j=0}^n$ and $(P_j)_{j=0}^m$ induce triangulations of the parallelograms $\mathcal{P}$ and $\mathcal{Q}$, as in Remark 3.5. Let $(P_j^*)_{j=0}^m$, B and Ω as above. Then the group $\Gamma = \langle A, B, t_{2Q_n}\rangle$ is a Kleinian group with fundamental domain Ω, and Γ is isomorphic to $\pi_1(M_\phi)$.*

In particular, $\mathbb{C}/\langle t_2, t_{2Q_n}\rangle$ is the cusp torus of M_ϕ.

Proof. All the faces of Ω are paired by gluing maps as follows:

$$\begin{aligned}
&A : (\infty, Q_{j-1}, Q_j)_\infty \mapsto (1/2, Q_j, Q_{j+1})_\infty, \quad j = 1, \dots, n-1,\\
&A : (\infty, Q_{n-1}, Q_n)_\infty \mapsto (1/2, Q_n, P_1^*)_\infty,\\
&B : (\infty, P_{j-1}^*, P_j^*)_\infty \mapsto (Q_n, P_j^*, P_{j+1}^*)_\infty, \quad j = 1, \dots, m-1,\\
&t_{2Q_n} : (\infty, 1/2, Q_0)_\infty \mapsto (\infty, P_m^*, Q_n),\\
&t_{2Q_n}^{-1} B : (\infty, P_{m-1}^*, P_m^*)_\infty \mapsto (Q_0, 1/2, Q_1)_\infty.
\end{aligned}$$

Also, the following edges are paired by gluing maps:

$$\begin{aligned}
&(\infty, Q_{j-1})_\infty \xrightarrow{A} (1/2, Q_j)_\infty \xrightarrow{A^{-1}} (\infty, Q_{j-1})_\infty, \quad j = 2, \dots, n,\\
&(\infty, P_{j-1}^*)_\infty \xrightarrow{B} (Q_n, P_j^*)_\infty \xrightarrow{B^{-1}} (\infty, P_{j-1}^*)_\infty, \quad j = 2, \dots, m-1,
\end{aligned}$$

and the sum of the two dihedral angles of Ω meeting at each pair is 2π. To check this, we denote by $\angle(z_1, z_2)_\infty$ the dihedral angle of Ω at the edge $(z_1, z_2)_\infty$ and by $\angle(z_1, z_2, z_3)$ the angle at z_2 in the Euclidean triangle (z_1, z_2, z_3). Then

$$\begin{aligned}\angle(1/2, Q_j)_\infty &= \angle(1/2, Q_{j-1}, Q_j) + \angle(1/2, Q_{j+1}, Q_j)\\ &= \angle(-1/2, Q_{j-1}, Q_{j-2}) + \angle(-1/2, Q_{j-1}, Q_j).\end{aligned}$$

Therefore $\angle(\infty, Q_{j-1})_\infty + \angle(1/2, Q_j)_\infty = 2\pi$ for $j = 1, \ldots, n$. The proof for the other pairs is similar.

There are three more equivalence classes of edges. The first one is:

$$(1/2, Q_0)_\infty \xrightarrow{t_{2Q_n}} (P_m^*, Q_n)_\infty \xrightarrow{B^{-1}} (P_{m-1}^*, \infty)_\infty \xrightarrow{t_{2Q_n}^{-1}B} (1/2, Q_0)_\infty,$$

and also the three dihedral angles meeting in this equivalence of edges adds up to 2π.

