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Palindromic and Colored Superdiagonal Compositions

Composiciones Superdiagonales Palíndromas y Coloreadas

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Abstract. A superdiagonal composition is one in which the i-th part or summand is of size greater than or equal to i. In this paper, we study the number of palindromic superdiagonal compositions and colored superdiagonal compositions. In particular, we give generating functions and explicit combinatorial formulas involving binomial coefficients and Stirling numbers of the first kind.

Key words and phrases. Compositions, palindromic compositions, colored compositions, generating functions, combinatorial identities.

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Resumen. Una composición superdiagonal es aquella composición en la que la i-ésima parte (o sumando) tiene un tamaño mayor o igual que i. En este artículo, estudiamos el número de composiciones superdiagonales palindrómicas y composiciones superdiagonales coloreadas. En particular, damos funciones generatrices y fórmulas combinatorias explícitas que involucran coeficientes binomiales y números de Stirling de la primera clase.

Palabras y frases clave. Composiciones, composiciones palíndromas, composiciones coloreadas, funciones generatrices, identidades combinatorias.

1. Introduction and Notation

A *composition* of a positive integer n is a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$. The summands σ_i

are called *parts* of the composition and n is referred to as the *weight* of σ. For example, the compositions of 4 are

$$(4), \quad (3,1), \quad (1,3), \quad (2,2), \quad (2,1,1), \quad (1,2,1), \quad (1,1,2), \quad (1,1,1,1).$$

A *palindromic* (or *self-inverse*) *composition* is one whose sequence of parts is the same whether it is read from left to right or right to left. For example, the palindromic compositions of 4 are

$$(4), \quad (2,2), \quad (1,2,1), \quad (1,1,1,1).$$

Hoggatt and Bicknell [8] studied palindromic compositions having parts in a subset of positive integers. In particular, they showed that the total number of palindromic compositions of n is given by $2^{\lfloor n/2 \rfloor}$. Moreover, the number of palindromic compositions of n with parts $\mathtt{1}$ and $\mathtt{2}$ are the interleaved Fibonacci sequence

$$1, \quad 1, \quad 2, \quad 1, \quad 3, \quad 2, \quad 5, \quad 3, \quad 8, \quad 5, \quad 13, \quad 8, \quad 21, \dots$$

The literature contains several generalizations and restrictions of the compositions. Much of them are related to the kind of parts or summands, for example compositions with even or odd parts [7, 8], with parts in arithmetical progressions [1, 9], compositions with colored parts [2, 4, 11], colored palindromic compositions [3, 6, 10], superdiagonal compositions [5], among other restrictions. For further information on compositions, we refer the reader to the text by Heubach and Mansour [7].

In this paper, we study *palindromic superdiagonal compositions*, that is a palindromic composition $(\sigma_1, \sigma_2, \dots, \sigma_\ell)$ of n, with the additional condition $\sigma_i \geq i$, for $i = 1, 2, \dots, \ell$. For example, the palindromic superdiagonal compositions of 10 are

$$(10), \quad (5,5), \quad (4,2,4), \quad (3,4,3).$$

Deutsch et al. [5] proved that the total number of superdiagonal compositions is given by the combinatorial sum:

$$\sum_{k \geq 1} \binom{n - \binom{k}{2} - 1}{k-1}.$$

Agarwal [2] generalized the concept of a composition by allowing the parts to come in various colors. By a *colored composition* of a positive integer n we mean a composition $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\ell)$ such that the part of size i can come in one of i different colors. The colors of the summand $\mathtt{i}$ are denoted by subscripts $i_1, i_2, \dots, i_i$ for each $i \geq 1$. For example, the colored compositions of 4 are given by

$$(4_1), \quad (4_2), \quad (4_3), \quad (4_4), \quad (3_1, 1_1), \quad (3_2, 1_1), \quad (3_3, 1_1), \quad (1_1, 3_1), \quad (1_1, 3_2),$$
$$(1_1, 3_3), \quad (2_1, 2_1), \quad (2_1, 2_2), \quad (2_2, 2_1), \quad (2_2, 2_2), \quad (2_1, 1_1, 1_1), \quad (2_2, 1_1, 1_1),$$
$$(1_1, 2_1, 1_1), \quad (1_1, 2_2, 1_1), \quad (1_1, 1_1, 2_1), \quad (1_1, 1_1, 2_2), \quad (1_1, 1_1, 1_1, 1_1).$$

