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A Study of Vector Measures

Un estudio de medidas vectoriales

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ABSTRACT. This article presents a fairly detailed exposition of Banach-space-valued measures or, in short, vector measures.

***Key words*:** Vector measure, measurability of vector-valued functions, modes of convergence, Bochner integral, Radon-Nikodým property.

RESUMEN. Este artículo presenta una exposición detallada de las medidas con valores en un espacio de Banach, o medidas vectoriales.

***Palabras clave*:** Medida vectorial, medibilidad de funciones con valores vectoriales, modos de convergencia, integral de Bochner, propiedad de Radon-Nikodým.

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1 Introduction

Vector measures are closely related to, and in many instances justified by, the study of Banach space theory. For instance, J. Diestel and J. J. Uhl, Jr. mention in their monograph [18], published in 1977, several early instances of the interplay between vector measures and Banach space theory. For the purpose of this article, the following paragraph, that highlights the origins of the Radon-Nikodým type theorems, is most relevant: “In 1936, J. A. Clarkson introduced the notion of uniform convexity to prove that absolutely continuous functions on a Euclidean space with values in a uniformly convex Banach space are the integrals of their derivatives. At the same time, Clarkson used vector measure theoretic ideas to prove that many familiar Banach spaces do not admit equivalent uniformly convex norms. N. Dunford and A. P. Morse, in 1936, introduced the notion of a boundedly complete basis to prove that absolutely continuous functions on a Euclidean space with values in a Banach space with a boundedly complete basis are the integrals of their derivatives. Shortly thereafter Dunford was able to recognize the Dunford-Morse theorem and the Clarkson theorem as genuine Radon-Nikodým theorems for the Bochner integral. This was the first Radon-Nikodým theorem for vector measures on abstract measure spaces".

In spite of these exciting beginnings, and except for a few notable instances, vector measures were quite forgotten in the forties and fifties. However, the monograph [19], written by N. Dinculeanu and published in 1967, inspired a renewed interest on the subject, particularly on the Radon-Nikodým theorem for the Bochner integral, and on the Orlicz-Pettis theorem (see, for instance, [18], Notes and Remarks, pp. 31-39). A search of the *Mathematical Reviews* for the relevant entries in the *2020 Mathematics Subject Classification*, shows that the study of vector measures is still of interest.

As for the organization of our exposition, it commences with a section that covers basic definitions and results on vector measures. The measurability of vector-valued functions is discussed in Section 3, and we dedicate Section 4 to the Bochner integral. In Section 5 we look at several modes of convergence, while in Section 6 we review the Radon-Nikodým property for the Bochner integral.

These sections amount to a fairly detailed study of vector measures. We prove quite a few results, and we provide abundant references, as well as numerous examples and plenty of commentary, including many of a historical nature.

The material covered in these sections uses results from the theory of real-valued measures that are found, for the most part, in any book on measure and integration. We refer often to [3] and [44]. Other sources are cited at the appropriate times.

2 Definitions and results pertaining to vector measures

We fix a measurable space (S, Σ) (see [3], p. 81), and a Banach space X with norm $\|\cdot\|$ which will always be real. In fact, all the linear spaces we consider will be real. This setting will suffice for much of our purposes. Reference [19], in particular, considers far more general settings.

Definition 1. A set function $m : \Sigma \to X$ is called a vector measure if

1. $m(\varnothing) = 0$.
2. For each pairwise disjoint family $\{A_j\}_{j\geq 1} \subseteq \Sigma$,

$$m\left(\bigcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} m(A_j). \tag{1}$$

Remark 1. The convergence of the series in (1) follows from the equality therein. Furthermore, for the same reason, the series converges unconditionally. However, a very deep result due to A. Dvoretsky and A. Rogers [21] asserts that unconditional convergence is equivalent to absolute convergence, exactly when the space X is a finite dimensional linear space. To be sure, absolute convergence implies unconditional convergence in any Banach space. The proof follows that of the real case (for the real case, see, for instance, [14], p. 45, Theorem 21).

Remark 2. When X in Definition 1 is the space $\mathbb{R}$, we refer to the vector measure m as signed measure. Let us mention that, generally, a signed measure is allowed to take one, and only one, of the values ∞ and $-\infty$ (see, for instance, [44], p. 20). However, in our context, a signed measure is always a particular case of a vector measure, that is, it has values in $\mathbb{R}$.

Following customary practice, we call a set function $m : \Sigma \to [0, \infty]$ satisfying the conditions in Definition 1, a measure. Let us observe that for a measure, the series on the right hand side of (1) may converge absolutely, or diverge to ∞.

In the few occasions in which we work with an specific measure, we will say so.

Remark 3. Definition 1 follows ([12], p. 357) and ([6], p. 99). Although we stick to this definition throughout, some references, for instance [18], make the distinction between countably additive vector measures, defined as in Definition 1, and finitely additive vector measures, where condition 2) is replaced by

2') For each finite and pairwise disjoint family $\{A_j\}_j \subseteq \Sigma$,

$$m\left(\bigcup_j A_j\right) = \sum_j m(A_j).$$

We refer to finitely additive vector measures as vector charges (for the real case, see [7]).

Let us observe that if m is a vector charge and A_1, A_2 are sets in Σ with $A_1 \subseteq A_2$, the equality $A_2 = A_1 \bigcup (A_2 \backslash A_1)$ implies that

$$m(A_2) - m(A_1) = m(A_2 \backslash A_1). \tag{2}$$

It should be clear that every vector measure is a vector charge. We will see shortly that the converse is not always true.

The proposition that follows collects several equivalent conditions for a vector charge to be a vector measure. These conditions appear in the literature, in various forms, when X is the space $\mathbb{R}$ of the real numbers.

Proposition 1. *Let $m : \Sigma \to X$ be a vector charge. Then, the following statements are equivalent:*

1. *The vector charge m is a vector measure.*
2. *If $\{A_j\}_{j\geq 1} \subseteq \Sigma$ and $A_j \subseteq A_{j+1}$ for all $j \geq 1$, then there exists*

$$\lim_{k\to\infty} m(A_k) = m\left(\bigcup_{j\geq 1} A_j\right).$$

3. *If* $\{A_j\}_{j\geq 1} \subseteq \Sigma$ *and* $A_{j+1} \subseteq A_j$ *for all* $j \geq 1$*, then there exists*

$$\lim_{k\to\infty} m(A_k) = m\left(\bigcap_{j\geq 1} A_j\right).$$

4. *If* $\{A_j\}_{j\geq 1} \subseteq \Sigma$*,* $A_{j+1} \subseteq A_j$ *for all* $j \geq 1$*, and* $\bigcap_{j\geq 1} A_j = \varnothing$*, then there exists*

$$\lim_{j\to\infty} m(A_j) = 0.$$

5. *If* $\{A_j\}_{j\geq 1} \subseteq \Sigma$ *are pairwise disjoint, then there exists*

$$\lim_{k\to\infty} m\left(\bigcup_{j\geq k} A_j\right) = 0.$$

Proof. To show that 1) $\Rightarrow$ 2), we fix $\{A_j\}_{j\geq 1} \subseteq \Sigma$ with $A_j \subseteq A_{j+1}$ for all $j \geq 1$, and we write

$$\bigcup_{j\geq 1} A_j = A_1 \bigcup \left(\bigcup_{j\geq 2} (A_j \backslash A_{j-1})\right).$$

If $B_1 = A_1$ and $B_j = A_j \backslash A_{j-1}$ for $j \geq 2$, the family $\{B_j\}_{j\geq 1} \subseteq \Sigma$ is pairwise disjoint and

$$\bigcup_{j\geq 1} B_j = \bigcup_{j\geq 1} A_j.$$

So,

$$\begin{aligned} m\left(\bigcup_{j\geq 1} A_j\right) &= m\left(\bigcup_{j\geq 1} B_j\right) = \sum_{j\geq 1} m(B_j) \\ &= \lim_{k\to\infty} \sum_{1\leq j\leq k} m(B_j) = m(A_1) + \lim_{k\to\infty} \sum_{2\leq j\leq k} (m(A_j) - m(A_{j-1})) \\ &= \lim_{k\to\infty} m(A_k). \end{aligned}$$

To prove 2) $\Rightarrow$ 3), we fix $\{A_j\}_{j\geq 1} \subseteq \Sigma$ with $A_{j+1} \subseteq A_j$ for all $j \geq 1$, and we write $B_j = A_1 \backslash A_j$, for all $j \geq 1$. Thus, $B_{j+1} \supseteq B_j$ for all $j \geq 1$. Then, according to 2), there exists

$$\lim_{k\to\infty} m(B_k) = m\left(\bigcup_{j\geq 1} B_j\right),$$

or, using (2),

$$\begin{aligned} m(A_1) - \lim_{k\to\infty} m(A_k) &= m\left(\bigcup_{j\geq 1} (A_1 \backslash A_j)\right) \\ &= m\left(A_1 \backslash \bigcap_{j\geq 1} A_j\right) = m(A_1) - m\left(\bigcap_{j\geq 1} A_j\right). \end{aligned}$$

It should be clear that 3) implies 4), since $m(\varnothing) = 0$.

To prove 4) $\Rightarrow$ 5), we fix any $\{A_j\}_{j\geq 1} \subseteq \Sigma$ pairwise disjoint, and we set $B_k = \bigcup_{j\geq k} A_j$. Then, $B_{k+1} \subseteq B_k$ for all $k \geq 1$ and

$$\bigcap_{k\geq 1} B_k = \bigcap_{k\geq 1} \bigcup_{j\geq k} A_j = \varnothing.$$

Thus, there exists

$$0 = \lim_{k\to\infty} m(B_k) = \lim_{k\to\infty} m\left(\bigcup_{j\geq k} A_j\right).$$

Finally, to prove 5) $\Rightarrow$ 1), we consider any family $\{A_j\}_{j\geq 1} \subseteq \Sigma$ pairwise disjoint. Then, for $k \geq 1$ fixed, using again (2),

$$\sum_{1\leq j\leq k} m(A_j) = m\left(\bigcup_{1\leq j\leq k} A_j\right) = m\left(\bigcup_{j\geq 1} A_j\right) - m\left(\bigcup_{j\geq k+1} A_j\right).$$

According to 5), there exists

$$\lim_{k\to\infty} m\left(\bigcup_{j\geq k+1} A_j\right) = 0,$$

so

$$m\left(\bigcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} m(A_j).$$

This completes the proof of the proposition. □

Example 1. *If I is the closed real interval $[0,1]$, we consider on I the Lebesgue measure space (see [3], p. 81, Example 1), consisting of the set I, the Lebesgue σ-algebra $\mathcal{L}_I$ on*

I, and the Lebesgue measure $\lambda : \mathcal{L}_I \to [0, \infty)$. Furthermore, for $1 \leq p < \infty$ fixed, we consider the Banach space $L^p(I)$, with the norm

$$\|f\|_{L^p(I)} = \left(\int_I |f(t)|^p \, dt \right)^{1/p}.$$

Then, we define the set function $m_p : \mathcal{L}_I \to L^p(I)$ as

$$m_p(A) = \overline{\chi}_A,$$

where $\overline{\chi}_A$ denotes the class in $L^p(I)$ of the characteristic function χ_A of the set $A \in \mathcal{L}_I$. We claim that m_p is a vector charge and, moreover, a vector measure. Indeed, it should be clear that $m_p(\varnothing) = 0$. As for condition 2) in Definition 1, if we fix $\{A_j\}_{j \geq 1} \subseteq \mathcal{L}_I$ pairwise disjoint,

$$\left\| m_p \left(\bigcup_{j \geq k} A_j \right) \right\|_{L^p(I)} = \left\| \overline{\chi}_{\bigcup_{j \geq k} A_j} \right\|_{L^p(I)} = \left(\lambda \left(\bigcup_{j \geq k} A_j \right) \right)^{1/p} \underset{k \to \infty}{\to} 0,$$

since $\left(\sum_{j \geq k} \lambda(A_j) \right)^{1/p} \to_{k \to \infty} 0$. Thus, according to Proposition 1, m_p is a vector measure.

The same set function, but with values in the Banach space $L^\infty(I)$ instead, is still a vector charge, denoted m_∞. In fact, if we fix any finite and pairwise disjoint family $\{A_j\}_j \subseteq \Sigma$,

$$m_\infty \left(\bigcup_j A_j \right) = \overline{\chi}_{\bigcup_j A_j} = \sum_j \overline{\chi}_{A_j} = \sum_j m_\infty(A_j).$$

However, m_∞ is not countably additive. We will prove it by showing that 4) in Proposition 1 does not hold. Indeed, let us consider the decreasing family of open intervals $\left\{ \left(0, \frac{1}{j}\right) \right\}_{j \geq 1}$. Since $\mathbb{R}$ is Archimedean, or, equivalently, since there are no non-zero infinitesimals in $\mathbb{R}$, we have

$$\bigcap_{j \geq 1} \left(0, \frac{1}{j}\right) = \varnothing.$$

So,

$$m_\infty \left(\bigcap_{j \geq 1} \left(0, \frac{1}{j}\right) \right) = 0.$$

However,

$$\left\| m_\infty \left(\left(0, \frac{1}{j}\right) \right) \right\|_{L^\infty(I)} = \left\| \overline{\chi}_{\left(0, \frac{1}{j}\right)} \right\|_{L^\infty(I)} = 1,$$

for all $j \geq 1$, showing that, indeed, 4) in Proposition 1 does not hold.

Proposition 2. *Every vector measure is bounded, meaning*

$$\sup_{A\in\Sigma}\|m(A)\|<\infty. \tag{3}$$

Proof. For each functional l in the topological dual X' of X, the composite set function $l\circ m:\Sigma\to\mathbb{R}$ is a signed measure. Thus, ([44], p. 30, Theorem 5), $l\circ m$ is a bounded signed measure, that is

$$\sup_{A\in\Sigma}|(l\circ m)(A)|<\infty,$$

where $|\cdot|$ denotes the absolute value in $\mathbb{R}$. So, if we apply Proposition 12 in ([43], p. 101), which follows from the Uniform Boundness Principle ([43], p. 95, Theorem 4), to the subset $\{m(A)\}_{A\in\Sigma}$ of X, we conclude that m is bounded, in the sense of (3).

This completes the proof of the proposition. □

Definition 2. The variation of the vector measure m, denoted $|m|$, is the set function $|m|:\Sigma\to[0,\infty]$ defined for each $A\in\Sigma$ as

$$|m|(A)=\sup\left\{\sum_j\|m(A_j)\|\right\}, \tag{4}$$

where the supremum is taken over all the finite partitions $\{A_j\}_j\subseteq\Sigma$, of A.

Remark 4. It should be clear that $|m|(\varnothing)=0$, so $|m|$ is never identically equal to infinity.

If we consider the partition consisting of one set, A itself, from (4) we get

$$\|m(A)\|\leq|m|(A), \tag{5}$$

for all $A\in\Sigma$.

Moreover, $|m|(A)=0$ for all $A\in\Sigma$ if, and only if, $m(A)=0$ for all $A\in\Sigma$.

When m is a signed measure, the variation $|m|$ can be defined as in Definition 2, or, equivalently, using the notions of positive variation and negative variation (see [3], pp. 84-85).

Lemma 1. *For a measure m, we have $|m|=m$, where $|m|$ is defined as in (4).*

Proof. If we fix $A\in\Sigma$,

$$|m|(A)=\sup_{\{A_j\}_j}\left\{\sum_j m(A_j)\right\}=m(A),$$

where $\{A_j\}_j\subseteq\Sigma$ is any finite partition of A. □

The variation of a vector measure is always a measure. We will prove this assertion shortly.

Definition 3. Given a vector measure m, the total variation of m is $|m|(S)$. We say that m has finite variation if $|m|(S) \in [0, \infty)$.

Example 2. *We claim that the vector measure m_1 defined in Example 1, has finite variation. In fact, if we fix $A \in \Sigma$ and $\{A_j\}_j \subseteq \Sigma$ is a finite partition of A, we have*

$$\sum_j \|m_1(A_j)\|_{L^1(I)} = \sum_j \left\|\overline{\chi}_{A_j}\right\|_{L^1(I)} = \sum_j \lambda(A_j) = \lambda(A).$$

Thus,

$$|m_1|(A) = \lambda(A).$$

In other words, the variation of m_1 is the Lebesgue measure λ.

In particular,

$$|m_1|(I) = 1.$$

Remark 5. As we will see in Remark 11, there are vector measures that do not have finite variation.

In ([19], pp. 32-33), it is observed that using partitions with more sets, does not alter Definition 2. Indeed,

Proposition 3. *If m is a vector measure, then, for each $A \in \Sigma$,*

$$|m|(A) = \sup\left\{\sum_{\alpha\in\Lambda} \|m(A_\alpha)\|\right\}, \tag{6}$$

where the supremum is taken over all the partitions $\{A_\alpha\}_{\alpha\in\Lambda} \subseteq \Sigma$ of A, for every set of indexes Λ.

Remark 6. Before going on to the proof of this proposition, let us recall that $\sum_{\alpha\in\Lambda} \|m(A_\alpha)\|$ is defined as

$$\sup_F \sum_{\alpha\in F} \|m(A_\alpha)\|, \tag{7}$$

where F varies over all the finite subsets of Λ. This generalized form of convergence is called *summability*.

The notion of summability for an arbitrary family $\{x_\alpha\}_{\alpha\in\Lambda}$ (see, for instance, [4], p. 138, Definition 9.1) was defined by E. H. Moore ([30], p. 63) as

$$\lim_F \sum\nolimits_{\alpha\in\Lambda} x_\alpha, \tag{8}$$

where the limit is taken with respect to the finite subsets of Λ, ordered, or directed, by inclusion. This is the Moore-Smith limit (see, for instance, [27], Chapter 2; [41]), defined by Moore and H. L. Smith in [31]. As a consequence of the definition of summability, x_α ends up being zero except for countably many values of α (see [4], p. 139, Theorem 9.1).

The Moore-Smith limit is used, for instance, to describe precisely the convergence of the Riemann sums (see, for instance, [2], Section 2). In our case, since we are working with a non-negative family, $\{\|m(A_\alpha)\|\}_{\alpha\in\Lambda}$, it should be clear that (7) is equivalent to the formal definition (8) of summability.

We are now ready to prove Proposition 3.