Another equivalence class is

$$\begin{aligned}&(Q_0, Q_1)_\infty \xrightarrow{A} (Q_1, Q_2)_\infty \xrightarrow{A} \cdots \xrightarrow{A} (Q_{n-1}, Q_n)_\infty \xrightarrow{A} (Q_n, P_1^*)_\infty\\ &\quad \xrightarrow{B^{-1}} (\infty, P_0^*)_\infty \xrightarrow{t_{2Q_n}} (\infty, P_m^*)_\infty \xrightarrow{t_{2Q_n}^{-1}B} (Q_0, Q_1)_\infty.\end{aligned}$$

Since

$$\sum_{j=1}^{n} \angle(Q_{j-1}, Q_j)_\infty = \angle(Q_0, 1/2, Q_n), \quad \text{and} \quad \angle(Q_n, P_1^*)_\infty = \angle(Q_n, 1/2, Q_n{+}1),$$

is easily seen that the dihedral angles of Ω meeting in this equivalence class of edges adds up to 2π.

The last equivalence class is

$$\begin{aligned}&(P_0^*, P_1^*)_\infty \xrightarrow{B} (P_1^*, P_2^*)_\infty \xrightarrow{B} \cdots \xrightarrow{B} (P_{m-1}^*, P_m^*)_\infty \xrightarrow{t_{2Q_n}^{-1}B} (1/2, Q_1)_\infty\\ &\quad \xrightarrow{A^{-1}} (\infty, Q_0)_\infty \xrightarrow{t_{2Q_n}} (\infty, Q_n)_\infty \xrightarrow{A} (P_0^*, P_1^*),\end{aligned}$$

and also is easily seen that the angle sum around this edge equivalence class is 2π.

By Poincare's Polyhedron Theorem (e.g. [3]), $\mathbb{H}^3/\Gamma$ is a complete hyperbolic manifold (t_{2Q_n} guarantees completeness because a Möbius transformation that commutes with t_{2Q_n} should be a translation). The fundamental group of this hyperbolic manifold has the following presentation

$$\Gamma = \langle A, B, t_{2Q_n} \mid At_{2Q_n}A^{-1}t_{2Q_n}^{-1}B^m \;=\; 1 \;=\; t_{2Q_n}^{-1}Bt_{2Q_n}B^{-1}A^n\rangle.$$

These relations can be written as

$$t_{2Q_n}At_{2Q_n}^{-1} = B^mA, \quad \text{and} \quad t_{2Q_n}A^{-n}Bt_{2Q_n}^{-1} = B,$$

respectively. Then the correspondence $(\alpha, \beta, t) \mapsto (A, A^{-n}B, t_{2Q_n})$ determines an isomorphism $G \to \Gamma$, where $G = \langle \alpha, \beta, t \mid t\alpha t^{-1} = (\alpha^n\beta)^m\alpha,\ t\beta t^{-1} = \alpha^n\beta\rangle$, but G is the fundamental group $\pi_1(M_\phi)$ because $(\alpha, \beta) \mapsto (\alpha, \alpha^n\beta)$ comes from the n-th iterate of a Dehn twist and $(\alpha, \alpha^n\beta) \mapsto ((\alpha^n\beta)^m\alpha, \alpha^n\beta)$ comes from the m-th iterate of another Dehn twist.

The last statement follows directly from the Mostow-Prasad Rigidity. ☑

4. Limit when $m \to \infty$ of cusp tori, for fixed n

This section aims to present a global description of the topological closure of the space of cusp tori corresponding to the monodromies R^nL^m, for arbitrary positive integers m and n.

Theorem 4.1. *The cusp torus corresponding to monodromy R^nL^m tends to a torus of the form $\mathbb{C}/\langle t_2, t_{i/\cosh(\ell/2)}\rangle$ as m tends to infinity, where ℓ satisfies*

$$\sinh^2 \frac{\ell}{2}\left(\frac{\cosh(n\ell/2) - 1}{\cosh(n\ell/2) + 1}\right) = -1. \tag{14}$$

Proof. By Theorem 3.7, the cusp torus corresponding to monodromy R^nL^m is given as a quotient of the form $\mathbb{C}/\langle t_2, t_{2Q_n}\rangle$ for a particular solution (ℓ, l) of Eq. (9).