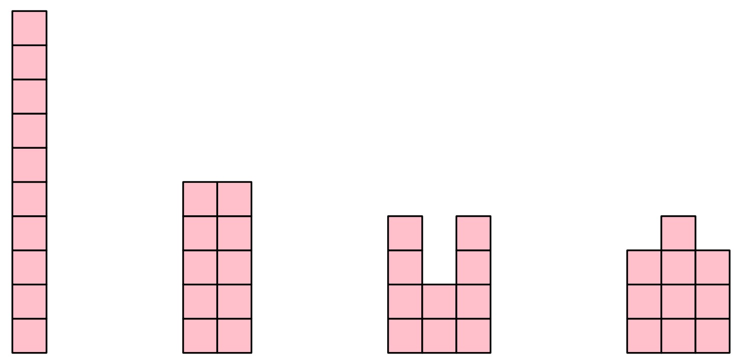
A *colored superdiagonal composition* is a colored composition such that the i-th part σ_i satisfies $\sigma_i \geq i$, for each $i \geq 1$. The colored superdiagonal compositions of 4 are

$$\begin{array}{l} (4_1), \quad (4_2), \quad (4_3), \quad (4_4), \quad (1_1, 3_1), \quad (1_1, 3_2), \quad (1_1, 3_3), \\ (2_1, 2_1), \quad (2_1, 2_2), \quad (2_2, 2_1), \quad (2_2, 2_2). \end{array} \tag{1}$$

The goal of this paper is to enumerate the palindromic superdiagonal compositions and colored superdiagonal compositions. These counting sequences are encoded by (ordinary) generating functions. Next, we expand the generating functions as power series to obtain explicit combinatorial formulas for the counting sequences.

2. Enumeration of the Palindromic Superdiagonal Compositions

Let $\mathcal{S}_{\mathsf{Pal}}$ denote the set of palindromic superdiagonal compositions. The composition $(\sigma_1, \sigma_2, \dots, \sigma_\ell)$ of n can be represented as a *bargraph* of ℓ columns, such that the i-th column contains σ_i cells for $1 \leq i \leq \ell$. For example, in Figure 1 we show the superdiagonal compositions of $n = 10$ with their bargraph representations.



(10) $(5, 5)$ $(4, 2, 4)$ $(3, 4, 3)$

FIGURE 1. Palindromic compositions of $n = 10$.

Let σ be a composition and let us denote the weight of σ by $|\sigma|$ and the number of parts of σ by $\rho(\sigma)$. Using these parameters, we introduce this bivariate generating function

$$S(x, y) := \sum_{\sigma \in \mathcal{S}_{\mathsf{Pal}}} x^{|\sigma|} y^{\rho(\sigma)}.$$

In Theorem 2.1 we give an expression for the generating function $S(x, y)$.

Theorem 2.1. *The bivariate generating function $S(x,y)$ is given by*

$$S(x,y)=\sum_{\sigma\in\mathcal{S}_{\mathsf{Pal}}} x^{|\sigma|}y^{\rho(\sigma)}=\sum_{m\geq 0}\left(\frac{x^{3m^2+m}}{(1-x^2)^m}y^{2m}+\frac{x^{3m^2+4m+1}}{(1-x)(1-x^2)^m}y^{2m+1}\right).$$

Proof. Let $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_{2m})$ be a palindromic superdiagonal composition of n with $2m$ parts. From the definition we have the condition $\sigma_i=\sigma_{2m+1-i}\geq 2m+1-i$, for $i=1,\dots,m$, see Figure 2 for a graphical representation of this case.

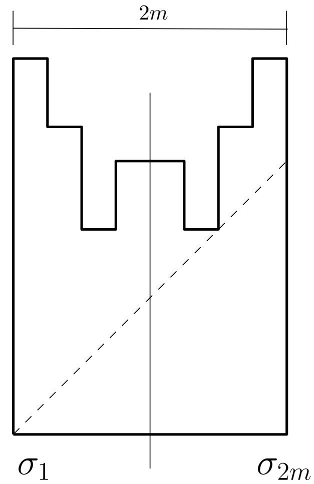


FIGURE 2. Decomposition of a palindromic superdiagonal composition.