Proof. The idea of the proof is simple, although there are minor technical details involved.

Let v_1 be the set function defined by the right-hand side of (6). It should be clear that $v_1(A) \geq |m|(A)$, for all $A \in \Sigma$. To prove the opposite inequality, let us first assume that $\nu_1(A)$ is finite. Then, given $\varepsilon > 0$, there exists a partition $\{A_\alpha^\varepsilon\}_{\alpha\in\Lambda_\varepsilon} \subseteq \Sigma$, of A and a finite subset F_ε of Λ_ε, so that

$$
\begin{aligned}
(v_1(A)-\varepsilon)-\varepsilon \leq \left(\sum_{\alpha\in\Lambda_\varepsilon}\|m(A_\alpha^\varepsilon)\|\right)-\varepsilon \leq \sum_{\alpha\in F_\varepsilon}\|m(A_\alpha^\varepsilon)\| + \left\|m\left(A\backslash\bigcup_{\alpha\in F_\varepsilon}A_\alpha^\varepsilon\right)\right\| \\
\leq |m|(A).
\end{aligned}
$$

Invoking once again the Archimedean nature of the real numbers, we conclude that

$$
v_1(A) \leq |m|(A).
$$

If $\nu_1(A)$ is infinite, given $M > 0$, there exists a partition $\left\{A_\alpha^M\right\}_{\alpha\in\Lambda_M} \subseteq \Sigma$, of A, so that

$$
M \leq \sum_{\alpha\in\Lambda_M}\left\|m(A_\alpha^M)\right\|.
$$

If $\sum\limits_{\alpha\in\Lambda_M}\left\|m(A_\alpha^M)\right\| = \infty$, then, for a finite subset F_M of Λ_M,

$$
\begin{aligned}
M &\leq \sum_{\alpha\in F_M}\left\|m(A_\alpha^M)\right\| \leq \sum_{\alpha\in F_M}\left\|m(A_\alpha^M)\right\| + \left\|m\left(A\backslash\bigcup_{\alpha\in F_M}A_\alpha^M\right)\right\| \\
&\leq |m|(A).
\end{aligned}
$$

So,

$$
\nu_1(A) = \infty = |m|(A).
$$

If, on the other hand, $\sum_{\alpha\in\Lambda_M}\left\|m(A_\alpha^M)\right\|$ is finite, given $\varepsilon > 0$, there exists a finite subset $F_{M,\varepsilon}$ of Λ_M so that

$$
\begin{aligned}
M-\varepsilon &\leq \left(\sum_{\alpha\in\Lambda_M}\left\|m(A_\alpha^M)\right\|\right)-\varepsilon \leq \sum_{\alpha\in F_{M,\varepsilon}}\left\|m(A_\alpha^M)\right\| \\
&\leq \sum_{\alpha\in F_M}\left\|m(A_\alpha^M)\right\| + \left\|m\left(A\backslash\bigcup_{\alpha\in F_{M,\varepsilon}}A_\alpha^M\right)\right\| \leq |m|(A).
\end{aligned}
$$

Once again,

$$\nu_1 (A) = \infty = |m| (A) .$$

This completes the proof of the proposition. □

Corollary 1. If m is a vector measure, then, for each $A \in \Sigma$,

$$|m| (A) = \sup \left\{ \sum_{j \geq 1} \|m(A_j)\| \right\} , \tag{9}$$

where the supremum is taken over all the countable partitions $\{A_j\}_{j\geq 1} \subseteq \Sigma$, of A.

Proof. Let us begin by observing that if we fix a partition $\{A_j\}_{j\geq 1} \subseteq \Sigma$, of A, then

$$\sum_{j \geq 1} \|m(A_j)\| = \sup_F \sum_{j \in F} \|m(A_j)\| , \tag{10}$$

where the left-hand side is interpreted, in the usual way, as the limit of the partial sums, while in the right-hand side F varies over all the finite subsets of $\mathbb{N}$, as in (7). That is to say, the right-hand side of (10) is interpreted in the sense of summability mentioned in Remark 6. The equality (10) holds because the convergence of the series $\sum_{j\geq 1} \|m(A_j)\|$ is equivalent, with the same sum, to its convergence in the sense of summability given by (7) (see, for instance, [23]). Let us emphasize that we are considering here sums with a countable number of terms.

Thus, if v_2 denotes the set function defined by the right-hand side of (9), we have

$$|m| (A) = v_1 (A) \geq v_2 (A) \geq |m| (A) ,$$

for all $A \in \Sigma$.

This completes the proof of the corollary. □

Remark 7. For a signed measure m, it is true (see [44], p. 30, Proposition 7 (iii)) that

$$|m| (A) = \sup \{ |m (B)| + |m (A \backslash B)| : B \in \Sigma, B \subseteq A \} . \tag{11}$$

However, the corresponding equality for a vector measure,

$$|m| (A) = \sup \{ \|m (B)\| + \|m (A \backslash B)\| : B \in \Sigma, B \subseteq A \} , \tag{12}$$

is not true, in general. Indeed, if it were true, as a consequence of (5) and (12), we could prove quite easily that

$$\sup \{ \|m (B)\| : B \in \Sigma, B \subseteq A \} \leq |m| (A) \leq 2 \sup \{ \|m (B)\| : B \in \Sigma, B \subseteq A \} .$$

In particular, we would have

$$\sup \{ \|m (B)\| : B \in \Sigma \} \leq |m| (S) \leq 2 \sup \{ \|m (B)\| : B \in \Sigma \} . \tag{13}$$

That is, we would have, for any vector measure, the equivalence between being bounded and having finite variation. This is not generally true, according to Proposition 2 and Remark 5. However, it is true when m is a signed measure (see [44], p. 31, Corollary 8).

Let us observe that the proof of (11) is based on the Jordan decomposition of m ([44], p. 30, Theorem 6) as the difference of two non-negative signed measures.

Now, we are ready to prove that the variation of a vector measure is, indeed, a measure.

Proposition 4. *The variation $|m|$ of a vector measure m is a measure.*

Proof. We prove this proposition by adapting the proof of Theorem 6.2 in ([39], p. 117).

It is clear that $|m|(\varnothing) = 0$. As for the countable additivity, let us fix any pairwise disjoint family $\{A_j\}_{j\geq 1} \subseteq \Sigma$. If $t_j \in \mathbb{R}$ satisfies $t_j < |m|(A_j)$, then, by the definition of $|m|$ and Corollary 1, there exists a countable partition $\{A_{j,k}\}_{k\geq 1} \subseteq \Sigma$, of A, so that

$$t_j < \sum_{k\geq 1} \|m(A_{j,k})\|,$$

for each $j \geq 1$.

Now, the family $\{A_{j,k}\}_{j,k\geq 1} \subseteq \Sigma$, is a countable partition of A. So,

$$\sum_{j\geq 1} t_j \leq \sum_{j\geq 1} \left(\sum_{k\geq 1} \|m(A_{j,k})\| \right) \underset{\text{(i)}}{=} \sum_{j,k\geq 1} \|m(A_{j,k})\| \leq |m|(A),$$

where the passage from iterated summation to double summation in equality (i) above, is a discrete version (see, for instance, [14], p. 46, Proposition 22) of Fubini's theorem for non-negative functions. So, we end up obtaining the inequality

$$\sum_{j\geq 1} t_j \leq |m|(A). \tag{14}$$

Now, taking the supremum of the left-hand side of (14) over all the suitable sequences $\{t_j\}_{j\geq 1}$, we obtain

$$\sum_{j\geq 1} |m|(A_j) \leq |m| \left(\bigcup_{j\geq 1} A_j \right).$$

To prove the converse direction, we consider an arbitrary countable partition $\{B_k\}_{k\geq 1} \subseteq \Sigma$ of $\bigcup_{j\geq 1} A_j$. Then, the family $\{A_j \cap B_k\}_{k\geq 1}$ is a countable partition of A_j and likewise, $\{A_j \cap B_k\}_{j\geq 1}$ is a countable partition of B_k. Thus,

$$\begin{aligned} \sum_{k\geq 1} \|m(B_k)\| &= \sum_{k\geq 1} \left\| \sum_{j\geq 1} m\left(A_j \bigcap B_k\right) \right\| \leq \sum_{k\geq 1}\sum_{j\geq 1} \left\| m\left(A_j \bigcap B_k\right) \right\| \\ &\underset{\text{(i)}}{=} \sum_{j\geq 1}\sum_{k\geq 1} \left\| m\left(A_j \bigcap B_k\right) \right\| \\ &\leq \sum_{j\geq 1} |m|(A_j), \end{aligned}$$

where the equality (i) above follows, again, from the discrete version of Fubini's theorem for non-negative functions.

Consequently,

$$|m|\left(\bigcup_{j\geq 1} A_j\right) = \sup_{\{B_k\}_{k\geq 1}} \sum_{k\geq 1} \|m(B_k)\| \leq \sum_{j\geq 1} |m|(A_j).$$

This completes the proof of the proposition. □

Definition 4. We say that a vector measure m has σ-finite variation if the measure $|m|$ has σ-finite variation. That is (see [3], p. 87, Definition 10), if there exists a countable covering $\{A_j\}_{j\geq 1}$ of S, so that $|m|(A_j)$ is finite for all $j \geq 1$. In other words, if the restriction $|m|/A_j$ of $|m|$ to A_j (see [3], p. 82, Definition 5), has finite variation.

Remark 8. Let us observe that $|m|/A = |m/A|$ for each $A \in \Sigma$. The proof of this assertion follows the proof for the real case (for the real case, see [3], p. 92, Lemma 4).

Remark 9. Of course, a vector measure of finite variation is trivially of σ-finite variation. We discuss in Remark 11 below, a non-trivial example of a vector measure of σ-finite variation.

Remark 10. As we have said before, the set functions we consider in this article will always have values in a real Banach space. Just to understand what might be different in the complex case, let us suppose, for a moment, that we fix a set function $m : \Sigma \to \mathbb{C}$, where $\mathbb{C}$ is the complex space of complex numbers. The function m is called a complex measure (see [39], p. 116), if $m(\varnothing) = 0$ and, for each countable and pairwise disjoint family $\{A_j\}_{j\geq 1} \subseteq \Sigma$,

$$m\left(\bigcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} m(A_j). \tag{15}$$

As in Definition 1, the convergence in $\mathbb{C}$ of the series in (15), is part of the definition. That is, it follows from the equality. As a consequence, the series converges unconditionally . Since $\mathbb{C}$ is a finite dimensional linear space, the series converges absolutely, as well.

The variation $|m|$ of a complex measure m is defined exactly as in Definition 2. However, it can be shown (see [39], p. 118, Theorem 6.4), that the resulting set function $|m|$, takes always finite values, in other words, $|m|(S) < \infty$. An examination of the proof of this result, shows how vital is to use the complex structure of $\mathbb{C}$.

Remark 11. Later on, we will work, for the most part, with vector measures of finite variation. To understand better the meaning of this assumption, we discuss now several examples.

We begin by developing an example suggested in ([12], p. 357, Appendix E, Exercise 5), of a vector measure that does not have finite variation.

We consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, where $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, ...\}$ and $\mathcal{P}(\mathbb{N})$ is the σ-algebra of all the subsets of $\mathbb{N}$. If l^2 is the Banach space of square-summable real sequences, we define $m : \mathcal{P}(\mathbb{N}) \to l^2$ as follows: Given $A \in \mathcal{P}(\mathbb{N})$, $m(A)$ is the sequence

$$k \to \begin{cases} \frac{1}{k} & \text{if } k \in A, \\ 0 & \text{if } k \notin A. \end{cases}$$

In other words,

$$m(A) = \left\{ \frac{1}{k} \chi_A(k) \right\}_{k \geq 1}.$$

The set function m defined in this manner, is a vector measure. Indeed, it should be clear that $m(\varnothing) = 0$. As for being countably additive, if $\{A_j\}_{j\geq 1} \subseteq \mathcal{P}(\mathbb{N})$ is pairwise disjoint,

$$\begin{aligned} m\left(\bigcup_{j\geq 1} A_j \right) &= \left\{ \frac{1}{k} \chi_{\bigcup_{j\geq 1} A_j}(k) \right\}_{k\geq 1} = \left\{ \frac{1}{k} \sum_{j\geq 1} \chi_{A_j}(k) \right\}_{k\geq 1} \\ &= \sum_{j\geq 1} m(A_j). \end{aligned}$$

Let us see now that this vector measure m is not of finite variation.

For $N \in \mathbb{N}$ fixed, the sets $\{j\}$ for $1 \leq j \leq N$ and $B_N = \{N+1, N+2, ...\}$ form a finite partition of $\mathbb{N}$. Then,

$$|m|(\mathbb{N}) \geq \sum_{1\leq j\leq N} \frac{1}{j} + \|m(B_N)\|_{l^2},$$

for every $N \in \mathbb{N}$, which implies that $|m|(\mathbb{N}) = \infty$.

Next, we consider a modification of an example presented in ([19], p. 40).

Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ be the measurable space considered before and let $\mathfrak{B}$ be the real Banach space of bounded functions $f : \mathbb{R} \to \mathbb{R}$, with the norm of the supremum. We define $m : \mathcal{P}(\mathbb{N}) \to \mathfrak{B}$ as

$$m(A) = \chi_A.$$

It should be clear that m is a vector measure and that

$$\|m(A)\|_{\mathfrak{B}} = 1,$$

for every $A \in \mathcal{P}(\mathbb{N})$. Moreover, if $A \in \mathcal{P}(\mathbb{N})$

$$|m|(A) \geq \sum_{j\in A} \|m(\{j\})\|_{\mathfrak{B}}.$$

That is to say, given $A \in \mathcal{P}(\mathbb{N})$, $|m|(A)$ equals infinity if, and only if, A is infinite.

We continue with two examples mentioned in p. 99 of [6].

For the first example, we fix again the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and we select a sequence $\{a_j\}_{j\geq 1}$ in X, so that the series $\sum_{j\geq 1} a_j$ converges unconditionally in X. Then, we define the set function $m : \mathcal{P}(\mathbb{N}) \to X$ as

$$m(A) = \begin{cases} \sum_{j\in A} a_j & \text{if } A \neq \varnothing \\ 0 & \text{if } A = \varnothing \end{cases}. \tag{16}$$

Thus defined, m is a vector measure. We claim that m has finite variation if, and only if, the series $\sum_{j\geq 1} a_j$ converges absolutely. In fact, let us start by assuming that the series $\sum_{j\geq 1} a_j$ converges absolutely. Then, given a finite partition $\{A_k\}_k \subseteq \mathcal{P}(\mathbb{N})$ of $\mathbb{N}$,

$$\sum_k \|m(A_k)\| = \sum_k \left\| \sum_{j\in A_k} a_j \right\| \leq \sum_{j\geq 1} \|a_j\|,$$

which implies that

$$|m|(\mathbb{N}) \leq \sum_{j\geq 1} \|a_j\|.$$

That is, m has finite variation.

Conversely, if m has finite variation,

$$\infty > |m|(\mathbb{N}) \geq \sum_{j=1}^{N} \|a_j\| + \left\| \sum_{j\geq N+1} a_j \right\| \geq \sum_{j=1}^{N} \|a_j\|,$$

for all $N \geq 1$, showing that the series $\sum_{j\geq 1} a_j$ converges absolutely.

If X has infinite linear dimension, according to Remark 1, the series $\sum_{j\geq 1} a_j$ might not converge absolutely. If this is the case, we can write, for instance, $\mathbb{N} = \bigcup_{j\geq 1} \{1, 2, ..., j\}$, so m has σ-finite variation, although it does not have finite variation.

The second example involves the vector measure m_p defined in Example 1 for $1 \leq p < \infty$. We claim that m_p has finite variation if, and only if, $p = 1$. In fact, let us start by observing that for $A \in \mathcal{L}_I$,

$$|m_p|(A) = \sup_{\{A_j\}_j} \left\{ \sum_j \|m_p(A_j)\|_{L^p(I)} \right\} = \sup_{\{A_j\}_j} \left\{ \sum_j (\lambda(A_j))^{1/p} \right\}. \tag{17}$$

That m_1 has finite variation was shown in Example 2.

If $1 < p < \infty$, let us fix $0 < \alpha < 1$ so that $\alpha p > 1$. Moreover, for $N \geq 2$, let us consider a family $\{I_j\}_{1 \leq jN}$ of N disjoint subintervals of I, each of length $\frac{1}{N^{\alpha p}}$. We observe that

$$\lambda \left(\bigcup_{1 \leq j \leq N} I_j \right) = \sum_{j=1}^{N} \lambda (I_j) = N^{1-\alpha p} < 1,$$

so, the family $\{I_j\}_{1 \leq j \leq N}$ jointly with the set $I \setminus \bigcup_{1 \leq j \leq N} I_j$ is a finite partition of I. According to (17),

$$\begin{aligned} |m_p| (I) &\geq \sum_{j=1}^{N} (\lambda (I_j))^{1/p} + \left(\lambda \left(I \setminus \bigcup_{1 \leq j \leq N} I_j \right) \right)^{1/p} \\ &\geq \sum_{j=1}^{N} (\lambda (I_j))^{1/p} = N \left(N^{-\alpha} \right) = N^{1-\alpha}, \end{aligned}$$

for all $N \geq 2$. Thus,

$$|m_p| (I) = \infty.$$

Actually, we can say quite a bit more about this last example. Indeed, as mentioned in ([6], (ii) in p. 99, and [18], p. 7, Example 16), for $1 < p < \infty$, $|m_p| (A) = \infty$ for every set $A \in \mathcal{L}_I$ with positive Lebesgue measure.

To see why this is the case, we need to introduce a few definitions and results, which we take from [3] and other sources. For more on the subject, we refer to Section 5 in [3] and the references therein.

Definition 5. A measure $\nu : \Sigma \to [0, \infty]$ is continuous if the singletons belong to Σ and they are ν-null, that is if $\{x\} \in \Sigma$ and $\nu (\{x\}) = 0$, for all $x \in S$.

Let us observe that the Lebesgue measure is continuous.

Remark 12. For future reference, let us say that Definition 5 carries over, without changes, to the case of a vector measure.