By Eq. (10), $\tau = 2Q_n$ satisfies

$$\tau^2 = (-2Q_0)^2 = \tanh^2 \frac{\ell}{2} + \tanh^2 \frac{l}{2} - 1. \tag{15}$$

As m tends to infinity the complex length l approaches zero (e.g. [11, Pivot Theorem]). Then

$$\tau^2 \to \tanh^2 \frac{\ell}{2} - 1 = \frac{-1}{\cosh^2(\ell/2)} \quad \text{as} \quad m \to \infty.$$

Note that our choice of ℓ in the proof of Proposition 3.4 implies $\nu\ell/2 = -n\ell/4$, then

$$\tau^2 = (-2Q_0)^2 = \tanh^2 \frac{\ell}{2} \tanh^2 \frac{n\ell}{4} = \tanh^2 \frac{\ell}{2}\left(\frac{\cosh(n\ell/2) - 1}{\cosh(n\ell/2) + 1}\right)$$

by (6). We conclude that

$$\tanh^2 \frac{\ell}{2}\left(\frac{\cosh(n\ell/2) - 1}{\cosh(n\ell/2) + 1}\right) = \frac{-1}{\cosh^2(\ell/2)}.$$ ☑

Eq. (14) can be written as

$$(x^2 - 1)\frac{T_n(x) - 1}{T_n(x) + 1} = -1 \tag{16}$$

in terms of $x = \cosh(\ell/2)$, where $T_n(x)$ denotes the Chebyshev polynomial of the first kind, which can be defined by the identity $T_n(\cosh\theta) = \cosh(n\theta)$ for all $n \in \mathbb{N}$. An equivalent definition is given by the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, for $n \in \{2, 3, \dots\}$, together with the initial elements $T_0(x) = 1$ and $T_1(x) = x$ (e.g. [9, Section 1.2.1]).

Example 4.2. For $n = 1$, Eq. (16) becomes $(x-1)^2 = -1$, (after removing the common factor $x+1$ because $\Re(\ell) \neq 0$ for a loxodromic map A), with solutions $x = 1 \pm i$. The limit torus $\mathbb{C}/\langle t_2, t_{i/x}\rangle$ is biholomorphic to $\mathbb{C}/\langle t_{2x}, t_i\rangle$. The torus $\mathbb{C}/\langle t_{2x}, t_i\rangle$ is isometric to $\mathbb{C}/\langle t_2, t_i\rangle$ for $x = 1 \pm i$.

Example 4.3. For $n = 2$, Eq. (16) becomes $(x^2-1)^2 = -x^2$. The four solutions to this equation are $x = (\pm\sqrt{3} \pm i)/2$. The limit torus $\mathbb{C}/\langle t_{2x}, t_i\rangle$ is isometric to $\mathbb{C}/\langle t_{\sqrt{3}}, t_i\rangle$ for all $x = (\pm\sqrt{3} \pm i)/2$.

Example 4.4. For $n = 3$, Eq. (16) becomes (after removing the common factor $x + 1$)

$$\frac{(x-1)^2(2x+1)^2}{(2x-1)^2} = -1. \tag{17}$$

The tori $\mathbb{C}/\langle t_2, t_{i/x}\rangle$ are not all biholomorphically equivalent, they depend on the chosen solution x of Eq. (17). An efficient way of numerically finding the limit torus is by the approximate tori for large m. The cusp torus $\mathbb{C}/\langle t_2, t_\tau\rangle$ with $\tau \approx 0.3624487 + 1.0563519i$ (Fig. 6) corresponds to monodromy R^3L^{1000}. On the other hand, $x_0 = (1 + 2i + \sqrt{5-4i})/4$ is a solution of Eq. (17) such that $i/x_0 \approx 0.36244966 + 1.05634686i$.

Example 4.5. The cusp torus corresponding to monodromy R^nL^m tends to torus $\mathbb{C}/\langle t_2, t_i\rangle$ as n and m tend to infinity. This follows from (15) by letting l and ℓ approach zero.