The columns i-th and $(2m+1-i)$-th contribute to the generating function the term

$$x^{2(2m+1-i)}y^2+x^{2(2m+2-i)}y^2+x^{2(2m+3-i)}y^2+\cdots=\frac{x^{2(2m+1-i)}y^2}{1-x^2},\ i=1,2,\dots,m.$$

Therefore the composition σ contributes to the generating function the term

$$\frac{x^{4m}y^2}{1-x^2}\frac{x^{4m-2}y^2}{1-x^2}\cdots\frac{x^{2m+2}y^2}{1-x^2}=\frac{x^{3m^2+m}y^{2m}}{(1-x^2)^m}.$$

If the number of parts is odd, that is $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_{2m-1})$, then from a similar argument the contribution to the generating function is given by

$$\frac{x^{3m^2+4m+1}}{(1-x)(1-x^2)^m}y^{2m+1}.$$

Summing in the above two cases over m we obtain the desired result. ☑

As a series expansion, the generating function $S(x,y)$ begins with

$$\begin{aligned}S(x,y) = 1 + xy + x^2y + x^3y + x^4\left(y^2+y\right) + x^5y + x^6\left(y^2+y\right)\\ + x^7y + x^8\left(y^3+y^2+y\right) + x^9\left(y^3+y\right) + \mathbf{x^{10}}\left(\mathbf{2y^3}+\mathbf{y^2}+\mathbf{y}\right)\\ + x^{11}\left(2y^3+y\right) + x^{12}\left(3y^3+y^2+y\right) + \cdots\end{aligned}$$

Notice that Figure 1 shows the palindromic superdiagonal compositions corresponding to the bold coefficient in the above series. Let $s(n)$ and $s(n,m)$ denote the number of palindromic superdiagonal compositions of n and the number of palindromic superdiagonal compositions of n with exactly m parts, respectively. Note that $s(n) = \sum_{m\geq 1} s(n,m)$. In Table 1 we show the first few values of the sequence $s(n,m)$.

$m \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	0	1	0	1	0	1	0	1	0	1	0
3	0	0	0	0	0	0	0	1	1	2	2	3	3
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
$m \backslash n$	14	15	16	17	18	19	20	21	22	23	24	25	26
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0	1	0	1	0	1
3	4	4	5	5	6	6	7	7	8	8	9	9	10
4	1	0	2	0	3	0	4	0	5	0	6	0	7
5	0	0	0	0	0	0	0	1	1	3	3	6	6

TABLE 1. Values of $s(n,m)$, for $1 \leq n \leq 26$ and $1 \leq m \leq 5$.

Setting $y = 1$ in Theorem 2.1 implies that the generating function for the total number of palindromic superdiagonal compositions is the following

$$S(x,1) = \sum_{n\geq 0} s(n)x^n = \sum_{m\geq 0} \frac{x^{3m^2+m}(1-x+x^{3m+1})}{(1-x)(1-x^2)^m}.$$

The values of the sequence $s(n)$, for $0 \leq n \leq 26$, are

1, 1, 1, 1, 2, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 9, 6, 11, 7, 13, 9, 16, 12, 20, 16, 25, ...

In Theorem 2.2 we give a combinatorial expression for the sequence $s(n,m)$. We set the convention, $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$.

Theorem 2.2. *The number of palindromic superdiagonal compositions*

(1) of $2n$ with $2k$ parts equals

$$s(2n,2k)=\binom{n-\binom{k+1}{2}-2\binom{k}{2}-1}{k-1},$$

(2) of either $2n$ or $2n-1$ with $2k-1$ parts equals

$$s(n,2k-1)=\binom{\lfloor\frac{n-3k^2}{2}\rfloor+2k-1}{k-1}.$$

Proof. From the proof of Theorem 2.1 we have

$$\begin{aligned}s(2n,2k)&=[x^{2n}]\frac{x^{3k^2+k}}{(1-x^2)^k}\\&=[x^{2n-3k^2-k}]\sum_{\ell\geq 0}\binom{k+\ell-1}{k-1}x^{2\ell}\\&=\binom{k+\frac{2n-3k^2-k}{2}-1}{k-1}\\&=\binom{n-\binom{k+1}{2}-2\binom{k}{2}-1}{k-1}.\end{aligned}$$

The combinatorial formula for $s(n,2k-1)$ is obtained in a similar manner. ☑

For example, $s(15,3)=s(15,2\cdot 2-1)=\binom{\lfloor\frac{15-3\cdot 2^2}{2}\rfloor+2\cdot 2-1}{2-1}=4$, the palindromic superdiagonal compositions being

$$(3,9,3),\quad (4,7,4),\quad (6,3,6),\quad (5,5,5).$$

3. Colored Superdiagonal Compositions

In this section we give the generating function for the total number of colored superdiagonal compositions. Remember that a composition $\sigma=(\sigma_1,\dots,\sigma_\ell)$ of n is a colored superdiagonal composition if $\sigma_i\geq i$ for all $1\leq i\leq\ell$, with the additional condition that if a part is of size i then it can come in one of i different colors. For example, $(3_2,2_1,5_3,5_2,6_6)$ is a colored superdiagonal composition of 21.