Definition 6. ([20], p. 645) Given a measure $\nu : \Sigma \to [0, \infty]$ and given $A \in \Sigma$, we say that A is a ν-atom if $\nu (A) > 0$ and for every Σ-measurable set $B \subseteq A$, is $\nu (B) = 0$ or $\nu (B) = \nu (A)$. We say that a measure is atomless if it does not have atoms.

Measures without atoms are usually called *non-atomic* or *not atomic*. However, the word "non-atomic" is used, with a different meaning, in measure theory (see, for instance, [12], p. 290), and the words "not atomic" have an specific meaning in computer programming [45].

If the singletons belong to Σ, every atomless measure ν is continuous. However, the converse is not generally true (see [3], p. 95, Remark 11).

Proposition 5. *Consider the Lebesgue measure space* $(\mathbb{R}^n, \mathcal{L}, \lambda)$ *(see [3], p. 81, Example 1). Given* $A \in \mathcal{L}$ *with* $\lambda (A) > 0$*, there exists an* $\mathcal{L}$*-measurable set* $B \subseteq A$ *so that* $0 < \lambda (B) < \lambda (A)$.

For the proof of this proposition, see p. 95 in [3].

Corollary 2. The Lebesgue measure is atomless.

Proposition 5 has a very interesting extension for finite measures that are atomless.

Proposition 6. *Let* (S, Σ) *be a measurable space. If* $\nu : \Sigma \to [0, \infty)$ *is a non identically zero atomless finite measure, for each real number c,* $0 < c < \nu(S)$*, there exists* $A \in \Sigma$ *so that* $\nu(A) = c$*.*

What this proposition says is that certain measures have the intermediate value property. For the somewhat lengthy proof, we refer to ([20], p. 645). The interesting point is that, for the class of atomless and finite measures, the word continuous should be interpreted as meaning that the measure takes a continuum of values.

The first version of Proposition 6 was proved by W. Sierpiński, in an article published in 1922 [40]. For a detailed account of Sierpiński's result, see p. 96 in [3].

Proposition 6 can be easily extended (see [3], p. 96, Remark 12) to non-identically zero and atomless σ-finite measures (see Definition 4).

Proposition 6 is not true, in general, when the measure ν has atoms (see [3], p. 96).

For much more on these matters, see ([19], p. 25, Section 2.9), where the property described in Proposition 6, is called, aptly, Darboux property.

We are now ready to take up the last assertion made in Remark 11.

Let us fix $A \in \mathcal{L}_I$ with $\lambda(A) > 0$ and let $a = \lambda(A)$. We claim that given $N \geq 1$, there is a finite partition $\{A_j\}_{1 \leq j \leq N} \subseteq \mathcal{L}_I$ of A, so that $\lambda(A_j) = \frac{a}{N}$.

Indeed, for $N = 1$, we can take $A_1 = A$. Next, according to Proposition 6, there exists an $\mathcal{L}_I$-measurable subset A_1 of A so that $\lambda(A_1) = \frac{a}{2}$. Since $\lambda(A \backslash A_1) = \frac{a}{2}$, the sets A_1 and $A \backslash A_1$ give the partition, for $N = 2$. If $N = 3$, it should be clear that we can pick $A_1 \subseteq A$ and $A_2 \subseteq A \backslash A_1$ so that the sets A_1, A_2 and $(A \backslash A_1) \backslash A_2$ give the partition. And so on. Then,

$$|m_p|(A) \geq \sum_{j=1}^{N} (\lambda(A_j))^{1/p} = a^{1/p} \sum_{j=1}^{N} N^{-1/p} = a^{1/p} N^{1-1/p},$$

so, $|m_p|(A) = \infty$.

As a consequence, the measure $|m_p|$ is not σ-finite.

Before stating the next result, let us observe that given vector measures m_1 and m_2 and real numbers α, β, by $\alpha m_1 + \beta m_2$ we mean the vector measure defined as

$$(\alpha m_1 + \beta m_2)(A) = \alpha(m_1(A)) + \beta(m_2(A)),$$

for every $A \in \Sigma$. It should be clear that

$$|\alpha m| = |\alpha|_{\mathbb{R}} |m|, \tag{18}$$

where $|\alpha|_{\mathbb{R}}$ is the absolute value of the real number α, and $|m|$ is the variation of m.

Lemma 2. *Given two vector measures* $m_1, m_2 : \Sigma \to X$*,*

$$|m_1 + m_2| \leq |m_1| + |m_2| \,. \tag{19}$$

Proof. Given $A \in \Sigma$ and given an arbitrary finite partition $\{A_j\}_j \subseteq \Sigma$, of A,

$$\begin{aligned} \sum_j \|(m_1 + m_2)(A_j)\| &\leq \sum_j \|m_1(A_j)\| + \sum_j \|m_2(A_j)\| \\ &\leq |m_1|(A) + |m_2|(A) \,. \end{aligned}$$

Thus, (19) holds.

This completes the proof of the lemma. □

Then, the following result should be clear:

Proposition 7. *The space* $\mathcal{M}_f$ *of all the vector measures* $m : \Sigma \to X$ *of finite variation is a normed linear space with the norm*

$$\|m\| = |m|(S) \,.$$

Definition 7. Two vector measures $m_1, m_2 : \Sigma \to X$ are mutually singular, denoted $m_1 \perp m_2$, if the measures $|m_1|$ and $|m_2|$ are mutually singular, also denoted $|m_1| \perp |m_2|$. That is to say, if there is a partition $S = E \bigcup F$, $E, F \in \Sigma$, such that $|m_1|(F) = 0$ and $|m_2|(E) = 0$. In particular, according with Lemma 1, a vector measure $m : \Sigma \to X$ and a measure $\nu : \Sigma \to [0, \infty]$ are mutually singular if the measures $|m|$ and ν are mutually singular.

Lemma 3. *Let* $m_1, m_2 : \Sigma \to X$ *be vector measures. If* $m_1 \perp \mu$ *and* $m_2 \perp \mu$*, then* $m_1 + m_2 \perp \mu$*.*

Proof. According to Definition 7, there are partitions $S = A \bigcup B$ and $S = E \bigcup F$ so that

$$\begin{aligned} |m_1|(A) &= \mu(B) = 0, \\ |m_2|(E) &= \mu(F) = 0. \end{aligned}$$

Then, let us consider the partition

$$S = \left(A \bigcap E\right) \bigcup \left(B \bigcup F\right)$$

So,

$$|m_1 + m_2| \left(A \bigcap E\right) \leq |m_1| \left(A \bigcap E\right) + |m_2| \left(A \bigcap E\right) = 0$$

while

$$\mu \left(B \bigcup F\right) \leq \mu(B) + \mu(F) = 0.$$

This completes the proof of the lemma. □

Lemma 4. *Let $m : \Sigma \to X$ be a vector measure and let $A \in \Sigma$. Then, the following statements are equivalent:*

1. $|m|(A) = 0$.
2. $m(A') = 0$ *for all* $A' \subseteq A$, $A' \in \Sigma$.

Proof. If $m(A') \neq 0$ for some $A' \subseteq A$, $A' \in \Sigma$,

$$|m|(A) \geq \|m(A')\| + \|m(A \backslash A')\| > 0.$$

Conversely, if 2) holds, from Definition 2, it should be clear that 1) holds.

This completes the proof of the lemma. □

As an immediate consequence of Lemma 4, we can state the following result:

Lemma 5. *Two vector measures m_1 and m_2 are mutually singular if, and only if, there exists a partition $S = A \bigcup B$, $A, B \in \Sigma$, such that*

$$\begin{aligned} m_1(B') &= 0 \text{ for all } B' \subseteq B,\ B' \in \Sigma, \\ m_2(A') &= 0 \text{ for all } A' \subseteq A,\ A' \in \Sigma. \end{aligned}$$

Lemma 5 allows us to restate Definition 7 with no reference to the variation of the vector measures m_1 and m_2.

Lemma 6. *If two vector measures $m_1, m_2 : \Sigma \to X$ are mutually singular,*

$$|m_1 + m_2| = |m_1| + |m_2|.$$

Proof. From Lemma 2, it suffices to prove that if m_1 and m_2 are mutually singular,

$$|m_1 + m_2| \geq |m_1| + |m_2|. \tag{20}$$

According to Definition 7, we can find a partition $X = E \cup F$, with $E, F \in \Sigma$, such that $|m_1|(F) = 0$ and $|m_2|(E) = 0$. It follows that if $A \in \Sigma$,

$$|m_1|(A) = |m_1|(A \bigcap E)$$

and

$$|m_2|(A) = |m_2|(A \bigcap F).$$

To prove (20), it will suffice to consider the following three cases: $|m_1|(A)$ and $|m_2|(A)$ are both finite, $|m_1|(A)$ is finite but $|m_2|(A) = \infty$, and $|m_1|(A) = |m_2|(A) = \infty$. We now prove (20) assuming that $|m_1|(A)$ is finite but $|m_2|(A) = \infty$.

So, given $\varepsilon > 0$ there is a partition$\{C_i\}_i$ of $A \cap E$ such that

$$\sum_i \|m_1(C_i)\| \geq |m_1|\,(A \cap E) - \varepsilon.$$

Likewise, given $M \geq 1$ there is a partition $\{D_j\}_j$ of $A \cap F$ such that

$$\sum_j \|m_2(D_j)\| \geq M.$$

So,

$$\begin{aligned} |m_1 + m_2|(A) &\geq \sum_i \|(m_1 + m_2)\,(C_i)\| + \sum_j \|(m_1 + m_2)\,(D_j)\| \\ &= \sum_i \|m_1(C_i)\| + \sum_j \|m_2(D_j)\| \\ &\geq |m_1|\,(A \cap E) - \varepsilon + M \end{aligned}$$

for every $\varepsilon >$ and every $M \geq 1$. Thus, we can conclude that

$$|m_1 + m_2|(A) = \infty = |m_1|(A) + |m_2|(A).$$

The proof of the other cases is similar.

This completes the proof of the lemma. □

Definition 8. Given two vector measures $m_1, m_2 : \Sigma \to X$, we say that m_1 is absolutely continuous with respect to m_2, denoted $m_1 \ll m_2$, if the measure $|m_1|$ is absolutely continuous with respect to the measure $|m_2|$, denoted $|m_1| \ll |m_2|$. This means (see [3], p. 86, Definition 8),

if $A \in \Sigma$ and $|m_2|\,(A) = 0$, then $|m_1|\,(A) = 0$ as well.

In particular, a vector measure $m : \Sigma \to X$ is absolutely continuous with respect to a measure $\mu : \Sigma \to [0, \infty]$ if $|m| \ll \mu$.

Remark 13. According to Lemma 4, $m_1 \ll m_2$ if, and only if, $A \in \Sigma$ and $m_2\,(A') = 0$ for all $A' \subseteq A$, $A' \in \Sigma$, implies that $m_1\,(A') = 0$ for all $A' \subseteq A$, $A' \in \Sigma$. That is to say, to define absolute continuity, we do not need to use the variation of the vector measures m_1 and m_2.

Definition 9. Given two vector measures $m_1, m_2 : \Sigma \to X$, we say that m_1 is m_2-continuous if there exists

$$\lim_{|m_2|(A) \to 0} m_1\,(A) = 0.$$

That is, for each $\varepsilon > 0$, there exists $\delta = \delta_\varepsilon > 0$ such that $A \in \Sigma$ and $|m_2|\,(A) < \delta$ imply $\|m_1\,(A)\| < \varepsilon$.

The following proposition compares absolute continuity of m_1 with respect of m_2 and m_2-continuity of m_1:

Proposition 8. *Let $m_1, m_2 : \Sigma \to X$ be vector measures.*

1. *If m_1 is m_2-continuous, then m_1 is absolutely continuous with respect to m_2.*
2. *If m_1 has finite variation, then m_1 is m_2-continuous if, and only if, m_1 is absolutely continuous with respect to m_2.*

Proof. To prove 1), let us fix $A \in \Sigma$ so that $|m_2|(A) = 0$. Then, for any $B \in \Sigma$, $B \subseteq A$, we have $\|m_1(B)\| < \varepsilon$, for every $\varepsilon > 0$. The Archimedean nature of $\mathbb{R}$ implies that $\|m_1(B)\| = 0$, so $m_1(B) = 0$. According to Lemma 4, $|m_1|(A) = 0$.

To prove 2), we only need to show that if m_1 is absolutely continuous with respect to m_2 then, m_1 is m_2-continuous. The proof of this implication reduces to work with the measure $|m_1|$ and the finite measure $|m_2|$, so we are back to the real case (for the proof of the real case see, for instance, [44], p. 132, Proposition 5 (ii)). □

Remark 14. Even for real-valued measures, 2) in Proposition 8 does not generally hold, when m_1 does not have finite variation (see, for instance, [44], p. 133, Example 6).

Proposition 9. *Let $m_1, m_2 : \Sigma \to X$ be vector measures. If $m_1 \ll m_2$ and $m_1 \perp m_2$, then m_1 is identically zero.*

Proof. Since m_1 and m_2 are mutually singular, there is a partition $S = E \bigcup F$, $E, F \in \Sigma$, such that $|m_1|(F) = 0$ and $|m_2|(E) = 0$. Since $m_1 \ll m_2$, we also have that $|m_1|(E) = 0$. Therefore, $|m_1|(A) = 0$ for all $A \in \Sigma$ or, equivalently, m_1 is the identically zero measure.

This completes the proof of the proposition. □

The use of the symbol $\perp$ in Definition 7, suggests a connection with some notion of orthogonality. This is, indeed, the case. We begin with the following definition:

Definition 10. ([26], p. 292) Given a real normed linear space $(X, \|\cdot\|)$, and given $u, v \in X$, we say that u is orthogonal to v, denoted $u \perp v$, if

$$\|u + v\| = \|u - v\| . \tag{21}$$

Let us observe that this definition gives a symmetric relation in u and v, so we can say that u and v are orthogonal.

The following result justifies the use of the word "orthogonal" in Definition 10.

Proposition 10. *([26], p. 292) Let $(X, \langle , \rangle)$ be a real inner product linear space. Then, given $u, v \in X$, $v \neq 0$, the following statements are equivalent:*

1. $\|u + v\| = \|u - v\|$.

2. $\langle u, v\rangle = 0$.

Proof. To begin, we observe that $\|\cdot\|$ is the norm associated with the inner product $\langle , \rangle$. That is to say,

$$\|u\|^2 = \langle u, u\rangle .$$

Now,

$$\begin{aligned}\|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle - \langle u,u\rangle + 2\langle u,v\rangle - \langle v,v\rangle \\ &= 4\langle u,v\rangle .\end{aligned}$$

Thus, $\|u+v\|^2 - \|u-v\|^2 = 0$ if, and only if, $\langle u, v\rangle = 0$.

This completes the proof of the proposition. □

Proposition 11. *If $m_1, m_2 \in \mathcal{M}_f$ and $m_1 \perp m_2$ in the sense that m_1 and m_2 are mutually singular, then $m_1 \perp m_2$ in the sense of Definition 10. However, the converse direction is not generally true, even in the real-valued case.*

Proof. According to Lemma 6, if m_1 and m_2 are mutually singular, then, $|m_1 + m_2| = |m_1| + |m_2|$. Likewise, $|m_1 - m_2| = |m_1| + |m_2|$. Thus,

$$\|m_1 + m_2\|_{\mathcal{M}_f} = \|m_1 - m_2\|_{\mathcal{M}_f} .$$

As for the converse, let $(I, \mathcal{L}_I, \lambda)$ be the Lebesgue measure space on the unit interval I. We consider, on $\mathcal{L}_I$, the Lebesgue measure λ and the signed measure $fd\lambda$, where

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1/2 \\ -1 & \text{if } 1/2 \leq t \leq 1 \end{cases} .$$

It should be clear that f is Lebesgue integrable. Moreover, $fd\lambda$ is λ-continuous, since (see, for instance, [44], p. 94, Exercise 26)

$$|fd\lambda|(A) = \int_A |f|\, d\lambda = \lambda(A) .$$

Thus, 1) in Proposition 8 tells us that $fd\lambda$ is absolutely continuous with respect to λ. Then according to Proposition 9, $fd\lambda$ and λ cannot be mutually singular, since f is not λ-a.e. zero. However,

$$\begin{aligned}|fd\lambda - \lambda|(I) &= \int_I |f-1|\, d\lambda = 2\lambda([1/2, 1]) = 1, \\ |fd\lambda + \lambda|(I) &= \int_I |f+1|\, d\lambda = 2\lambda([0, 1/2)) = 1.\end{aligned}$$

This completes the proof of the proposition. □

For more on orthogonality, we refer to ([3], Section 6) and the references therein.

Theorem 1. *(Lebesgue Decomposition) Let* (S, Σ, μ) *be a measure space (see [3], p. 81), where the measure* μ *is finite. Let* $m : \Sigma \to X$ *be a vector measure with finite variation (see Definition 3). Then, there exist unique vector measures* $m_a, m_s : \Sigma \to X$ *of finite variation such that*

$$\begin{aligned} m &= m_a + m_s, \\ m_a &\ll \mu, \, m_s \perp \mu. \end{aligned}$$

For the proof of this theorem, we refer to ([19], p. 189, Theorem 7) and ([18], p. 31, Theorem 9). There are, in fact, several versions of this theorem, with various degrees of generality (see, for instance, [11], [16], [13], [19], as well as the references mentioned in [18], p. 39).

For the genesis of the Lebesgue decomposition theorem for vector measures and the like, we refer to ([18], p. 39).

3 Measurability of vector-valued functions

In this section, we will need to refer often to the measurability of real-valued functions, with respect to a fix measure space (S, Σ, μ). For clarity and to avoid repetitions, we will refer to real-valued functions that are Σ-measurable (for the definition see, for instance, [44], p. 71, Definition 1 and p. 78, Corollary 3) as *measurable*. In the few occasions in which we work with an specific measure space, we will say so.

For now, we fix a measure space (S, Σ, μ) which we take to be *complete*. That is to say, if $A \in \Sigma$ and $\mu(A) = 0$ or, in other words, A is μ-null, then, every subset of A also belongs to Σ and, thus, is μ-null. Moreover, as in ([6], p. 100), we assume that the measure μ is σ-finite.

Under these conditions, we will say, from now on, that we have fixed a complete and σ-finite measure space. Other sources work with a finite measure μ (see, for instance, [18], p. 41).