Fig. 7 is an illustration of the closure of the space of all cusp tori corresponding to monodromy R^nL^m. The symmetry shown in Fig. 7 is due to triangulations corresponding to R^nL^m and R^mL^n differ by complex conjugation, as we explain now.

If (ℓ, l) is a solution of Eq. (9), another solution is the complex conjugate $(\bar{\ell}, \bar{l})$ because $h(\bar{z}) = \overline{h(z)}$ for every hyperbolic function. Moreover, the sequences $(Q_j)_{j=0}^{n+1}$ and $(P_j)_{j=0}^{m+1}$ determined by (ℓ, l) and $(\bar{\ell}, \bar{l})$ differ by complex conjugation. In fact, the triangulations of R^nL^m and R^mL^n differ by a complex conjugation and relabeling. More precisely, exchanging m and n in Eq. (9) has the effect on the sequences of reversing their order. For example, if $(Q_j)_{j=0}^{n+1}$ satisfies the similarity condition of the Section 3.1, then $P_j := Q_{n-j}$, $j \in \{0, \dots, n+1\}$, satisfies the similarity condition of the Section 3.2.

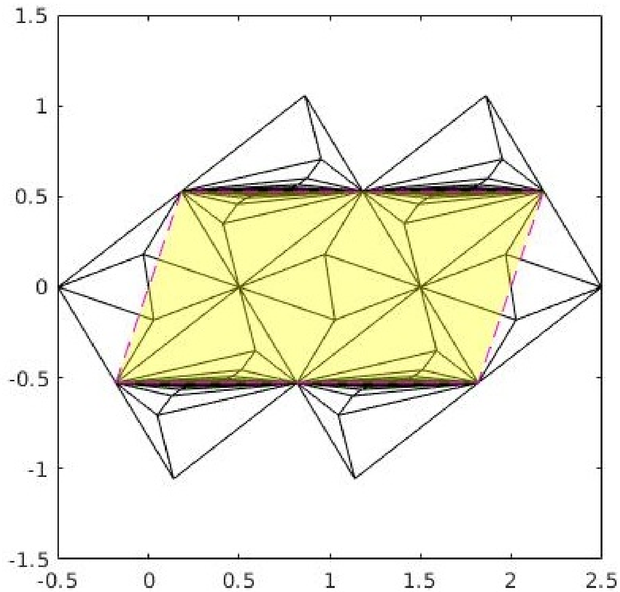

FIGURE 6. Cusp torus corresponding to monodromy R^3L^{1000}.

5. Arbitrary monodromy

The purpose of this section is to sketch briefly an adaptation of some techniques in Section 3 to arbitrary monodromy $R^{n_1}L^{m_1}R^{n_2}L^{m_2}\cdots R^{n_k}L^{m_k}$.

For each $i \in \{1, \ldots, k\}$, consider a sequence of complex numbers $(Q_{(i,j)})_{j=0}^{n_i+1}$ such that the triangles $(1/2, Q_{(i,j+1)}, Q_{(i,j)})$ and $(-1/2, Q_{(i,j-1)}, Q_{(i,j)})$ are similar for $j = 1, 2, \ldots, n_i$; and a sequence of complex numbers $(P_{(i,j)})_{j=0}^{m_i+1}$ such that $(-1/2, P_{(i,j+1)}, P_{(i,j)})$ is similar to $(1/2, P_{(i,j-1)}, P_{(i,j)})$ for $j = 1, 2, \ldots, m_i$.

Each sequence $(Q_{(i,j)})_{j=0}^{n_i+1}$ or $(P_{(i,j)})_{j=0}^{m_i+1}$ is determined by their first two terms. The Möbius maps

$$A_i(z) = \frac{z + a_i}{2z + 1} \qquad \text{and} \qquad B_i(z) = \frac{z - b_i}{-2z + 1},$$

where $a_i := 2Q_{(i,1)}Q_{(i,0)} - Q_{(i,0)} + Q_{(i,1)}$ and $b_i := 2P_{(i,1)}P_{(i,0)} + P_{(i,0)} - P_{(i,1)}$, satisfy $Q_{(i,j)} = A_i(Q_{(i,j-1)})$ and $P_{(i,j)} = B_i(P_{(i,j-1)})$.