Let $c(n)$ denote the number of colored superdiagonal compositions of n. In Theorem 3.3 below we give the generating function for this sequence. We need some definitions and one lemma.

Given integers $n,k\geq 0$, let $\left[{n \atop k}\right]$ denote the (unsigned) *Stirling numbers of the first kind*, which are defined as connection constants in the polynomial identity

$$x(x+1)\cdots(x+(n-1))=\sum_{k=0}^{n}\left[{n \atop k}\right]x^k. \tag{2}$$

The Stirling numbers of the first kind give the number of permutations on n elements with k cycles. It is also known that this sequence satisfies the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \text{ for } n \geq 1 \text{ and } k \geq 1, \tag{3}$$

with the initial conditions $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 1$ and $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right] = 0$ for $n \geq 1$.

Let n be a non-negative integer. We introduce the polynomials defined by

$$Q_n(x) := \prod_{\ell=1}^{n}(\ell - (\ell-1)x), \quad n \geq 1, \tag{4}$$

with the initial value $Q_0(x) = 1$. The first six polynomials are

$$\begin{aligned}
&Q_0(x) = 1, \quad Q_1(x) = 1, \quad Q_2(x) = -x + 2, \quad Q_3(x) = 2x^2 - 7x + 6,\\
&Q_4(x) = -6x^3 + 29x^2 - 46x + 24,\\
&Q_5(x) = 24x^4 - 146x^3 + 329x^2 - 326x + 120,\\
&Q_6(x) = -120x^5 + 874x^4 - 2521x^3 + 3604x^2 - 2556x + 720.
\end{aligned}$$

Notice that $Q_n(x)$ is a polynomial of degree $n-1$. Moreover, from (4) we have

$$Q_n(x) = nQ_{n-1}(x) - (n-1)xQ_{n-1}(x), \quad n \geq 1.$$

Lemma 3.1. *The polynomials $Q_n(x)$ can be expressed as*

$$Q_n(x) = \sum_{k=0}^{n}(-1)^k T(n,k)x^k,$$

where

$$T(n,k) = \sum_{i=0}^{n-k}\binom{n-i}{k}\begin{bmatrix} n \\ i \end{bmatrix}.$$

Proof. From (2), we have the equality

$$\prod_{k=0}^{n-1}(x-k) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^{n-k}x^k.$$

Therefore, for all positive integer n we have

$$\begin{aligned}
Q_n(x) &= \prod_{\ell=1}^{n}(\ell - (\ell-1)x) = \prod_{\ell=1}^{n}(1 - (\ell-1)(x-1))\\
&= (x-1)^n \prod_{\ell=1}^{n}\left(\frac{1}{1-x} - \ell + 1\right) = (x-1)^n \prod_{\ell=0}^{n-1}\left(\frac{1}{1-x} - \ell\right)\\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^{n-k}(x-1)^{n-k}.
\end{aligned}$$

Hence

$$[x^k]Q_n(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{n-i}[x^k](x-1)^{n-i} = (-1)^k \sum_{i=0}^{n} \binom{n-i}{k} \begin{bmatrix} n \\ i \end{bmatrix}$$

and $T(n,k) = \sum_{i=0}^{n-k} \binom{n-i}{k} \begin{bmatrix} n \\ i \end{bmatrix}$. ☑

The first few values of the sequence $T(n,k)$ are

$$[T(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 7 & 2 & 0 & 0 & 0 & 0 \\ 24 & 46 & 29 & 6 & 0 & 0 & 0 \\ 120 & 326 & 329 & 146 & 24 & 0 & 0 \\ 720 & 2556 & 3604 & 2521 & 874 & 120 & 0 \\ 5040 & 22212 & 40564 & 39271 & 21244 & 6084 & 720 \end{pmatrix}$$

This array corresponds with the sequence A059364 in [12]. In Theorem 3.2 below we give a combinatorial interpretation for the sequence $T(n,k)$. We will assume that permutations are expressed in *standard cycle form*, i.e., minimal elements first within each cycle, with cycles arranged left-to-right in ascending order of minimal elements. For example, the permutation 146285937 has the standard cycle form (1)(24)(3658)(79). Notice that we also write the fixed points (cycles of size 1). The symbol $\bar{i}$ in a permutation is called an *overline element*.