We begin with the following definition:

Definition 11. A function $\varphi : S \to X$ is a vector-valued simple function if there exists a finite family $\{M_j\}_j \subseteq \Sigma$ of pairwise disjoint sets of finite measure and a finite family $\{x_j\}_j \subseteq X$ so that

$$\varphi = \sum_j x_j \chi_{M_j}.$$

We recall that χ_{M_j} is the characteristic function of the set M_j.

The family of vector-valued simple functions $\varphi : S \to X$ is a real linear space.

Lemma 7. *If* $\varphi : S \to X$ *is a vector-valued simple function, the function* $\|\varphi\| : S \to \mathbb{R}$ *defined as* $\|\varphi\|(t) = \|\varphi(t)\|$ *is a simple function (for the definition, see, for instance, [44], p. 78).*

Proof. If $\varphi = \sum_j x_j \chi_{M_j}$, then

$$\|\varphi\| = \sum_j \|x_j\| \chi_{M_j}.$$

Thus, $\|\varphi\|$ is simple.

This completes the proof of the lemma. □

Let us observe that when X is the space $\mathbb{R}$, the notions of vector-valued simple function and simple function coincide.

Definition 12. Following, for instance, ([25], p. 72, Definition 3.5.4; [43], p. 336, Definition 3), we call a function $f : S \to X$ strongly measurable if there is a sequence $\{\varphi_k\}_{k\geq 1}$ of vector-valued simple functions so that $\varphi_k \to f$, μ-a.e. on S, as $k \to \infty$. As usual, by μ-a.e. we mean that $\varphi_k(t) \to_{k\to\infty} f(t)$ in X, except for t in a μ-null set.

Remark 15. Functions satisfying Definition 12 are sometimes called μ-measurable (see, for instance, ([18], p. 41, Definition1).

We observe that, by Definition 12, every vector-valued simple function is strongly measurable. Moreover, strongly measurable functions form a real linear space.

Lemma 8. *If $f : S \to X$ is strongly measurable, the function $\|f\| : S \to \mathbb{R}$ defined as $\|f\|(t) = \|f(t)\|$, is measurable.*

Proof. Let $\{\varphi_k\}_{k\geq 1}$ be a sequence of μ-simple functions so that $\varphi_k \to f$, μ-a.e. on S, as $k \to \infty$. Then,

$$\|\varphi_k\| \underset{k\to\infty}{\to} \|f\|, \ \mu\text{-a.e. on } S,$$

so, $\|f\|$ is measurable (see, for instance, [44], p. 74, Corollary 15).

This completes the proof of the lemma. □

Definition 13. A function $f : S \to X$ is weakly measurable if, for each functional l in the topological dual X' of X, the real-valued function $l \circ f$ is measurable.

Remark 16. It should be clear from the definitions, that every strongly measurable function is weakly measurable. However, the converse is not generally true. Indeed, here is an example, taken from ([43], p. 337, Example 5):

If I denotes the unit interval $[0, 1]$, we consider the space $l^2(I)$ of those functions $f : I \to \mathbb{R}$ that are zero except for t in a countable subset of I and for which $\sum_{t\in I} (f(t))^2$ is finite. The space $l^2(I)$ is a Hilbert space (see, for instance, [4], p. 144) and the family $\{e_t\}_{t\in I}$ defined as

$$e_t(s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases},$$

is an orthonormal basis for $l^2(I)$.

Let $(I, \mathcal{L}_I, \lambda)$ be the Lebesgue measure space on I as in Example 1, and let $A \subseteq I$ be a subset that is not Lebesgue measurable (see, for instance, [44], p. 11, Example 1). We define the function $g : I \to l^2(I)$ as

$$g(t) = \begin{cases} 0 & \text{if } t \notin A \\ e_t & \text{if } t \in A \end{cases} .$$

Then, we claim that g is weakly measurable, but not strongly measurable. In fact, since the dual of $l^2(I)$ is isomorphic, as inner product space, to $l^2(I)$ (see, for instance, [44], p. 239, Theorem 19), given $h \in \left(l^2(I)\right)'$, h can be represented as $\sum_{t \in I} \langle h, e(t) \rangle e(t)$, with $\langle h, e(t) \rangle = 0$ except for t in a countable subset of I (see, for instance, [44], p. 242, Theorem 24 v) and p. 240, Proposition 21). Moreover,

$$h(g(t)) = \begin{cases} 0 & \text{if } t \notin A \\ \langle h, e_t \rangle & \text{if } t \in A \end{cases} .$$

So, the real-valued function $t \to h(g(t))$ is Lebesgue measurable because it is zero except for t in a countable subset of I. Thus, g is weakly measurable.

On the other hand, the real-valued function $t \to \|g(t)\|_{l^2(I)}$ is the characteristic function of A. So, according to Lemma 8, the function g is not strongly measurable.

Let us mention that $l^2(I)$ can be formally described using the notion of summability defined in (8). Equivalently, $l^2(I)$ is the space $L^2(I)$ (see, for instance, [44], p. 209) when the measure space consists of the interval I, the σ-algebra $\mathcal{P}(I)$ of all the subsets of I and the counting measure $\varkappa$ (for the definition see, for instance, [44], p. 23, Example 7).

Definition 14. A function $f : S \to X$ is separably-valued if $f(S)$ is separable. That is to say, if $f(S)$ has a countable subset that is dense in $f(S)$. The function f is almost separably-valued if there is a μ-null set $N \in \Sigma$ so that $f(S \backslash N)$ is separable.

Let us observe that the function g defined in Remark 16 is not even almost separably-valued. Indeed,

$$\|e_t - e_{t'}\|^2_{l^2(I)} = \sum_{s \in I} \left(e_t(s) - e_{t'}(s)\right)^2 = \begin{cases} 0 & \text{if } t = t' \\ 2 & \text{if } t \neq t' \end{cases} ,$$

so, given a λ-null set N, the only way for the image $g(I \backslash N)$ to be separable in $l^2(I)$ is to be countable. But the countability of $\{e_t\}_{t \in I \backslash N}$ implies that $I \backslash N$ is countable. So, I should be λ-null, while we know that $\lambda(I) = 1$.

The next result provides the link between strong and weak measurability. It is due to B. J. Pettis ([33], p. 278, Theorem 1.1).

Theorem 2. *(Pettis measurability theorem) (for the proof see, for instance, [18], p. 72, Theorem 3.5.3; [43], p. 339, Theorem 9) A function is strongly measurable if, and only if, it is weakly measurable and almost separably-valued.*

Although we have stated Theorem 2 in a way that agrees with the current literature, we should mention that Pettis calls separably-valued what is now called almost separably-valued.

In view of Theorem 2, it should be clear why the function defined in Remark 16, is not strongly measurable.

The following result is an immediate consequence of Theorem 2:

Corollary 3. If the space X is separable, weakly measurability and strong measurability are equivalent.

Remark 17. The conclusion of Lemma 8 holds for a weakly measurable function (see, for instance, [25], p. 72, Theorem 3.5.2; [43], p. 338, Proposition 6), when the space X has a countable determining, or norming, set, Λ, in X'. Let us recall that $\Lambda \subseteq X'$ is determining or norming for X if for each $x \in X$

$$\|x\| = \sup\{|l(x)| : l \in \Lambda\}. \tag{22}$$

Such a set can be constructed from any dense subset of X (see, for instance, [43], p. 81, Proposition 11). This is another way of reaching the conclusion in Corollary 3.

Since we always have (see, for instance, [43], p. 79, Corollary 5)

$$\|x\| = \sup\{|l(x)| : l \in X',\ \|l\| \leq 1\},$$

the unit ball in X' is always a norming set for X.

Remark 18. It would be desirable and, indeed, it is true, that the μ-a.e. limit of a sequence of strongly measurable functions is also strongly measurable. For a direct, albeit somewhat technical, proof, we refer to ([43], p. 339, Proposition 8). We shall postpone the proof until we have defined the integral of a vector-valued function. In this way, we will be able to give a fairly elementary proof.

In the next result, we begin to see the convenience of working with a measure μ that is σ-finite.

Lemma 9. *Let $f : S \to X$ be a function. Then the following statements are equivalent:*

1. *For each $A \in \Sigma$ with finite measure, there exists a sequence $\{\varphi_k\}_{k\geq 1}$ of vector-valued simple functions such that, for each $k \geq 1$ the function φ_k is zero in $S\backslash A$ and $\varphi_k \to f$, μ-a.e. on A, as $k \to \infty$.*
2. *The function f is strongly measurable.*

Proof. Let us prove 1) $\Rightarrow$ 2). Since μ is σ-finite, we can write

$$S = \bigcup_{j\geq 1} A_j, \tag{23}$$

where $\{A_j\}_{j\geq 1} \subseteq \Sigma$ and $\mu(A_j)$ is finite for all $j \geq 1$. Moreover, we can disjoint the sets A_j by defining inductively $B_1 = A_1$ and $B_{k+1} = A_{k+1}\backslash \bigcup_{1\leq j\leq k+1} A_j$. Thus, we assume that the family $\{A_j\}_{j\geq 1}$ is already pairwise disjoint.

By hypothesis, for each $j \geq 1$, there exists a sequence $\left\{\varphi_k^j\right\}_{k\geq 1}$ of vector-valued simple functions vanishing outside A_j, that converges to f, μ-a.e. on A_j, as $k \to \infty$. For each $j \geq 1$, let $N_j \subseteq A_j$ be a μ-null set such that $\varphi_k^j(t) \to_{k\to\infty} f(t)$ for each $t \in A_j \backslash N_j$. Moreover, let $N = \bigcup_{j\geq 1} N_j$. The set N is a μ-null subset of S, since $\mu(N) \leq \sum_{j\geq 1} \mu(N_j)$. This inequality can be proved disjointing the sets N_j.

Now, for each $k \geq 1$ fixed, we consider

$$\varphi_k = \sum_{j=1}^{k} \varphi_k^j.$$

The function φ_k is vector-valued simple. Moreover, we claim that for each $t \in S \backslash N$, $\varphi_k(t) \to f(t)$ in X, as $k \to \infty$. In fact, If we fix $t \in S\backslash N$, $t \in A_j \backslash N_j$, for a unique $j = j_t \geq 1$. Thus, $\varphi_k(t) = \varphi_k^j(t)$ for $k \geq j$, so $\varphi_k(t) \to f(t)$ in X, as $k \to \infty$.

As for the proof of 2) $\Rightarrow$ 1), if $\varphi_k(t) \to f(t)$ in X as $k \to \infty$ for $t \in S\backslash N$, the set N being μ-null, it should be clear that $\varphi_k(t) \to f(t)$ in X as $k \to \infty$, for $t \in A \backslash (N \bigcap A)$. Let

$$\varphi_k^A(t) = \left\{ \begin{array}{cc} \varphi_k(t) & \text{if } t \in A \\ 0 & \text{if } t \in S\backslash A \end{array} \right\}.$$

Then, the sequence $\left\{\varphi_k^A\right\}_{k\geq 1}$ converges to f, μ-a.e. on A, as $k \to \infty$.

This completes the proof of the lemma. □

Remark 19. If 1) in Lemma 9 holds for a fixed $A \in \Sigma$, we say that f is strongly measurable on A.

Thus, Lemma 9 tells us that f is strongly measurable if, and only if, it is strongly measurable on every $A \in \Sigma$ with finite measure. In particular, f is strongly measurable if, and only if, it is strongly measurable on every set of a partition of S as in (23).

Lemma 10. *Consider two functions, $f : S \to \mathbb{R}$, and $g : S \to X$, such that f is measurable and g is strongly measurable. Then, the function $fg : S \to X$ defined as $(fg)(t) = f(t)\,g(t)$ is strongly measurable.*

Proof. Let $\left\{\varphi_j\right\}_{j\geq 1}$ be a sequence of simple functions converging to f, μ-a.e. on S, and let $\left\{\psi_j\right\}_{j\geq 1}$be a sequence of vector-valued simple functions converging to g, μ-a.e. on S, as $j \to \infty$. Then it should be clear that, for each $j \geq 1$, the function $\varphi_j \psi_j$ is a vector-valued simple function and the sequence $\left\{\varphi_j \psi_j\right\}_{j\geq 1}$ converges to fg, μ-a.e. on S, as $j \to \infty$.

This completes the proof of the lemma. □

Remark 20. Dinculeanu defines a function $f : S \to X$ to be strongly measurable if (see [19], p. 89, Definition 4)

1. For each open set $G \subseteq X$, the set $f^{-1}(G)$ belongs to Σ.

2. The function f is almost separably-valued on sets of finite measure.

Moreover, Dinculeanu proves (see [19], p. 92, Proposition 12) the following result:

A function $f : S \to X$ is strongly measurable if, and only if, the following two conditions are verified:

1. The function f is weakly measurable.
2. The function f is almost separably-valued.

Since the measure space (S, Σ, μ) is σ-finite, Pettis measurability theorem implies that Dinculeanu's definition of strong measurability is equivalent to Definition 12.

Let us observe that the set G in 1) of Remark 20, can be assumed to be closed or a Borel set (for the definition see, for instance, [44], p. 39) in X.

Remark 21. In view of Remark 20, when X is the space $\mathbb{R}$, strong measurability coincides with measurability.

For the next two results, we assume that S is a topological space and that Σ contains the open subsets of S or, equivalently, that Σ contains the Borel σ-algebra.

Lemma 11. *If $f : S \to X$ is continuous, then f is weakly measurable.*

Proof. For each $l \in X'$, the topological dual of X, the real-valued function $l \circ f$ is continuous, so it is measurable (see, for instance, [39], p. 12).

This completes the proof of the lemma. □

Lemma 12. *If S is separable and $f : S \to X$ is continuous, then f is strongly measurable.*

Proof. On account of Lemma 11 and Theorem 2, it suffices to prove that f is separably valued. In fact let D be a countable subset of S that is dense in S. We claim that $\{f(s)\}_{s \in D}$ is dense in $f(S)$. In fact, we fix $x_0 \in f(S)$ and $\varepsilon > 0$. Then, $x_0 = f(s_0)$ for some $s_0 \in S$. By continuity, there is $\delta = \delta_\varepsilon > 0$ so that for each s in the open ball $B(s_0, \delta)$, $\|f(s) - f(s_0)\| < \varepsilon$. In particular, there exists $s_1 \in D$ so that $s_1 \in B(s_0, \delta)$. Then,

$$\|f(s_1) - x_0\| = \|f(s_1) - f(s_0)\| < \varepsilon.$$

This completes the proof of the lemma. □

There is a version of Lemma 11 for weakly continuous functions (see [25], p. 73).

4 The Bochner integral

We fix a measure space (S, Σ, μ) which, as in the previous section, we assume to be complete. Moreover, we take the measure μ to be σ-finite. We still refer to real-valued functions that are Σ-measurable as measurable. Likewise, real-valued functions that are Lebesgue integrable with respect to μ (for the definition see, for instance, [44], p. 85), will be called *integrable*. We fix a Banach space X with norm $\|\cdot\|$. On occasion, we will need to identify the space and its norm, so we will write $\|\cdot\|_X$.

We are now ready to define an integral, with respect to the measure μ, of certain strongly measurable functions. More specifically, we will develop the integral defined by S. Bochner in an article published in *Fundamenta Mathematica* in 1933 [8].

Let us begin by defining the integral of a vector-valued simple function.

Definition 15. Given a vector-valued simple function $\varphi = \sum_j x_j \chi_{A_j}$ as in Definition 11, we define the Bochner integral of φ on S as

$$\int_S \varphi d\mu = \sum_j x_j \mu(A_j).$$

Lemma 13. *The following statements are true:*

1. *Definition 15 does not depend on the representation of φ as in Definition 11.*
2. *The Bochner integral defines, on the linear space of vector-valued simple functions, a linear operator with values in X.*
3. $$\left\| \int_S \varphi d\mu \right\| \leq \sum_j \|x_j\| \mu(A_j) = \underset{(i)}{\int_S \|\varphi\| d\mu},$$

   where (i) is the Lebesgue integral of the real-valued function $\|\varphi\|$, with respect to the measure μ.

The proof of this lemma is straightforward and it will be omitted.

Definition 16. A function f is Bochner integrable if the following two conditions hold:

1. The function f is strongly measurable.
2. There exists a sequence $\{\varphi_k\}_{k\geq 1}$ of vector-valued simple functions such that

$$\varphi_k \underset{k\to\infty}{\to} f, \ \mu\text{-a.e. on } S,$$

$$\int_S \|f - \varphi_k\| d\mu \underset{k\to\infty}{\to} 0. \tag{24}$$

The integral in (24) makes sense since $\|f - \varphi_k\|$ is measurable according to Lemma 24, and it is a non-negative function (see, for instance, [44], p. 83).

Lemma 14. *Given a function f and a sequence $\{\varphi_k\}_{k\geq 1}$ satisfying the conditions in Definition 16,*

1. *the sequence $\left\{\int_S \varphi_k d\mu\right\}_{k\geq 1}$ is a Cauchy sequence in X.*
2. *the limit in X, as $k \to \infty$, of the sequence $\left\{\int_S \varphi_k d\mu\right\}_{k\geq 1}$ does not depend on the sequence $\{\varphi_k\}_{k\geq 1}$ satisfying 2) in Definition 16.*

Proof. To prove 1), we observe that, according to Lemma 13,

$$\left\|\int_S \varphi_l d\mu - \int_S \varphi_k d\mu\right\| \leq \int_S \|\varphi_l - \varphi_k\| \, d\mu \leq \int_S \|\varphi_l - f\| \, d\mu + \int_S \|f - \varphi_k\| \, d\mu \underset{l,k\to\infty}{\to} 0.$$

To prove 2), let $\{\psi_k\}_{k\geq 1}$ be another sequence of vector-valued simple functions satisfying 2) of Definition 16. Then, according to 1), we know that the sequence $\left\{\int_S \psi_k d\mu\right\}_{k\geq 1}$ is also a Cauchy sequence in X. Moreover,

$$\left\|\int_S \psi_k d\mu - \int_S \varphi_k d\mu\right\| \leq \int_S \|\psi_k - \varphi_k\| \, d\mu \leq \int_S \|\psi_k - f\| \, d\mu + \int_S \|f - \varphi_k\| \, d\mu \underset{k\to\infty}{\to} 0,$$

which shows that 2) holds.