Each sequence $(Q_{(i,j)})_{j=0}^{n_i+1}$ and $(P_{(i,j)})_{j=0}^{m_i+1}$ determines respectively a quadrilateral $\mathcal{Q}_i$ and $\mathcal{P}_i$, as in Section 3.4. All of these quadrilaterals have a diagonal of length 1, and therefore we can again consider $-1/2$ and $1/2$ as two of their opposite vertices. To join quadrilaterals $\mathcal{Q}_i$ and $\mathcal{P}_i$ or $\mathcal{P}_i$ and $\mathcal{Q}_{i+1}$, we require the following conditions (see Fig. 8):

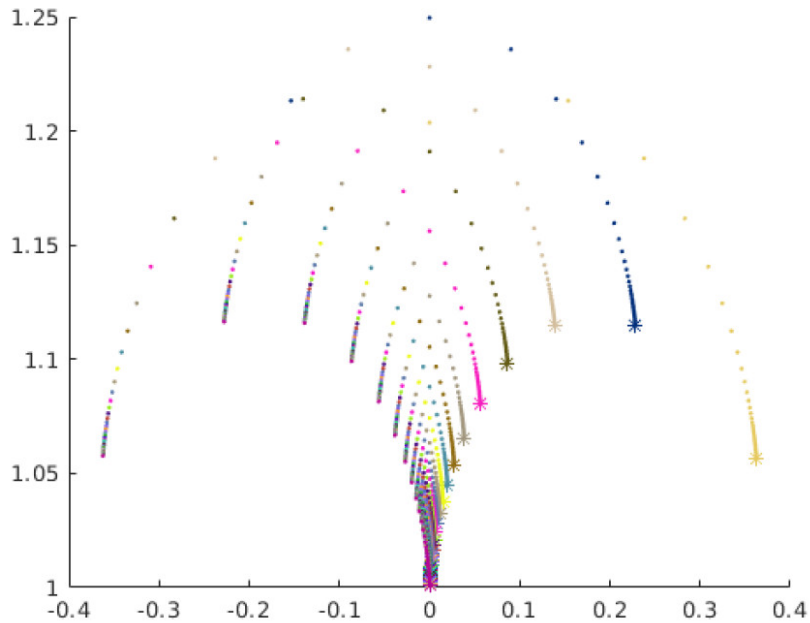

FIGURE 7. Each point represents the τ of a cusp torus $\mathbb{C}/\langle t_2, t_\tau\rangle$, and each asterisk is a limit point of the set of all such points τ. We marked with the same color each τ associated with a family $\mathcal{F}_n := \{R^n L^m : m \in \mathbb{N}\}$ and its corresponding limit point. These limit points lie in the right half-plane. The limit points of families $\mathcal{G}_n := \{R^m L^n : m \in \mathbb{N}\}$ lie in the left half-plane, but they are not marked.

- $(-1/2, P_{(i,\,0)}, P_{(i,\,1)})$ must be a translation of $(Q_{(i,\,n_i)}, 1/2, Q_{(i,\,n_i+1)})$ and
- $(Q_{(i+1,\,1)}, Q_{(i+1,\,0)}, 1/2)$ must be a translation of $(P_{(i,\,m_i+1)}, -1/2, P_{(i,\,m_i)})$,

for $i \in \{1, 2, \ldots, k\}$.

In order to write any term of any sequence as a function of $Q_{(1,0)}$ and $Q_{(1,1)}$, it is sufficient to write $P_{(i,0)}$ and $P_{(i,1)}$ as a function of some terms of $(Q_{(i,j)})_{j=0}^{n_i+1}$, and $Q_{(i,0)}$ and $Q_{(i,1)}$ as a function of some terms of $(P_{(i-1,j)})_{j=0}^{n_{i-1}+1}$.