Theorem 3.2. *The sequence $T(n,k)$ counts the number of permutations on n elements such that in the standard cycle form there are k non-minimal overline elements.*

Proof. Let π be a permutation on n elements with i cycles ($0 \leq i \leq n$). Notice that there are $\begin{bmatrix} n \\ i \end{bmatrix}$ of these permutations. The permutation π has i minimal elements. From the remaining $n-i$ elements, we have to choose the k overline elements in $\binom{n-i}{k}$ ways. Summing over i the count is now

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \binom{n-i}{k} = T(n,k).$$

☑

For example, $T(4,2)=29$, where the permutations are

$$\begin{array}{lllllll}
(1)(2\,\overline{3}\,\overline{4}), & (1)(2\,\overline{4}\,\overline{3}), & (1\,\overline{2})(3\,\overline{4}), & (1\,\overline{2}\,\overline{3})(4), & (1\,\overline{2}\,\overline{3}\,4), & (1\,\overline{2}\,3\,\overline{4}), & (1\,2\,\overline{3}\,\overline{4}),\\
(1\,\overline{2}\,\overline{4}\,3), & (1\,\overline{2}\,4\,\overline{3}), & (1\,2\,\overline{4}\,\overline{3}), & (1\,\overline{2}\,\overline{4})(3), & (1\,\overline{3}\,\overline{2})(4), & (1\,\overline{3}\,\overline{4}\,2), & (1\,\overline{3}\,4\,\overline{2}),\\
(1\,3\,\overline{4}\,\overline{2}), & (1\,\overline{3}\,\overline{4})(2), & (1\,\overline{3})(2\,\overline{4}), & (1\,\overline{3}\,\overline{2}\,4), & (1\,\overline{3}\,2\,\overline{4}), & (1\,3\,\overline{2}\,\overline{4}), & (1\,\overline{4}\,\overline{3}\,2),\\
(1\,\overline{4}\,3\,\overline{2}), & (1\,4\,\overline{3}\,\overline{2}) & (1\,\overline{4}\,\overline{2})(3), & (1\,\overline{4}\,\overline{3})(2), & (1\,\overline{4}\,\overline{2}\,3), & (1\,\overline{4}\,2\,\overline{3}), & (1\,4\,\overline{2}\,\overline{3}),\\
(1\,\overline{4})(2\,\overline{3}). & & & & & &
\end{array}$$

Theorem 3.3. *The generating function for the number of colored superdiagonal compositions is*

$$C(x)=\sum_{m\geq 0}\frac{x^{\binom{m+1}{2}}}{(1-x)^{2m}}Q_m(x).$$

Proof. Let $\sigma=(\sigma_1,\dots,\sigma_m)$ be a non-empty colored superdiagonal composition of n with m parts. If $\sigma_i=\ell$ with $\ell\geq i$, then σ_i contributes to the generating function the term ℓx^ℓ, for $\ell\geq i$ and $1\leq i\leq m$. Let $C_m(x)$ be the generating function of the colored superdiagonal compositions with m parts. Then we have the following expression

$$\begin{aligned}
C_m(x) &= \left(\sum_{i\geq 1} ix^i\right)\left(\sum_{i\geq 2} ix^i\right)\cdots\left(\sum_{i\geq m} ix^i\right)\\
&= \frac{x}{(1-x)^2}\frac{(2-x)x^2}{(1-x)^2}\cdots\frac{(m-(m-1)x)x^m}{(1-x)^2}\\
&= \frac{x^{\binom{m+1}{2}}}{(1-x)^{2m}}\prod_{\ell=1}^{m}(\ell-(\ell-1)x)\\
&= \frac{x^{\binom{m+1}{2}}}{(1-x)^{2m}}Q_m(x).
\end{aligned}$$

Finally, summing the last expression over $m\geq 0$, we get the desired result. ☑

From Theorem 3.3 and Lemma 3.1 we obtain the following corollary.

Corollary 3.4. *The number of colored superdiagonal compositions of* n *is given by*

$$c(n)=\sum_{m,\ell\geq 0}\binom{2m+\ell-1}{\ell}T\left(m,n-\binom{m+1}{2}-\ell\right).$$

The first few values of the sequence $c(n)$ are

$$1,\quad 1,\quad 2,\quad 5,\quad \mathbf{11},\quad 21,\quad 42,\quad 86,\quad 171,\quad 322,\quad 596,\dots$$

Notice that Equation (1) shows the colored superdiagonal compositions corresponding to the bold term in the above sequence. The sequence $c(n)$ does not appear in [12].

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