This completes the proof of the lemma. □

Definition 17. Given a function $f : S \to X$, satisfying Definition 16, the limit of $\int_S \varphi_k d\mu$ as $k \to \infty$, is called the Bochner integral of f on S, with respect to the measure μ, denoted $\int_S f d\mu$. When there is no ambiguity as to the measure we are using to integrate f on S, we will just refer to the Bochner integral of f and we will say that f is Bochner integrable.

It should be clear from the context, whether $\int_S f d\mu$ refers to the Bochner integral of a vector-valued function, or to the Lebesgue integral of a real-valued function.

Remark 22. Given a Bochner integrable function $f : S \to X$ and given $A \in \Sigma$, it should be clear that the function $f\chi_A$ is Bochner integrable as well. We define

$$\int_A f d\mu = \int_S f\chi_A d\mu.$$

Lemma 15. *Given a Bochner integrable function $f : S \to X$ and given $A, B \in \Sigma$ disjoint,*

$$\int_{A\bigcup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

Proof. Using Definition 16 and Lemma 14, we can write

$$\begin{aligned}\int_A f d\mu + \int_B f d\mu &= \lim_{j\to\infty}\int_S \varphi_j \chi_A d\mu + \lim_{j\to\infty}\int_S \varphi_j \chi_B d\mu \\ &= \lim_{j\to\infty}\int \varphi_j \chi_{A\bigcup B} d\mu = \int_{A\bigcup B} f d\mu.\end{aligned}$$

We observe that

$$\begin{aligned}\int_S \left\|\varphi_j\chi_A - f\chi_A\right\| d\mu &\le \int_S \left\|\varphi_j - f\right\| d\mu \underset{j\to\infty}{\to} 0, \\ \int_S \left\|\varphi_j\chi_B - f\chi_B\right\| d\mu &\le \int_S \left\|\varphi_j - f\right\| d\mu \underset{j\to\infty}{\to} 0,\end{aligned}$$

thus,

$$\begin{aligned}&\int_S \left\|\varphi_j\chi_{A\bigcup B} - f\chi_{A\bigcup B}\right\| d\mu \\ \le\ &\int_S \left\|\varphi_j\chi_A - f\chi_A\right\| d\mu + \int_S \left\|\varphi_j\chi_B - f\chi_B\right\| d\mu \underset{j\to\infty}{\to} 0.\end{aligned}$$

This completes the proof of the lemma. □

Remark 23. Lemma 15 says that given a Bochner integrable function $f : S \to X$, the vector-valued set function

$$\Sigma \ni A \to \int_A f d\mu \in X$$

is finitely additive.

Lemma 16. *The following statements hold:*

1. *If f is Bochner integrable, then the real-valued function $\|f\|$ is integrable. Moreover,*

$$\left\|\int_S f d\mu\right\| \le \int_S \|f\|\, d\mu. \tag{25}$$

2. *The functions $f : S \to X$ that are Bochner integrable form a real linear space.*

Proof. To prove 1), we pick a sequence $\{\varphi_k\}_{k\ge 1}$ as in Definition 16. Then, there exists $k_0 \ge 0$ so that

$$\int_S \left\|f - \varphi_{k_0}\right\| d\mu \le 1.$$

Then,

$$\int_S \|f\|\, d\mu \le 1 + \sum_j \left\|x_j^{k_0}\right\| \mu\left(M_j^{k_0}\right).$$

As for the inequality (25), from 3) in Lemma 13 and 2) in Definition [8],

$$\begin{aligned}\left\|\int_S f d\mu\right\| &= \left\|\lim_{k\to\infty}\int_S \varphi_k d\mu\right\| = \lim_{k\to\infty}\left\|\int_S \varphi_k d\mu\right\| \\ &\leq \lim_{k\to\infty}\int_S \|\varphi_k\|\, d\mu = \int_S \|f\|\, d\mu + \lim_{k\to\infty}\int_S \|f-\varphi_k\|\, d\mu = \int_S \|f\|\, d\mu.\end{aligned}$$

The proof of 2) is straightforward and will be omitted.

This completes the proof of the lemma. □

The converse of 1) in Lemma 16 does not hold generally, even when X is the space $\mathbb{R}$. To see it, let us consider the measure space $(I, \mathcal{L}_I, \lambda)$ (see Example 1). If we fix $A \subset [0,1]$ that is not Lebesgue measurable and define $f : [0,1] \to \mathbb{R}$ as $f(t) = 1$ if $t \in A$, $f(t) = -1$ if $t \in [0,1] \setminus A$, according to Remark 20, f is not Lebesgue measurable, while $|f|$ is.

Definition 18.

$$B^1(S) = \{f : S \to X : f \text{ is Bochner integrable}\}.$$

We write $B^1(S, X)$, if we wish to emphasize the space X.

Theorem 3. *The space $B^1(S)$ becomes a complete semi-normed linear space defining*

$$\|f\|_{B^1(S)} = \int_S \|f\|\, d\mu.$$

Proof. Taking into account Lemma 16 and well known properties of the Lebesgue integral, it should be clear that $\left(B^1(S), \|\cdot\|_{B^1(S)}\right)$ is a semi-normed linear space. Let us see that it is complete.

Let $\{f_j\}_{j\geq 1}$ be a Cauchy sequence in $B^1(S)$. That is to say,

$$\int_S \|f_j - f_k\|\, d\mu \underset{j,k\to\infty}{\rightarrow} 0.$$

By Definition 16, for each $j \geq 1$, there is a vector-valued simple function φ_j so that

$$\int_S \left\|f_j - \varphi_j\right\| d\mu \leq \frac{1}{j}. \tag{26}$$

Then,

$$\begin{aligned}\int_S \left\|\varphi_j - \varphi_k\right\| d\mu &\leq \int_S \left\|f_j - \varphi_j\right\| d\mu + \int_S \|f_k - \varphi_k\|\, d\mu + \int_S \|f_j - f_k\|\, d\mu \\ &\leq \frac{1}{j} + \frac{1}{k} + \int_S \|f_j - f_k\|\, d\mu \underset{j,k\to\infty}{\rightarrow} 0.\end{aligned}$$

So, for each $l \geq 1$, there is $K = K_l \geq 1$ such that

$$\int_S \left\|\varphi_j - \varphi_k\right\| d\mu \leq \frac{1}{2^l}, \tag{27}$$

for all $k, j \geq K$.

Let $\left\{\varphi_{j_l}\right\}_{l\geq 1}$ be a subsequence of $\left\{\varphi_j\right\}_{j\geq 1}$ so that

$$\int_S \left\|\varphi_{j_{l+1}} - \varphi_{j_l}\right\| d\mu \leq \frac{1}{2^l}.$$

Then, the series $\sum_{l\geq 1} \int_S \left\|\varphi_{k_{l+1}} - \varphi_{k_l}\right\| d\mu$ converges. By Fatou's lemma (see, for instance, [44], p. 87, Theorem 15), the series $\sum_{l\geq 1} \left\|\varphi_{k_{l+1}} - \varphi_{k_l}\right\|$ is integrable. As a consequence, it converges μ-a.e. on S (see, for instance, [44], p. 8, Proposition 14). Since X is complete,

$$\varphi_{k_{L+1}} = \varphi_{k_1} + \sum_{l=1}^{L} \left(\varphi_{k_{l+1}} - \varphi_{k_l}\right)$$

converges μ-a.e. on S (see, for instance, [44], p. 167, Theorem 4). That is to say, there is a function $f : S \to X$ so that $\varphi_{k_{L+1}} \to_{L\to\infty} f$, μ-a.e. on S. Thus, the function f is strongly measurable, by Definition 12. Let us see that f also satisfies 2) in Definition 16.

$$\begin{aligned}\int_S \left\|\varphi_{k_{L+1}} - f\right\| d\mu &= \int_S \left\|\sum_{l\geq L+1} \left(\varphi_{k_{l+1}} - \varphi_{k_l}\right)\right\| d\mu \\ &\leq \int_S \sum_{l\geq L+1} \left\|\varphi_{k_{l+1}} - \varphi_{k_l}\right\| d\mu \\ &\underset{\text{(i)}}{\leq} \sum_{l\geq L+1} \int_S \left\|\varphi_{k_{l+1}} - \varphi_{k_l}\right\| d\mu \underset{L\to\infty}{\to} 0,\end{aligned} \tag{28}$$

where we have used in (i) Fatou's lemma. So, f is Bochner integrable.

Finally, $f_k \to_{k\to\infty} f$ in $B^1(S)$. In fact,

$$\int_S \|f_k - f\| d\mu \leq \underbrace{\int_S \|f_k - \varphi_k\| d\mu}_{\text{(i)}} + \underbrace{\int_S \left\|\varphi_k - \varphi_{k_l}\right\| d\mu}_{\text{(ii)}} + \underbrace{\int_S \left\|\varphi_{k_l} - f\right\| d\mu}_{\text{(iii)}}$$

Let us fix $\varepsilon > 0$. According to (26), (i) is $\leq \varepsilon$ for $k \geq K_1 = K_{1,\varepsilon}$. According to (27), (ii) is $\leq \varepsilon$ for $k, k_l \geq K_2 = K_{2,\varepsilon}$. Since $\left\{\varphi_{j_l}\right\}_{l\geq 1}$ is a subsequence of $\left\{\varphi_j\right\}_{j\geq 1}$, there is $L_1 = L_{1,\varepsilon} \geq 1$ so that $l \geq L_1$ implies $k_l \geq K_2$. Let $K = K_\varepsilon = \max\{K_1, K_2\}$. As for (iii), according to (28), there exists $L_2 = L_{2,\varepsilon} \geq 1$ such that (iii) is $\leq \varepsilon$ for $l \geq L_2$. Thus, for $l \geq \max\{L_1, L_2\}$ and $k \geq K$, all three terms are $\leq \varepsilon$.

This completes the proof of the theorem. □

Remark 24. The proof of Theorem 3 is significantly more involved that its counterpart for the space $L^1(S)$ of Lebesgue integrable functions. Indeed, in $B^1(S)$ we do not have a Monotone Convergence Theorem (for the real-valued case see, for instance, [44], p. 83, Theorem 7). This theorem plays a key role in the case of $L^1(S)$ (see, for instance, [44], p. 208, Theorem 5).

Remark 25. It should be clear that given $f, g : S \to X$ that are equal μ-a.e. on S, the strong measurability or the Bochner integrability of one of them is equivalent, respectively, to the strong measurability or the Bochner integrability of the other. Moreover,

$$\int_S f d\mu = \int_S g d\mu.$$

Equality μ-a.e. on S defines an equivalence relation on $B^1(S)$. As a consequence, the quotient of $B^1(S)$ by this relation becomes a Banach space (see, for instance, [44], p. 181, Section 5.4). Nevertheless, for now we will continue working in $B^1(S)$ as given by Definition 18.

We must acknowledge that in doing so, we create a minor discrepancy with the space $L^p(I)$, which we have used before and we will use in future sections. Indeed, since we think of $L^p(I)$ as a Banach space, its elements are really classes of equivalence. However, in the instances in which $L^p(I)$ appears, we always end up working with a particular function, so, really, the discrepancy can go unnoticed.

Theorem 4. *(Dominated Convergence Theorem for the Bochner integral) Let $\{f_j\}_{j\geq 1}$ be a sequence in $B^1(S)$ that satisfies the following two conditions:*

1. *There is an integrable function $g : S \to [0, \infty)$ such that*

$$\|f_j\| \leq g,$$

   μ-a.e. on S.

2. *There is a function $f : S \to X$ such that*

$$f_j \underset{j\to\infty}{\to} f,$$

   μ-a.e. on S.

Then, $f \in B^1(S)$ and $f_j \to_{j\to\infty} f$ in $B^1(S)$.

Proof. For any $k \geq 1$ fixed,

$$f_j - f_k \underset{j\to\infty}{\to} f - f_k,$$

μ-a.e. on S. Therefore,

$$\|f_j - f_k\| \underset{j\to\infty}{\to} \|f - f_k\|,$$

μ-a.e. on S. As a consequence, the function $\|f - f_k\|$ is measurable. Moreover,

$$\|f - f_k\| \underset{k\to\infty}{\to} 0,$$

μ-a.e. on S and

$$\|f - f_k\| \leq 2g,$$

μ-a.e. on S. We can now use Lebesgue's Dominated Convergence Theorem (see, for instance, [44], p. 87, Theorem 16) to conclude that there is

$$\lim_{k\to\infty} \int_S \|f - f_k\| \, d\mu = 0.$$

As a consequence, there is

$$\lim_{j,k\to\infty} \int_S \|f_j - f_k\| \, d\mu = 0.$$

Using Theorem 3, there is a function $\widetilde{f} \in B^1(S)$ such that the sequence $\{f_j\}_{j\geq 1}$ converges to $\widetilde{f}$ in $B^1(S)$. That is to say, there is

$$\lim_{j\to\infty} \int_S \left\|f_j - \widetilde{f}\right\| d\mu = 0.$$

Then,

$$\int_S \left\|f - \widetilde{f}\right\| d\mu \leq \int_S \|f - f_j\| \, d\mu + \int_S \left\|f_j - \widetilde{f}\right\| d\mu \underset{j\to\infty}{\to} 0.$$

So, $f = \widetilde{f}$, μ-a.e. on S. According to Remark 25, $f \in B^1(S)$ and $f_j \to_{j\to\infty} f$ in $B^1(S)$.

This completes the proof of the theorem. $\square$

Remark 26. As a consequence of (25) and Theorem 4, we have

$$\int_S f_j d\mu \underset{j\to\infty}{\to} \int_S f d\mu.$$

For strongly measurable functions we have the following characterization of Bochner integrability, which is analogous to the characterization of integrability for real-valued functions (see, for instance, [44], p. 85, Proposition 11):

Proposition 12. *Let $f : S \to X$ be strongly measurable. Then, the following statements are equivalent:*

1. *The function f is Bochner integrable.*
2. *The function $\|f\|$ is integrable.*

Proof. In view of 1) in Lemma 16, we only need to prove that 2) implies 1).

By definition of strong measurability, there is a sequence $\{\varphi_j\}_{j\geq 1}$ of vector-valued simple functions so that $\varphi_j \to_{j\to\infty} f$, μ-a.e. on S. For each $j \geq 1$, let

$$S_j = \left\{t \in S : \left\|\varphi_j(t)\right\| \leq 2\left\|f(t)\right\|\right\}.$$

By definition, $S_j \in \Sigma$. We now consider $\psi_j = \varphi_j \chi_{S_j}$. It should be clear that each ψ_j is a vector-valued simple function. Moreover, $\left\|\psi_j\right\| \leq 2\left\|f\right\|$ for all $j \geq 1$ and $\psi_j \to_{j\to\infty} f$, μ-a.e. on S. To see the last assertion, it suffices to observe that there is a μ-null set N so that

$$S = \left(\bigcup_{j\geq 1} S_j\right) \bigcup N.$$

Then, using Theorem 4 we conclude that f is Bochner integrable.

This completes the proof of Proposition 12. □

Remark 27. The proof of 2) ⇒ 1) in Proposition 12, gives a sequence $\{\psi_j\}_{j\geq 1}$ of vector-valued simple functions satisfying both conditions in Definition 16.

Corollary 4. Let us assume that S is a separable topological space, and that Σ contains the open subsets of S. That is to say, it contains the Borel σ-algebra. Then, continuity implies Bochner integrability.

Proof. It follows from Lemma 12 and Proposition 12. □

The next result that follows is an immediate consequence of Remark 19 and Theorem 4.

Corollary 5. If $f : S \to X$ is strongly measurable and bounded, then it is Bochner integrable over every set $A \in \Sigma$ of finite measure.

Remark 28. According to Remark 21, Proposition 12 and Proposition 11 in ([44], p. 85), the Bochner integral is the Lebesgue integral, for real-valued functions.

We are now ready to prove the statement made in Remark 18:

Proposition 13. *Let $\{f_j\}_{j\geq 1}$ be a sequence of strongly measurable functions. If $\{f_j\}_{j\geq 1}$ converges, μ-a.e. on S, to a function $f : S \to X$, then f is strongly measurable.*

Proof. It should be clear that for each $j \geq 1$, the function $\frac{1}{1+\|f_j\|}$ is measurable. Then, according to Lemma 10, the function $\frac{f_j}{1+\|f_j\|}$ is strongly measurable, for each $j \geq 1$. Moreover,

$$\left\| \frac{f_j}{1 + \|f_j\|} \right\| \leq 1$$

and

$$\frac{f_j}{1 + \|f_j\|} \underset{j \to \infty}{\to} \frac{f}{1 + \|f\|},$$

μ-a.e. on S. Thus, Corollary 5 and Theorem 4 imply that $\frac{f}{1+\|f\|}$ is Bochner integrable over every set $A \in \Sigma$ of finite measure. In particular, $\frac{f}{1+\|f\|}$ is strongly measurable on every set $A \in \Sigma$ of finite measure. Then, Lemma 9 and Remark 19, tell us that $\frac{f}{1+\|f\|}$ is strongly measurable on S, and, thus, f is strongly measurable on S.

This completes the proof of the proposition. □

Theorem 5. *Let X, Y be Banach spaces and let $T \in L(X, Y)$, the Banach space of those operators $T : X \to Y$, that are linear and continuous, with the operator norm $\|\cdot\|$. Moreover,let $f : S \to X$.*

Then, the following statements are true:

1. *The application $f \to T \circ f$ preserves strong measurability.*
2. *The application $f \to T \circ f$ is linear and continuous from $B^1(S, X)$ into $B^1(S, Y)$.*
3. $$T\left(\int_S f d\mu\right) = \int_S (T \circ f)\, d\mu,$$

   for every $f \in B^1(S, X)$.

Proof. If f is the pointwise limit in X, μ-a.e. on S, of a sequence $\{\varphi_k\}_{k\geq 1}$ of vector-valued simple functions, with values in X, then, $\{T \circ \varphi_k\}_{k\geq 1}$ is a sequence of vector-valued simple functions, with values in Y, that converges pointwise to $T \circ f$ in Y, μ-a.e. on S. This proves 1).