Lemma 5.1. *For $i \in \{1, 2, \ldots, k\}$,*

(1) $P_{(i,\,0)} = -Q_{(i,\,n_i)}$ *and* $Q_{(i+1,\,0)} = -P_{(i,\,m_i)}$,

(2) $P_{(i,\,1)} = Q_{(i,\,n_i+1)} - 1/2 - Q_{(i,\,n_i)}$ *and* $Q_{(i+1,\,1)} = P_{(i,\,m_i+1)} + 1/2 - P_{(i,\,m_i)}$,

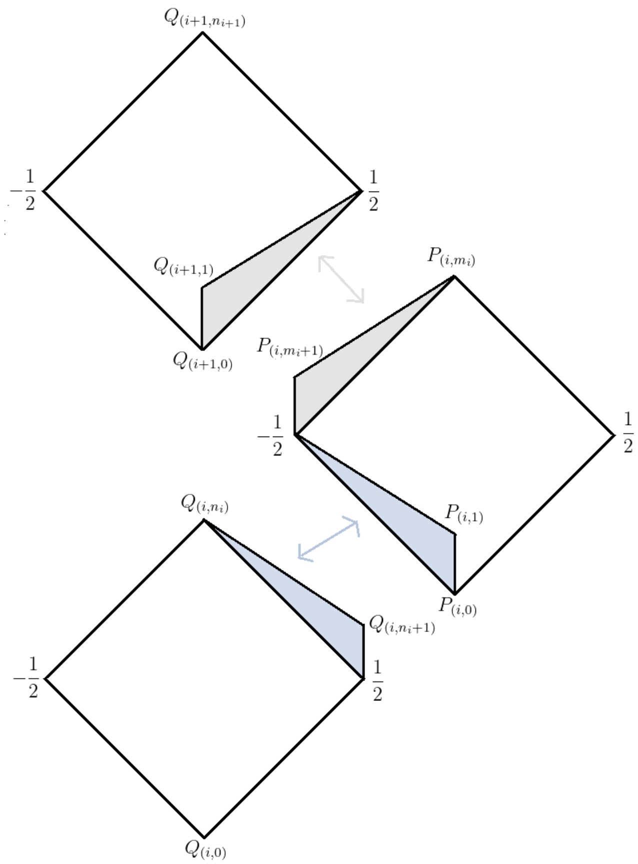

FIGURE 8

where $Q_{(k+1,\,0)} := Q_{(1,\,0)}$ *and* $Q_{(k+1,\,1)} := Q_{(1,\,1)}$.

***Proof*.** We have

$$\begin{aligned} P_{(i,\,0)} + 1/2 &= 1/2 - Q_{(i,\,n_i)}, \\ Q_{(i+1,\,0)} - 1/2 &= -1/2 - P_{(i,\,m_i)}, \\ P_{(i,\,1)} - P_{(i,\,0)} &= Q_{(i,\,n_i+1)} - 1/2, \\ Q_{(i+1,\,1)} - Q_{(i+1,\,0)} &= P_{(i,\,m_i+1)} + 1/2. \end{aligned}$$

From the first two equations, we conclude the first sentence. Substituting the first two equations into the last two, we obtain the second sentence. ☑

We conclude that the triangulation of the cusp torus corresponding to monodromy $R^{n_1}L^{m_1}R^{n_2}L^{m_2}\cdots\ R^{n_k}L^{m_k}$ can be constructed from $Q_{(1,\,0)}$ and

$Q_{(1,1)}$, which are solutions of the system

$$\begin{aligned} Q_{(1,0)} &= -P_{(k,n_k)}, \\ Q_{(1,1)} &= P_{(k,n_k+1)} + 1/2 + Q_{(1,0)}, \end{aligned}$$

where $P_{(k,n_k)}$ and $P_{(k,n_k+1)}$ are written as functions of $Q_{(1,0)}$ and $Q_{(1,1)}$.

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