As for 2), it should be clear that the application is linear. Now, given $f \in B^1(S, X)$, by 1), $T \circ f$ is strongly measurable. Thus, the real-valued function $\|(T \circ f)(t)\|_Y$ is measurable. Since

$$\|(T \circ f)(t)\|_Y \leq \|T\| \, \|f(t)\|_X,$$

for $t \in S$, where $\|T\|$ denotes the operator norm, we conclude, according to Proposition 12, that $T \circ f \in B^1(S, Y)$. This proves 2).

Finally, to prove 3), we use Definition 16 and Proposition 12. Let $\{\varphi_k\}_{k\geq 1}$ be a sequence of vector-valued simple functions, with values in X, as in Definition 16. Then, $\{T\circ\varphi_k\}_{k\geq 1}$ is a sequence of vector-valued simple functions, with values in Y, such that

$$T\circ\varphi_k \underset{k\to\infty}{\rightarrow} T\circ f,$$

pointwise in Y, μ-a.e. on S. Moreover, since

$$\|T\circ\varphi_k - T\circ f\|_Y = \|T\circ(\varphi_k - f)\|_Y \leq \|T\|\,\|\varphi_k - f\|_X\,,$$

we have

$$\int_S \|T\circ\varphi_k - T\circ f\|_Y\,d\mu \leq \|T\| \int_S \|\varphi_k - f\|_X\,d\mu \underset{k\to\infty}{\rightarrow} 0.$$

Thus, the sequence $\{T\circ\varphi_k\}_{k\geq 1}$ satisfies the conditions in Definition 16, with respect to $T\circ f$. Consequently, $T\circ f\in B^1(S,Y)$ and, according to Definition 17,

$$\int_S (T\circ\varphi_k)\,d\mu \rightarrow_{k\to\infty} \int_S (T\circ f)\,d\mu,$$

in Y. Finally,

$$T\left(\int_S \varphi_k d\mu\right) = \int_S (T\circ\varphi_k)\,d\mu,$$

for all $k\geq 1$ and

$$T\left(\int_S \varphi_k d\mu\right) \underset{k\to\infty}{\rightarrow} T\left(\int_S f d\mu\right),$$

which gives us 3).

This completes the proof of the theorem. □

Remark 29. As a consequence of Theorem 5, it should be clear that

$$T\left(\int_A f d\mu\right) = \int_A (T\circ f)\,d\mu, \tag{29}$$

for every $A\in\Sigma$.

There is a version of (29) for linear and closed operators (see [18], p. 47, Theorem 6). For the definition of closed operator see, for instance, ([4], p. 261, Definition 16.1).

Remark 30. When X and Y are the space $\mathbb{R}$, Theorem 5 does not say anything, because a linear and continuous operator from $\mathbb{R}$ into $\mathbb{R}$ is given as the multiplication by a fixed real number. In this sense, we can say that Theorem 5 is of interest when the spaces X and Y have more than one linear dimension.

The following result is an application of Theorem 5, when the space X has a countable norming set. We postpone the consideration of the general case until Section 6.

Corollary 6. Let us suppose that the space X has a countable norming set. Then, if $f \in B^1(S)$ and

$$\int_A f d\mu = 0$$

for every $A \in \Sigma$, the function f is zero, μ-a.e. on S.

Proof. According to Theorem 5 and Remark 29, given $l \in X'$,

$$\int_A (l \circ f)\, d\mu = 0,$$

for every $A \in \Sigma$. Then (see, for instance, [44], p. 88, Proposition 18), there exists a μ-null set N_l so that $(l \circ f)(t) = 0$ for $t \in S \backslash N_l$. Now, let $* \subseteq X'$ be a countable norming set for X, and let $N = \cup_{l \in *} N_l$. Then, for $t \in S \backslash N$,

$$\|f(t)\| = \sup\{|l(x)| : l \in \Lambda\} = 0.$$

This completes the proof of the corollary. □

Remark 31. According to Remark 17, every separable space X has a countable norming set.

Essentially, we conclude here our presentation of the Bochner integral. We will only add a few words about the vector-valued counterparts of the spaces $L^p(S)$.

We introduce them in the following definitions:

Definition 19. For $1 < p < \infty$, the space $B^p(S)$ consists of those strongly measurable functions $f : S \to X$ such that $\|f(\cdot)\|^p$ belongs to $L^1(S)$.

Definition 20. An function $f : S \to X$ is said to be essentially bounded if there is a μ-null set $N \in \Sigma$ and a real number $C > 0$ such that $\|f(t)\| \leq C$ for $t \in S \backslash N$.

The space $B^\infty(S)$ consists of those functions functions $f : S \to X$ that are strongly measurable and essentially bounded.

Proposition 14. *For $1 < p < \infty$, the space $B^p(S)$ becomes a complete semi-normed linear space if we define*

$$\|f\|_{B^p(S)} = \left(\int_S \|f\|^p \, d\mu \right)^{1/p}.$$

Proposition 15. *The space $B^\infty(S)$ becomes a complete semi-normed linear space if we define*

$$\|f\|_{B^\infty(S)} = \inf\{C > 0 : \|f(t)\| \geq C \textit{ for } t \notin N,\ \mu\textit{-null set}\}.$$

The proofs of these propositions are fairly straightforward applications of previously used techniques.

Nevertheless, for these propositions and for other results that we are omitting, notably Fubini's theorem for the Bochner integral, we refer to [25] and [46].

Remark 32. The Bochner integral can be viewed as a strong approach to the integration of vector-valued functions. There are weak approaches as well, for instance, the Pettis integral, defined by Pettis in [33] (see also [43], p. 334, Section 26.3). For other approaches to integrating vector-valued functions see, for instance, ([25], p. 62, Section 3.3) and the references therein. For a history of the integration of vector-valued functions, we refer to ([25], p. 62, Section 3.3) and, specially, to [24] and [5].

We now continue with a section dedicated to reviewing, and comparing, several modes of convergence.

5 Modes of convergence

So far, we have come across pointwise convergence, μ-a.e. convergence, and the convergence in $B^1(S)$ which may be called *convergence in μ-mean*.

We now want to extend to vector-valued functions, the notion of convergence in μ-measure and to compare it with the modes of convergence just mentioned. Our purpose is mostly to collect definitions and results that will be used later. For the most part, this section trivially follows the real-valued case, so many proofs will be omitted.

Definition 21. The sequence $\{f_j\}_{j\geq 1}$ of strongly measurable functions converges in μ-measure to the strongly measurable function f if, for each $\alpha > 0$, there exists

$$\lim_{j\to\infty} \mu\left(\{t \in S : \|f_j(t) - f(t)\| > \alpha\}\right) = 0.$$

This definition is exactly the definition on the real-valued case, with norm instead of absolute value (see, for instance, [44], p. 107).

Proposition 16. *The following statements are true:*

1. *Let $f_j, f, g_j, g : S \to X$ be strongly measurable for all $j \geq 1$ and let $a, b \in \mathbb{R}$. Then, if $f_j \to_{j\to\infty} f$ in μ-measure and $g_j \to_{j\to\infty} g$ in μ-measure, then*
$$af_j + bg_j \underset{j\to\infty}{\to} af + bg,$$
in μ-measure.
2. *Let $f_j : S \to X$ be strongly measurable for all $j \geq 1$, and let $g_j : S \to \mathbb{R}$ be measurable for all $j \geq 1$. Then, if $f_j \to_{j\to\infty} 0$ in μ-measure and $g_j \to_{j\to\infty} 0$ in μ-measure,*
$$f_j g_j \underset{j\to\infty}{\to} 0,$$
in μ-measure.
3. *Assume that μ is a finite measure. Let $f_j, f : S \to X$ be strongly measurable for all $j \geq 1$, and let $g : S \to \mathbb{R}$ be measurable. Then, if $f_j \to_{j\to\infty} f$ in μ-measure,*
$$f_j g \underset{j\to\infty}{\to} fg,$$
in μ-measure.

4. *Assume again that μ is a finite measure. Let $f_j, f : S \to X$ be strongly measurable for all $j \geq 1$, and let $g_j, g : S \to \mathbb{R}$ be measurable for all $j \geq 1$. Then, if $f_j \to_{j\to\infty} f$ in μ-measure and $g_j \to_{j\to\infty} f$ in μ-measure,*

$$f_j g_j \underset{j\to\infty}{\to} fg,$$

in μ-measure.

5. *Let $f_j, f^1, f^2 : S \to X$ be strongly measurable for all $j \geq 1$. Then, if $f_j \to_{j\to\infty} f^1$ in μ-measure and $f_j \to_{j\to\infty} f^2$ in μ-measure, $f^1 = f^2$ μ-a.e. on S.*

Proof. The proof follows, word by word, the real-valued case, if we replace absolute values with norms, in the appropriate places (for the proof of the real-valued case, see, for instance, [44], p. 107, Proposition 1). □

Remark 33. That the measure μ be finite is a necessary condition for 3) and 4) to hold, even in the real-valued case (see, for instance, [44], p. 108, Example 2).

Proposition 17. *(Tchebyshev's inequality) Given $f \in B^1(S)$, the following inequality holds:*

$$\mu(\{t \in S : \|f(t)\| > \alpha\}) \leq \frac{1}{\alpha} \int_{\{t\in S:\|f(t)\|>\alpha\}} \|f\| \, d\mu,$$

for each $\alpha > 0$.

Proof.

$$\begin{aligned} \alpha\mu(\{t \in S : \|f(t)\| > \alpha\}) &= \int_{\{t\in S:\|f(t)\|>\alpha\}} \alpha d\mu \\ &\leq \int_{\{t\in S:\|f(t)\|>\alpha\}} \|f\| \, d\mu. \end{aligned}$$

This completes the proof of the proposition. □

The following result is essentially a reformulation of Tchebyshev's inequality:

Proposition 18. *Convergence in μ-mean implies convergence in μ-measure.*

Proof. Let $f_j \to_{j\to\infty} f$ in $B^1(S)$. Then, according to Proposition 17, for $\alpha > 0$ fix, we can write

$$\mu(\{t \in S : \|f_j(t) - f(t)\| > \alpha\}) \leq \frac{1}{\alpha} \int_S \|f_j - f\| \, d\mu \underset{j\to\infty}{\to} 0.$$

This completes the proof of the corollary. □

Remark 34. The converse implication in Corollary 18 is not generally true, even in the real-valued case (see, for instance, [44], p. 112).

The proof of the following result follows the proof in the real-valued case (see, for instance, [44], p. 112 and p. 108, Theorem 3).

Proposition 19. *Convergence in μ-measure does not generally imply μ-a.e. convergence, even in the real-valued case. However, if $f_j \to_{j\to\infty} f$ in μ-measure, there is a subsequence $\{f_{j_k}\}_{k\geq 1}$ that converges to f μ-a.e. on S.*

Corollary 7. If $f_j \to_{j\to\infty} f$ in $B^1(S)$, there is a subsequence $\{f_{j_k}\}_{k\geq 1}$ that converges to f μ-a.e. on S.

Remark 35. Convergence μ-a.e. does not imply, in general, convergence in μ-mean or convergence in μ-measure, even in the real-valued case (see, for instance, [44], p. 112).

Definition 22. Let

$$B^0(S) = \{f : S \to X : f \text{ is strongly measurable}\}.$$

According to Remark 15, $B^0(S)$ is a real linear space. When it is necessary to identify the Banach space, we will write $B^0(S, X)$.

Proposition 20. *When the measure μ is finite, $B^0(S)$ becomes a semi-metric space with the semi-metric*

$$d_{B^0}(f, g) = \int_S \frac{\|f-g\|}{1+\|f-g\|} d\mu.$$

Proof. It follows the real-valued case (see, for instance, [44], p. 110, Lemma 7). □

Proposition 21. *We assume that the measure μ is finite. Then, given a sequence $\{f_j\}_{j\geq 1}$ and a function f in $B^0(S)$, the following statements are equivalent:*

1. *$f_j \to_{j\to\infty} f$ in μ-measure.*
2. *$d_{B^0}(f_j, f) \to_{j\to\infty} 0$.*

Proof. It follows the real-valued case (see, for instance, [44], p. 110, Theorem 9 (i)). □

Proposition 22. *Again, we assume that the measure μ is finite. Then, $\left(B^0(S), d_{B^0}\right)$ is a complete semi-metric space.*

Proof. It follows the real-valued case (see, for instance, [44], p. 109, Theorem 6 and Theorem 9). □

When the measure μ is finite, instead of Remark 35, we have the following positive results:

Proposition 23. *If the measure μ is finite, the following statements hold:*

1. *Convergence μ-a.e. implies convergence in μ-measure.*

2. *Convergence μ-a.e. implies convergence in μ-mean.*

Proof. We fix a sequence $\{f_j\}_{j\geq 1}$ and a function f, in $B^0(S)$, so that $f_j \to_{j\to\infty} f$, μ-a.e. on S. Then,

$$\frac{\|f_j(t) - f(t)\|}{1 + \|f_j(t) - f(t)\|} \underset{j\to\infty}{\to} 0,$$

for μ-a.a. $t \in S$. Let us recall that μ-a.a. $t \in S$ means for $t \in S\backslash N$, where N is a μ-null set. Moreover,

$$\frac{\|f_j(t) - f(t)\|}{1 + \|f_j(t) - f(t)\|} \leq 1,$$

for μ-a.a. $t \in S$ and for all $j \geq 1$. Since μ is a finite measure, Lebesgue's Dominated Convergence Theorem tell us that

$$d_{B^0}(f_j, f) \to_{j\to\infty} 0.$$

That is to say, according to Proposition 23, $f_j \to_{j\to\infty} f$ in μ-measure. So, we have 1).

To prove 2), we use again Lebesgue's Dominated Convergence Theorem, to conclude that there exists

$$\lim_{j\to\infty} \int_S \|f_j - f\| \, d\mu = 0.$$

Thus, we have 2).

This completes the proof of the proposition. □

There are other forms of convergence, for example, those related to uniform convergence, that we will not consider here (for the real-case see, for instance, [44], p. 112). For a complete analysis and comparison of modes of convergence in the real-valued case, see ([32], p. 237).

6 The Radon-Nikodým Property for the Bochner integral

It is often said that the Bochner integral is, for the most part, "just" the Lebesgue integral, with norms instead of absolute values. While this assertion might appear to be true some times, other times becomes a gross underestimation of the Bochner integral. Nowhere is this underestimation more patent than when one considers the problem of representing the action of a vector measure as an integral. This is the subject we are about to take up.

Once again, we fix a complete and σ-finite measure space (S, Σ, μ). We also fix a Banach space X.

We begin with a definition.

Definition 23. Given $f \in B^1(S)$, let $m_f : \Sigma \to X$ be the vector-valued set function defined as

$$m_f(A) = \int_A f d\mu.$$

Theorem 6. *The following statements hold:*

1. *There exists*
$$\lim_{\mu(A)\to 0} m_f(A) = 0.$$
2. *The set function m_f is a vector measure.*
3. *The vector measure m_f has finite variation and*
$$|m_f|(A) = \int_A \|f\|\, d\mu, \tag{30}$$
for every $A \in \Sigma$.

Proof. To prove 1), we use Remark 22 and 1) in Lemma 16, to obtain

$$\|m_f(A)\| = \left\|\int_A f d\mu\right\| \leq \int_A \|f\|\, d\mu \underset{\mu(A)\to 0}{\to} 0,$$

according, for instance to ([44], p. 88, Theorem 17).

Let us prove 2). By convention, if no other way, $m_f(\varnothing) = 0$. Next, let $\{A_j\}_{j\geq 1} \subseteq \Sigma$ be a family of pairwise disjoint sets. Then,

$$\begin{aligned}
\left\| m_f\left(\bigcup_{j\geq 1} A_j\right) - \sum_{j=1}^{k} m_f(A_j)\right\| &= \left\|\int_{\bigcup_{j\geq 1} A_j} f d\mu - \sum_{j=1}^{k}\int_{A_j} f\right\| \\
&= \left\|\int_{\bigcup_{j\geq 1} A_j} f d\mu - \int_{\bigcup_{1\leq j\leq k} A_j} f d\mu\right\| \\
&= \left\|\int_{\bigcup_{j\geq k+1} A_j} f d\mu\right\| \underset{k\to\infty}{\to} 0,
\end{aligned}$$

since $\mu\left(\bigcup_{j\geq k+1} A\right) \to_{k\to\infty} 0$, according to Proposition 1, which applies, word by word, to the case of a measure $\mu : \Sigma \to [0,\infty]$.

To prove 3), let $\{A_j\}_j \subseteq \Sigma$ be any finite partition of S. Then,

$$\sum_j \|m_f(A_j)\| = \sum_j \left\|\int_{A_j} f d\mu\right\| \underset{\text{(i)}}{\leq} \sum_j \int_{A_j} \|f\|\, d\mu = \int_S \|f\|\, d\mu,$$

where in (i) we have used Remark 22 and 1) in Lemma 16. Thus,

$$|m_f|(S) \leq \int_S \|f\|\, d\mu,$$

which proves that m_f has finite variation.

To prove (30), we explain in detail the proof sketched in ([18], p. 46, Theorem 4 iv)).

For starters, we take $A \in \Sigma$ and we fix a sequence $\left\{\varphi_j\right\}_{j\geq 1}$ of vector-valued simple functions, so that

$$\int_A \left\|\varphi_j - f\right\| d\mu \underset{j\to\infty}{\to} 0.$$

That is, for each $\varepsilon > 0$ fixed, there exists $J = J_\varepsilon \geq 1$ such that

$$\int_A \left\|\varphi_j - f\right\| d\mu < \varepsilon, \tag{31}$$

for $j \geq J$. We fix $j = J$ and write

$$\varphi_J = \sum_{k=1}^{K^J} x^J \chi_{AJ}.$$

For the same fixed $\varepsilon > 0$, there exists a finite partition $\{B_l^\varepsilon\}_{1\leq l\leq L^\varepsilon} \subseteq \Sigma$, of S, so that

$$|m_f|(A) - \varepsilon < \sum_{l=1}^{L^\varepsilon} \|m_f(B_l^\varepsilon)\|. \tag{32}$$

Since the sets A_k^J, jointly with the set $A\backslash \bigcup_{1\leq j\leq K^j} A_k^J$ form a partition of A, we can write

$$\begin{aligned}
\sum_{l=1}^{L^\varepsilon} \|m_f(B_l^\varepsilon)\| &= \sum_{l=1}^{L^\varepsilon} \left\| m_f \left[\left(\left(\bigcup_{j=1}^{K^J} A_k^J \right) \bigcap B_l^\varepsilon \right) \bigcup \left(\left(A \backslash \bigcup_{j=1}^{K^{Jj}} A^J \right) \bigcap B_l^\varepsilon \right) \right] \right\| \\
&= \sum_{l=1}^{L^\varepsilon} \left\| \left(\sum_{k=1}^{K^J} m_f\left(A_k^J \bigcap B_l^\varepsilon \right) \right) + m_f\left(\left(A \backslash \bigcup_{j=1}^{K^J} A_k^J \right) \bigcap B_l^\varepsilon \right) \right\| \\
&\leq \sum_{l=1}^{L^\varepsilon} \sum_{k=1}^{K^J} \left\| m_f\left(A_k^J \bigcap B_l^\varepsilon \right) \right\| \\
&\quad + \sum_{l=1}^{L^\varepsilon} \left\| m_f\left(\left(A \backslash \bigcup_{j=1}^{K^J} A_k^J \right) \bigcap B_l^\varepsilon \right) \right\|.
\end{aligned} \tag{33}$$

Thus, (32) also holds for the new partition, which is a refinement of the partition $\{B_l^\varepsilon\}_{1\leq l\leq L^\varepsilon}$. For brevity, let us denote this refinement $\mathcal{P}$ and let us call C the sets in $\mathcal{P}$.

We have,

$$\begin{aligned}
\sum_{C\in\mathcal{P}}\left\|m_{\varphi_J}(C)\right\| &= \sum_{l=1}^{L^\varepsilon}\sum_{k=1}^{K^J}\left\|m_{\varphi_J}\left(A_k^J\bigcap B_l^\varepsilon\right)\right\| \\
&\quad+\sum_{l=1}^{L^\varepsilon}\left\|m_{\varphi_J}\left(\left(A\backslash\bigcup_{k=1}^{K^J}A_k^J\right)\bigcap B_l^\varepsilon\right)\right\| \\
&= \sum_{l=1}^{L^\varepsilon}\sum_{k=1}^{K^j}\left\|\sum_{m=1}^{K^J}x_m^J\mu\left(A_k^J\bigcap B_l^\varepsilon\bigcap A_m^J\right)\right\| \\
&\quad+\sum_{l=1}^{L^\varepsilon}\left\|x_m^J\mu\left(A_m^J\bigcap\left(A\backslash\bigcup_{j=1}^{K^J}A_k^J\right)\bigcap B_l^\varepsilon\right)\right\| \\
&= \sum_{l=1}^{L^\varepsilon}\sum_{k=1}^{K^J}\left\|x_m^J\mu\left(A_k^J\bigcap B_l^\varepsilon\right)\right\| = \sum_{l=1}^{L^\varepsilon}\sum_{k=1}^{K^J}\left\|x_m^J\right\|\mu\left(A_k^J\bigcap B_l^\varepsilon\right) \\
&= \int_A\left\|\varphi_J\right\|d\mu.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\left|m_f\right|(A)-\int_A\|f\|\,d\mu\right| &\leq \underbrace{\left|m_f\right|(A)-\sum_{C\in\mathcal{P}}\left\|m_f(C)\right\|}_{\text{(i)}} \\
&\quad+\underbrace{\left|\sum_{C\in\mathcal{P}}\left\|m_f(C)\right\|-\sum_{C\in\mathcal{P}}\left\|m_{\varphi_J}(C)\right\|\right|}_{\text{(ii)}} \\
&\quad+\underbrace{\left|\int_A\left\|\varphi_J\right\|d\mu-\int_A\|f\|\,d\mu\right|}_{\text{(iii)}}.
\end{aligned}$$

According to (33), (i) is $<\varepsilon$. As for (ii), it can be bound by

$$\begin{aligned}
\sum_{C\in\mathcal{P}}\left|\left\|m_f(C)\right\|-\left\|m_{\varphi_J}(C)\right\|\right| &\leq \sum_{C\in\mathcal{P}}\left\|m_f(C)-m_{\varphi_J}(C)\right\| \\
&= \sum_{C\in\mathcal{P}}\left\|\int_C f\,d\mu-\int_C\varphi_J\,d\mu\right\| \\
&\leq \int_A\left\|f-\varphi_J\right\|d\mu<\varepsilon,
\end{aligned}$$

according to (31).

Likewise, we bound (iii) by

$$\int_A \|f - \varphi_J\| \, d\mu < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen and $\mathbb{R}$ has the Archimedean Property, we conclude that (30) is true.

This completes the proof of the theorem. □

As a consequence of Theorem 6, we present now the general version of Corollary 6.

Corollary 8. If $f \in B^1(S)$ and $\int_A f d\mu = 0$ for all $A \in \Sigma$, $f = 0$ μ-a.e. on S.

Proof. Since

$$m_f(A) = \int_A f d\mu = 0,$$

for every $A \in \Sigma$, according to 3) in Theorem 6, and

$$\int_S \|f\| \, d\mu = |m_f|(S) = 0.$$

Thus, we can conclude (see, for instance, [44], p. 84, Proposition 10) that there exists a μ-null set N in Σ, so that $\|f\|(t) = 0$, for $t \in S \backslash N$.

This completes the proof of the corollary. □

Corollary 9. The vector measure m_f in Definition 23 and Theorem 6, is absolutely continuous with respect to μ.

Proof. Since 1) in Theorem 6 shows that m_f is μ-continuous, the proof of this corollary follows immediately from Definition 9 and 1) in Proposition 8. □

Remark 36. The series $\sum_{j\geq 1} m_f(A_j)$ converges, not only unconditionally (see Remark 1), but also absolutely. In fact,

$$\begin{aligned} \sum_{j\geq 1} \|m_f(A_j)\| &= \sum_{j\geq 1} \left\| \int_{A_j} f d\mu \right\| \leq \sum_{j\geq 1} \int_{A_j} \|f\| \, d\mu \\ &= \int_{\bigcup_{j\geq 1} A_j} \|f\| \, d\mu \leq \int_S \|f\| \, d\mu. \end{aligned}$$

Let us recall that this is not the case for an arbitrary vector measure, unless X has finite linear dimension (see Remark 1 and (16) in Remark 11).

It is natural to ask whether, given a vector measure $m : \Sigma \to X$, there is a function $f_m \in B^1(S)$ so that

$$m(A) = \int_A f_m d\mu, \tag{34}$$

for every $A \in \Sigma$.

On account of Theorem 6, for m to be written as in (34), m would need, at least, to have finite variation and to be μ-continuous. For a vector-measure that has finite variation, μ-continuity is equivalent to absolute continuity with respect to μ (see, Proposition 8). So, we can, equivalently, say that m needs to be absolutely continuous with respect to μ.

For such a vector measure, the answer to the question of representability is, generally, no. Before giving an example, we need to prove an auxiliary known result, which admits various formulations (see, for instance, [1], p. 29, Lemma 4).

Lemma 17. *Let $F : I \to B^1\left(I, L^1(I)\right)$ be a strongly measurable function. Then, there exists $f \in L^1(I \times I)$ and a λ-null set $N \subseteq I$ so that $F = f(t, \cdot)$, for $t \in I \backslash N$.*

Proof. Let us first observe that $L^1(I \times I)$ is defined with respect to the measure space $(I \times I, \mathcal{L}_{I\times I}, \lambda \times \lambda)$, where $\mathcal{L}_{I\times I}$ is the σ-algebra of the Lebesgue measurable subsets of the unit square $I \times I$, and the measure $\lambda \times \lambda$, defined on $\mathcal{L}_{I\times I}$, is the (completed) product measure, (see, for instance, [44], p. 118), that is to say, the Lebesgue measure on $\mathcal{L}_{I\times I}$.

By Definition 12, there exists a sequence $\left\{\varphi_j\right\}_{j\geq 1}$ of vector-valued simple functions so that $\varphi_j \to F$, μ-a.e. on I, as $j \to \infty$. That is, for each $j \geq 1$, we can write

$$\varphi_j(t, s) = \sum_{k=1}^{K^j} x_k^j(s)\chi_{A_k^j}(t),$$

with $x_k^j \in L^1(I)$, and

$$\int_I \left|\varphi_j(t) - F(t)\right| d\lambda \underset{j\to\infty}{\to} 0, \tag{35}$$

for $t \in I\backslash N_1$, where N_1 is a λ-null set. Let us observe that, in fact, we have selected a representative x_k^j, of the class x_k^j in the Banach space $L^1(I)$. It should be clear that the function $\varphi_j(t, s)$ belongs to $L^1(I \times I)$.

For each $j \geq 1$, let A_j be the measurable set defined as

$$A_j = \left\{t \in I : \left\|\varphi_j(t)\right\|_{L^1(I)} \leq 2\left\|F(t)\right\|_{L^1(I)}\right\}.$$

Set

$$\psi_j = \varphi_j \chi_{A_j}.$$

We claim that $\left\{\psi_j\right\}_{j\geq 1}$ is a Cauchy sequence in $L^1(I \times I)$. Indeed, from (35), we can say that there is

$$\lim_{j,m\to\infty} \int_I \left|\psi_j(t, s) - \psi_m(t, s)\right| d\lambda_s = 0,$$

for $t \in I \backslash N_1$. For $l \geq 1$, let

$$B_l = \left\{ t \in I : \|F(t)\|_{L^1(I)} \leq l \right\}.$$

Then,

$$\int_I \left|\psi_j(t,s) - \psi_m(t,s)\right| d\lambda \leq 4l,$$

for all $j, m \geq 1$, $t \in B_l$. Using Theorem 4 on the sequence $\left\{\psi_j - \psi_m\right\}_{j,m\geq 1}$, there is

$$\lim_{j,m\to\infty} \int_{B_l} \left(\int_I \left|\psi_j(t,s) - \psi_m(t,s)\right| d\lambda_s \right) d\lambda_t = 0,$$

for each $l \geq 1$. So, there is a function $f^l \in L^1(I \times I)$ so that

$$\psi_j \underset{j\to\infty}{\to} f^l$$

in $L^1(B_l \times I)$ for each $l \geq 1$. Moreover, $f^{l+1}/B_l \times I = f^l$, $(\lambda \times \lambda)$-a.e.

Now, we define a function f, unique up to a $(\lambda \times \lambda)$-null set, as $f = f^l$ on $B^l \times I$. It should be clear that f is $(\lambda \times \lambda)$-measurable. Moreover, there exists

$$\lim_{j\to\infty} \int_{B^l \times I} \left|\psi_j - f\right| d(\lambda \times \lambda) \underset{\text{(i)}}{=} \lim_{j\to\infty} \int_{B^l} \left(\int_I \left|\psi_j(t,s) - f(t,s)\right| d\lambda_s \right) d\lambda_t = 0,$$

for each $l \geq 1$, where the equality (i) follows from Fubini's Theorem (see, for instance, [44], p. 119, Theorem 3). Thus (see, Corollary 7), there is a subsequence $\left\{\psi_{j_k}\right\}_{k\geq 1}$ and a λ-null set $N_2 \subseteq I$, so that

$$\int_I \left|\psi_{j_k}(t,s) - f(t,s)\right| d\lambda_s \underset{j_k\to\infty}{\to} 0,$$

for $t \in I \backslash N_2$. According to (35),

$$\int_I \left|\psi_j - F(t)\right| d\lambda \underset{j\to\infty}{\to} 0,$$

for $t \in I \backslash N_1$, so, we conclude that $f(t,s) = F(t)(s)$, λ-a.e. in I, for each $t \in I \backslash (N_1 \bigcup N_2)$.

This completes the proof of the lemma. $\square$

Now, we are ready to present the example, which is mentioned, with some differences, in several sources (see, for instance, [6], p. 103):

Example 3. *We consider the Lebesgue measure space $(I, \mathcal{L}_I, \lambda)$ and the measure $m_1 : \mathcal{L}_I \to L^1(I)$ defined as $m_1(A) = \chi_A$ (see Example 1). According to Example 2, m_1 has finite variation and, since $|m_1| = \lambda$, the vector measure m_1 is obviously absolutely* *continuous with respect to* μ*. Let us suppose that there is a function* $f_{m_1} \in B^1\left(I, L^1(I)\right)$ *so that*

$$m_1(A) = \int_A f_{m_1} d\lambda, \tag{36}$$

for each $A \in \Sigma$*.*

According to Lemma 17, we can view f_{m_1} *as a function* $f_{m_1}(s,t)$ *defined and Lebesgue integrable on the unit square* $I \times I$*. Thus, we can write*

$$\chi_A(t) = \int_A f_{m_1}(t,s)\, d\lambda_s.$$

Now, given $A, B \in \mathcal{L}_I$*, disjoint,*

$$\int_{A\times B} f_{m_1} d(\lambda \times \lambda) = \int_B \chi_A(t)\, d\lambda_t = \lambda\left(A \bigcap B\right) = 0.$$

So, the function f_{m_1} *ends up being zero outside the diagonal in* $I \times I$ *(see, for instance, [44], p. 88, Proposition 18). That is to say,* f_{m_1} *is zero,* $(\lambda \times \lambda)$*-a.e., which goes against the assumed representation (36).*

Remark 37. If c_0 denotes the Banach space of real sequences that converge to zero, with the $\sup$ norm, an example involving trigonometric series (see [18], p. 60, Example 1) or ([6], p. 103, (ii)), shows that vector measures with values in c_0 are not, generally, representable. A third example, can be seen in ([6], p. 103, (iii)), showing the same for the Banach space $C(I)$ of real valued functions that are continuous on I, with the $\sup$ norm. For a discussion on the role played by atoms (see Definition 6) in the representability of measures, see ([18], p. 61).

When m is a signed measure, the question of whether it can be represented as an integral, has an affirmative answer, given by the Radon-Nikodým theorem, which we state as follows:

Theorem 7. *Let* m *be a signed measure* $m : \Sigma \to \mathbb{R}$ *that is absolutely continuous with respect to* μ*. Then, there exists* $f = f_m \in L^1(S)$ *so that (34) holds for every* $A \in \Sigma$*.*

Let us recall that a signed measure $m : \Sigma \to \mathbb{R}$ is always bounded (see, for instance, [44], p. 30, Theorem 5). Moreover, a signed measure $m : \Sigma \to \mathbb{R}$ is bounded if, and only if, it has finite variation (see, for instance, [44], p. 31, Corollary 8). Thus, in the statement of Theorem 7, absolute continuity is equivalent to μ-continuity (see, for instance, [44], p. 132, Proposition 5 (ii)).

For the proof of Theorem 7 as stated, see, for instance, ([38], p. 238, Theorem 23). Let us recall that μ is a σ-finite measure. Without this assumption, the conclusion of Theorem 7 is not generally true (see, for instance, [44], p. 134, Example 9). For slightly different versions of Theorem 7 and their proofs see, for instance, ([44], p. 133, Theorem 8), ([12], p. 132, Theorem 4.2.2 and p. 135, Theorem 4.2.3), ([39], p. 121, Theorem 6.10) and, of course, many other sources.

H. Lebesgue proved the first version of Theorem 7 in 1904 [28]. In fact, the evolution of the Radon-Nikodým theorem for measures, runs parallel to that of the Lebesgue decomposition theorem. For a fairly detailed account of these historical matters, see ([3], pp. 89-90; [19], p. 414, Notes and Remarks 11), and the references therein.

In the words of Diestel and Uhl ([18], p. 51), "The failure of the Radon-Nikodým theorem for the Bochner integral is not to be interpreted as a negative aspect of the Bochner integral. Indeed, the failure ... has powerful repercussions in operator theory, the geometry of Banach spaces, duality theory for Banach spaces, vector-valued probability theory and integration theory itself." We may add that the failure has been, indeed, very fruitful.

So, instead of looking for the Radon-Nikodým theorem for the Bochner integral, we need to talk about the Radon-Nikodým property for the Bochner integral, which is the title of this section.

Definition 24. A Banach space X has the Radon-Nikodým property with respect to a complete and σ-finite measure space (S, Σ, μ), if given a vector measure $m : \Sigma \to X$ that has finite variation and is absolutely continuous with respect to μ, there exists $f_m \in B^1(S)$ so that (34) holds.

A Banach space X has the Radon-Nikodým property, in short RNp, if it has the RNp with respect to every complete and σ-finite measure space (S, Σ, μ).

Remark 38. When the function f_m in Definition 24 exists, it is unique up to a μ-null set. In fact, if there are two such functions, $f_{m,1}$ and $f_{m,2}$, their difference should represent the zero measure, so, (see (30)),

$$\int_S \|f_{m,1} - f_{m,2}\| \, d\mu = 0,$$

and the conclusion follows.

When m and f are related by

$$m(A) = \int_A f d\mu,$$

for all $A \in \Sigma$, it is usual to write

$$m = f d\mu$$

and to refer to f as the *Radon-Nikodým derivative of* m *with respect to* μ, denoted $\frac{dm}{d\mu}$. In this case, we refer to m as the *indefinite integral of* f. Of course, the inspiration for this terminology comes from Calculus. At least in the case of signed measures, the terminology is justified by properties that echo those encountered in Calculus (see, for instance, [38], p.241; [44], p. 135).

Proposition 24. *Fix a complete and σ-finite measure space (S, Σ, μ) and assume that the Banach space X has the Radon-Nikodým property with respect to (S, Σ, μ). Let*

$$\mathcal{M}_{f,a} = \{m : \Sigma \to X : m \text{ is a vector measure of finite variation and } m \ll \mu\}.$$

Then, the following statements hold:

1. *The space $\mathcal{M}_{f,a}$ is a normed linear subspace of $\mathcal{M}_f$ (for the definition, see Proposition 7).*
2. *The map $\Lambda : B^1(S) \to \mathcal{M}_{f,a}$, defined as $\Lambda(f) = fd\mu$, is an isometric semi-isomorphism. That is to say, it is linear, surjective, $|fd\mu|(S) = \|f\|_{B^1(S)}$ and $\Lambda(f) = 0$ implies that $f = 0$, μ-a.e. on S.*
3. *The space $\mathcal{M}_{f,a}$ is Banach.*

Proof. The proof of 1) is an immediate consequence of Proposition 7. As for 2), it follows from Definition 24 and Theorem 6, since given $m \in \mathcal{M}_{f,a}$,

$$\|m\| = \|f_m\|_{B^1(S)}.$$

Finally, if $\{m_j\}_{j\geq 1}$ is a Cauchy sequence in $\mathcal{M}_{f,a}$, the sequence $\{f_{m_j}\}_{j\geq 1}$ is Cauchy in $B^1(S)$, so it converges to some $f \in B^1(S)$. Therefore, $fd\mu \in B^1(S)$ and

$$\|m_j - fd\mu\| = \left\|f_{m_j} - f\right\|_{B^1(S)} \underset{j\to\infty}{\to} 0,$$

which proves 3).

This completes the proof of the proposition. □

When X is the real space $\mathbb{R}$, we can say quite a bit more.

Proposition 25. *The linear space*

$$\mathcal{M}_{\mathbb{R}} = \{m : \Sigma \to \mathbb{R} : m \text{ is a signed measure}\},$$

is complete with the norm $\|m\| = |m|(S)$.

Proof. We give a fairly direct proof of this known result.

First, let us observe that $m \in \mathcal{M}_{\mathbb{R}}$ being finite, automatically implies that m has finite variation (see Remark 7). Next, let $\{m_j\}_{j\geq 1}$ be a Cauchy sequence in $\mathcal{M}_{\mathbb{R}}$. That is to say, given $\varepsilon > 0$, there is $J = J_\varepsilon \geq 1$ so that

$$\|m_j - m_k\| < \varepsilon,$$

for $j, l \geq J$. According to (5), this implies that,

$$\underset{\text{(i)}}{|m_j(A) - m_k(A)|} \leq \underset{\text{(ii)}}{|m_j - m_k|}(S) \underset{j,k\to\infty}{\to} 0, \tag{37}$$

for every $A \in \Sigma$, so the sequence $\{m_j\}_{j\geq 1}$ of set functions from Σ to $\mathbb{R}$ is Cauchy, uniformly on Σ. Let us observe that $|\cdot|$ in (i) is the absolute value in $\mathbb{R}$, while $|\cdot|$ in (ii) indicates the variation of the signed measure $m_j - m_k$. This slight ambiguity should not cause any trouble.

From (37), the real sequence $\{m_j(A)\}_{j\geq 1}$ is a Cauchy sequence in $\mathbb{R}$ for each $A \in \Sigma$, so it has limit. We define a set function $m : \Sigma \to \mathbb{R}$ as

$$m(A) = \lim_{j\to\infty} m_j(A). \tag{38}$$

Since the sequence $\{m_j\}_{j\geq 1}$ is Cauchy, uniformly on Σ, the limit in (38) is uniform on Σ. We claim that the set function m is a signed measure. First, it is clear that $m(\emptyset) = 0$. To prove that m is countably additive, we begin by observing that m is finitely additive, by its definition. In fact, if $\{A_l\}_l$ is any finite family of pairwise disjoint sets in Σ,

$$\begin{aligned} m\left(\bigcup_l A_l\right) &= \lim_{j\to\infty} m_j\left(\bigcup_l A_l\right) = \lim_{j\to\infty}\sum_l m_j(A_l) \\ &= \sum_l \lim_{j\to\infty} m_j(A_l) = \sum_l m(A_l). \end{aligned}$$

Next, let us consider any countable family of pairwise disjoint sets in Σ, $\{A_l\}_{l\geq 1}$, and let us write $A = \bigcup_{l\geq 1} A_l$. For $L \geq 1$ and $j \geq 1$ to be chosen later, we can write

$$\begin{aligned} \left|m(A) - \sum_{l=1}^{L} m(A_l)\right| &\leq |m(A) - m_j(A)| + \left|\sum_{l=1}^{L}(m_j(A_l) - m(A_l))\right| \\ + \left|\sum_{l\geq L+1} m_j(A_l)\right| &= (i) + (ii) + (iii). \end{aligned}$$

Let us estimate each of these three terms.

For (i), $|m(A) - m_j(A)| < \varepsilon/3$, for $j \geq J = J_\varepsilon$, independently of $A \in \Sigma$. We then fix $j = J$ in the other two terms. For (iii), since $\sum_{l\geq 1} m_J(A_l)$ converges, to $m_J(A)$, $\left|\sum_{l\geq L+1} m_J(A_l)\right| < \varepsilon/3$, for $L \geq L_0$. Finally, for $L \geq L_0$, we can write (ii) as

$$\left|\sum_{l=1}^{L}(m_J(A_l) - m(A_l))\right| = \left|(m_J - m)\left(\bigcup_{l=1}^{L} A_l\right)\right| < \varepsilon/3,$$

since we already observed that the convergence is uniform on Σ. Thus, m is a signed measure and, hence, it has finite variation. That is to say, $m \in \mathcal{M}_{\mathbb{R}}$.

The last step is to prove that $\{m_j\}_{j\geq 1}$ converges to m in $\mathcal{M}_{\mathbb{R}}$, for which we use the right hand side of the following inequality (see (13) in Remark 7):

$$\sup_{A\in\Sigma} |m_j(A) - m(A)| \leq |m_j - m|(S) \leq 2\sup_{A\in\Sigma} |m_j(A) - m(A)|.$$

If we use again the uniform convergence of $\{m_j\}_{j\geq 1}$ to m, given $\varepsilon > 0$, we have $|m_j(A) - m(A)| < \varepsilon/2$, for $j \geq J = J_\varepsilon$ and for all $A \in \Sigma$. This completes the proof of the proposition. □

So far, we haven't said much about when the Radon-Nikodým property holds or doesn't hold. We dedicate the rest of the section to remedy this deficiency.

Remark 39. According to Remark 37, Example 3 shows that $L^1(I)$ does not have the Radon-Nikodým property. The Banach space c_0 does not have it either (see [18], p. 60, Example 1; [6], p. 103, (ii)). Example (iii) in ([6], p. 103), as well as Example 8 in ([18], p. 73), show that $C(I)$ does not have the Radon-Nikodým property.

The question of when a Banach space has the Radon-Nikodým property has grown into a huge subject, with vast repercussions, on which we do not intend to dwell for long. To show how diverse the answers to the question of representability can be, we state two of these answers.

Theorem 8. *([37]) Let (S, Σ, μ) be a complete and σ-finite measure space. For a vector measure $m : \Sigma \to X$, the following statements, 1) and 2), are equivalent:*

1. *There exists a function $f \in B^1(S)$, such that $m = f d\mu$.*
2. *The vector measure m satisfies the following conditions:*
   (a) *$m \ll \mu$.*
   (b) *m has finite variation.*
   (c) *For each $A \in \Sigma$ with $0 < \mu(A) < \infty$, there exists $E \subseteq A$, $E \in \Sigma$ and a compact set $K \subseteq X$ not containing zero, so that $\mu(E) > 0$ and for all $E' \subseteq E$, $E' \in \Sigma$, the set $m(E')$ is contained in the cone generated by K.*

Remark 40. When the space X has finite linear dimension, 2)c) in Theorem 8 is satisfied by any vector measure $m : \Sigma \to X$. In fact, when we choose

$$K = \{x \in X : \|x\| = 1\},$$

the cone generated by K, which is defined as

$$\{\lambda x : x \in K, \lambda \geq 0\},$$

is X, so 2) c) holds. That is to say, Theorem 8 becomes the familiar Radon-Nikodým theorem, when X is finite dimensional.

Theorem 8 is considered the first general Radon-Nikodým theorem for the Bochner integral. By general we mean that, although X needs to satisfy a certain geometric condition, it is not a particular type of Banach space.

As for the promised second answer to the question of representability, it appears as Theorem 5.21 in ([6], p. 112). Benyamini attributes it to S. Qian [35]. To simplify matters, of the two statements that are proved to be equivalent to the Radon-Nikodým property, we only mention one, which involves differentiability. The other equivalent statement involves the notion of ε-differentiability (see [6], p. 111, Definition 5.19).

Theorem 9. *([35]) Let X be a Banach space. Then, the following two assertions are equivalent:*

1. *X has the Radon-Nikodým property.*
2. *Every absolutely continuous function $F : [0,1] \to X$ is differentiable almost everywhere.*

Absolute continuity of a vector-valued function is defined in the same way as for a real-valued function (for the real case see, for instance, [44], p. 160, Definition 1). That is also the case for the notion of differentiability (see, for instance, [15], Chapter VIII).

Remark 41. The Banach space l^1 of absolutely summable real sequences $\{x_j\}_{j\geq 1}$ with the norm $\sum_{j\geq 1} |x_j|$, has the Radon-Nikodým property (see [18], p. 64). That is also the case for separable Hilbert spaces (see [18], p. 64, Theorem 6; p. 67) and for Banach spaces that are duals and separable ([18], p. 79, Theorem 1).

Many results on vector measure representability, run parallel to the representability of certain linear and continuous operators. This is a very interesting principle, which, at least for some spaces, has been known for a long time (see, for instance, ([19], p. 415, Notes and Remarks 12 and the references therein).

Indeed, if (S, Σ, μ) is a complete and finite measure space, the Riesz representation theorem tells us that the topological dual of $L^1(S)$ is $L^\infty(S)$, while Theorem 7 gives us the representability of certain signed measures. The same two results can be stated for vector-valued functions and measures. Specifically, ([18], p. 59),

Theorem 10. *(Riesz representation theorem) If X is a Banach space, each linear and continuous operator $T : L^1(S) \to X$ can be represented as*

$$T(f) = \int_S fg d\mu, \tag{39}$$

for a function $g \in B^\infty(S)$.

Theorem 11. *(Radon-Nikodým theorem) Given a vector measure $m : \Sigma \to X$ that has finite variation and is absolutely continuous with respect to μ, there exists $g \in B^1(S)$ so that*

$$m(A) = \int_A g d\mu,$$

for all $A \in \Sigma$.

Diestel and Uhl discuss (see [18], pp. 60-61) several examples that illustrate the interplay between these two theorems. Furthermore, they prove, among others, the following result (see [18], p. 63, Theorem 5), that makes this interplay official:

Theorem 12. *Let X be a Banach space and let (S, Σ, μ) be a complete and finite measure space. Then, X has the Radon-Nikodým property with respect to (S, Σ, μ) if, and only if, each linear and continuous $T : L^1(S) \to X$ is representable as in (39).*

Benyamini states (see [6], p. 123) a version of Theorem 12 in terms of the topological dual of $B^p(I)$, for $1 \leq p < \infty$: It is always the case that $B^q(I, X')$ is isometrically embedded in the topological dual $(B^p(I, X))'$, where $\frac{1}{p} + \frac{1}{q} = 1$ and X'. That this embedding is onto if X has the Radon-Nikodým property, is due to S. Bochner and A. E. Taylor [9]. For a detailed discussion, see ([18], Chapter III).

For a historical account of the Riesz representation theorem, see [22].

There is a large body of work dedicated to the Radon-Nikodým property in all its manifestations. Besides the three monographs already mentioned and their extensive bibliographies, we mention, for instance, [10], [34], [16], [29], [17], [42], and the references therein.

The article [17] presents, in great detail, the evolution of the Radon-Nikodým property, in the vector-valued setting.

Thus, we conclude the study of vector measures we set to write.

References

[1] J. Alvarez, On the inversion of pseudo-differential operators, *Studia Math.*, Vol. 64 (1979) 25-32.

[2] J. Alvarez, C. Espinoza-Villalva and M. Guzmán-Partida, The integrating factor method in Banach spaces, *Sahand Communications in Mathematical Analysis*, Vol. 11, No. 1 (Summer 2018) 115-132, scma.maragheh.ac.ir/article_31559.html

[3] J. Alvarez and M. Guzmán-Partida, About and beyond the Lebesgue decomposition of a signed measure, *Lecturas Matemáticas*, Colombian Mathematical Society, Vol. 37, No. 2 (2016) 79-103. http://scm.org.co/archivos/revista/Articulos/1195.pdf

[4] G. Bachman and L. Narici, *Functional Analysis*, Academic Press 1966.

[5] R. G. Bartle, A general bilinear vector integral, *Studia Math.*, Vol. 15 (1956) 337-352.

[6] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Volume I*, American Mathematical Society 2000.

[7] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges: A Study of Finitely Additive Measures*, Academic Press 1983

[8] S. Bochner, Integration von Funkionen, deren Werte die Elemente eines Vectorraumes sind, *Fund. Math.*, Vol. 20 (1933) 262-276.

[9] S. Bochner and A. E. Taylor, Linear functionals on certain spaces of abstractly valued functions, *Ann. of Math.*, Vol. 39 (1938) 913-944.

[10] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým property*, Lect. Notes in Math. 993, Springer 1983.

[11] J. K. Brooks, Decomposition theorems for vector measures, *Proc. Amer. Math. Soc.*, Vol. 21, No. 1 (April 1969) 27-29.

[12] D. L. Cohn, *Measure Theory*, Birkhäuser 1997.

[13] T. P. Dence, A Lebesgue decomposition for vector valued additive set functions, *Pacific J. Math.*, Vol. 57, No. 1 (1975) 91-98. https://msp.org/pjm/1975/57-1/pjm-v57-n1-p10-p.pdf

[14] J. DePree and C. Swartz, *Introduction to Real Analysis*, John Wiley & Sons 1988.

[15] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press 1960.

[16] J. Diestel and B. Faires, On vector measures, *Trans. Amer. Math. Soc.*, Vol. 198 (1974) 253-271. www.ams.org/journals/tran/1974-198-00/S0002-9947-1974-0350420-8/S0002-9947-1974-0350420-8.pdf

[17] J. Diestel and J. J. Uhl, Jr., The Radon-Nikodym theorem for Banach space valued measures, *Rocky Mountain J. Math.*, Vol. 6, No. 1 (Winter 1976) 1-46. https://projecteuclid.org/download/pdf_1/euclid.rmjm/1250130381.

[18] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys, Vol. 15, Amer. Math. Soc. 1977.

[19] N. Dinculeanu, *Vector Measures*, Pergamon Press 1967.

[20] R. M. Dudley and R. Norvaiša, *Concrete Functional Calculus*, Springer 2013.

[21] A. Dvoretsky and C. A. Rogers, Absolute and unconditional convergence in normed linear spaces, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 36 (1950) 192-197. http://www.ncbi.nlm.nih.gov/pmc/articles/PMC1063160

[22] J. D. Gray, The shaping of the Riesz representation theorem, *Arch. History Exact Sci.*, Vol. 31 (1984) 127-187.

[23] T. H. Hildebrandt, Unconditional convergence in normed vector spaces, *Bull. Amer. Math. Soc.*, Vol. 46 (1940) 959-962. www.ams.org/journals/bull/1940-46-12/S0002-9904-1940-07344-6/S0002-9904-1940-07344-6.pdf

[24] T. H. Hildebrandt, Integration in abstract spaces, *Bull. Amer. Math. Soc.*, Vol. 59 (1953) 111-139. https://projecteuclid.org/download/pdf_1/euclid.bams/1183517761

[25] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups, Revised and Expanded Edition*, American Mathematical Society 1957.

[26] R. C. James, Orthogonality in normed linear spaces, *Duke Math. J.*, Vol. 12 (1945) 291-302.

[27] J. L. Kelley, *General Topology*, D. Van Nostrand 1955.

[28] H. Lebesgue, *Leçons sur l'Intégration et la Recherche des Functions Primitives*, Gauthier-Villards 1904.

[29] S. Moedomo and J. J. Uhl, Jr., Radon-Nikodým theorems for the Bochner and Pettis integrals, *Pacific. J. Math.* Vol. 38, No. 2 (1971) 531-536. https://msp.org/pjm/1971/38-2/pjm-v38-n2-p23-p.pdf

[30] E. H. Moore, General Analysis, *Memoirs Amer. Phil. Soc.*, Vol. 1, No. 2 (1939).

[31] E. H. Moore and H. L. Smith, A general theory of limits, *Amer. J. Math.*, Vol. 44 (1922) 102-121. http://www.jstor.org/stable/pdf/2370388.pdf

[32] M. Munroe, *Introduction to Measure and Integration*, Addison-Wesley, 1953.

[33] B. J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc.*, Vol. 44, No. 2 (September 1938) 277-304. https://www.ams.org/journals/tran/1938-044-02/S0002-9947-1938-1501970-8/S0002-9947-1938-1501970-8.pdf

[34] R. R. Phelps, Convex Functions, Monotone Operators and Differentiability, Second Edition, *Lect. Notes in Math.* 1364, Springer 1993.

[35] S. Qian, Nowhere differentiable Lipschitz maps and the Radon-Nikodým property, *J. Math. Anal. Appl.*, Vol. 185 (1994) 613-616.

[36] J. R. Retherford, Review of *Vector Measures* by J. Diestel and J. J. Uhl, Jr., Math. Surveys, Vol. 15, Amer. Math. Soc. 1977, *Bulletin Amer. Math. Soc.* 84, No. 4 (July 1978). http://projecteuclid.org/euclid.bams/1183540941

[37] M. Rieffel, The Radon-Nikodym theorem for the Bochner integral, *Trans. Amer. Math. Soc.* Vol. 131 (1968) 466-487. https://www.ams.org/journals/tran/1968-131-02/S0002-9947-1968-0222245-2/S0002-9947-1968-0222245-2.pdf

[38] H. L. Royden, *Real Analysis, Second Edition*, MacMillan Publishing Co. 1968.

[39] W. Rudin, *Real and Complex Analysis, Third Edition*, McGraw-Hill 1987.

[40] W. Sierpiński, Sur les fonctions d'ensemble additives et continues, *Fund. Math.*, Vol. 3 No. 1 (1922) 240-246. http://matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3125.pdf

[41] H. L. Smith, A general theory of limits, *National Mathematics Magazine*, Vol. 12, No. 8 (May 1938) 371-379.

[42] K. Sundaresan, The Radon-Nikodym theorem for Lebesgue-Bochner function spaces, *J. Funct. Anal.*, Vol. 24 (1977) 276-279. `https://core.ac.uk/download/pdf/82724172.pdf`

[43] C. Swartz, *An Introduction to Functional Analysis*, Marcel Dekker 1992.

[44] C. Swartz, *Measure, Integration and Function Spaces*, World Scientific 1994.

[45] *Wikipedia*. https://en.wikipedia.org/wiki/Linearizability

[46] A. C. Zaanen, *Integration*, North-Holland 1967.

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