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A self-contained guide to Frécon's theorem

Una guía autocontenida al teorema de Frécon

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Abstract. A streamlined exposition of Frécon's theorem on non-existence of bad groups of Morley rank 3. Systematising ideas by Poizat and Wagner, we avoid incidence geometries and use group actions instead; the proof becomes short and completely elementary.

Key words and phrases. groups of finite Morley rank, bad groups.

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Resumen. Presentamos una breve demostración depurada del teorema de Frécon sobre la no existencia de grupos malos de rango de Morley 3. Abstrayendo ideas de Poizat y Wagner, evitamos el uso de las geometrías de incidencia. En su lugar usamos acciones de grupos; así la demostración se torna verdaderamente elemental y concisa.

Palabras y frases clave. Grupos de rango de Morley finito, grupos malos.

1. Introduction and prerequisites

History

Groups of finite Morley rank are the model-theoretic generalisation of affine algebraic groups over algebraically closed fields. Although their original definition was influenced by Morley's analysis of uncountable categoricity, which accounts for their name, they can be described purely algebraically after Borovik and Poizat: a *ranked group* is a group equipped with a dimension function on its definable sets, which behaves like the Zariski dimension in algebraic geometry.

Details can be found in the classical book [1]. The main open question in the field is the following.

Algebraicity Conjecture (Cherlin-Zilber). An infinite simple ranked group G is isomorphic, as a group, to $\Gamma(\mathbb{K})$ for some algebraic group Γ and some algebraically closed field $\mathbb{K}$.

In the seminal [2], where he formulated the conjecture, Cherlin undertook the classification of connected groups of small rank. If G has rank 1, a theorem by Reineke [7] already implied abelianity. In rank 2, Cherlin proved that G is soluble; in rank 3, that G is either soluble or simple. In the rank 3, simple case, Cherlin saw two subcases:

- either G has a definable subgroup of rank 2; in that case he could show $G \simeq \mathrm{PGL}_2(\mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field interpretable in G,
- or G is what Cherlin called a *bad group*: a simple group of rank 3 in which all definable, proper subgroups have rank ≤ 1.

Cherlin could not eliminate such 'bad groups' of rank 3 and the problem remained open until 2016. Frécon [4] proved non-existence of bad groups of rank 3, thus deriving the Cherlin-Zilber Algebraicity Conjecture in rank 3. Then Poizat and Wagner [6, 8, 5] provided rewritings of Frécon's proof, extending his non-existence result to a whole class of configurations with rank $2n + 1$.

Our article reformulates the latter proof, but avoids any reference to Frécon's planes or to incidence geometries. Instead our language is action-theoretic, with focus on involutions as one should expect. Clearly our results are not original. But our exposition shows the actual computational contents of the argument. Our point is that focus on planes brings no clarity as it hides more algebraic phenomena.

Prerequisites

We expect our reader to be familiar with basic model-theoretic notions. Here *definable* sets are truly interpretable with parameters. In a ranked groups, such sets bear an integer-valued dimension which enables basic fibre computations. We use the following facts on ranked groups.

Facts. Let G be a ranked group.

(1) There is a notion of genericity on definable sets and on points, a notion obeying the usual rules of algebraic geometry.

(2) There are notions of degree (for definable sets) and connected component (for definable *subgroups*); a definable subgroup is connected iff it has degree 1.

(3) Unique divisibility: if G has no involutions, then every $g \in G$ has a unique square root [1, exercises 11 and 12 of § 5.1].

(4) If G is simple and acts definably on a set of rank and degree 1, then $G \simeq \mathrm{PGL}_2(\mathbb{K})$ (Hrushovski) [1, Theorem 11.98].

Theorem 1.1 (Borovik, Corredor, Delahan, Nesin, Poizat; [1, Theorem 13.3])**.** *Let G be a simple, ranked group in which all definable, connected, soluble subgroups are nilpotent; call* bad *such a group and* Borel subgroups *its maximal, connected, soluble subgroups. Then:*

(5.a) *Borel subgroups are self-normalising, conjugated,* TI *(distinct conjugates meet trivially), and* covering *(viz. their union is G);*

(5.b) *for $g \neq 1$ there is a unique Borel subgroup $B = B(g)$ containing g; if $g \in Z(B)$ then $B = C_G(g)$;*

(5.c) *G has no definable, involutive automorphisms and in particular no involutions.*

The recent [3] generalises the absence of involutions, and also explains and unifies other phenomena; however it does not generalise the lack of involutive automorphisms.

Remark

One may view the Frécon-Poizat-Wagner result as a reinforcement of a folklore observation (left as an exercise): let G be a simple group with an almost self-normalising, TI subgroup of rank $n > 0$. If $\mathrm{rk}\, G \leq 2n$ then equality holds, and G has an involution.

2. Short proof of the theorem by Frécon and Poizat–Wagner

Definition 2.1. Let G be a group. A *$*$-bi-G-set* is a set Ω equipped with an action of $G \times G$ and an involutive bijection $* : \Omega \to \Omega$ satisfying $((a, b) \cdot \omega)^* = (b, a) \cdot \omega^*$.

Notice that Ω need not be definable; and neither does the action. However $G \times G$ is a group structure (possibly not a pure group); an important condition is when all stabilisers $\mathrm{Stab}_{G\times G}(\omega)$ are definable subgroups. If so we say that the $*$-bi-G-set *has definable stabilisers.*

Examples 2.2.

(i) G acts on itself by left- and right-translation (with an inverse on the right), and star is inversion, viz $(a, b) \cdot g = agb^{-1}$ and $g^* = g^{-1}$;

(ii) The latter induces an action of G on $\mathcal{P}(G)$ and on $\mathcal{P}_{\mathrm{def}}(G)$, the class of all definable subsets of G: $(a,b)\cdot X = aXb^{-1}$ and $X^* = X^{-1} = \{x^{-1} : x \in X\}$.

If G is a ranked group, both the action and the star are compatible with the equivalence relation on definable subsets: $X \sim Y$ if $\mathrm{rk}(X \triangle Y) < \mathrm{rk}\, X$. (By convention $\mathrm{rk}\, \varnothing = -\infty$; put $[\varnothing]_\sim = \{\varnothing\}$ to deal with the empty set).

(iii) Therefore $\Omega = \mathcal{P}_{\mathrm{def}}(G)/\sim$ is yet another $*$-bi-G-set. One easily sees that although Ω is not a definable set, stabilisers and orbits of its points are definable. The only points of $\mathcal{P}_{\mathrm{def}}(G)/\sim$ fixed under $G \times G$ are $[\varnothing]_\sim$ and $[G]_\sim$.

Definition 2.3 (after a reading of Poizat)**.** Let Ω be a $*$-bi-G-set. For $g \in G$, the *symmetry* through g is the involutive bijection of Ω given by $\sigma_g(\omega) = (g, g^{-1}) \cdot \omega^*$.

In the above action on $\mathcal{P}_{\mathrm{def}}(G)/\sim$, these functions are uniformly defined.

Definition 2.4. A subset $X \subseteq G$ is *non-confined* if it is contained in no finite union definable, proper cosets.

Lemma 2.5. The following is inconsistent: G is a simple ranked group with no definable, involutive automorphisms; Ω is a $*$-bi-G-set with definable stabilisers; $\omega \in \Omega$ is not fixed under $G \times G$ but $\Sigma(\omega) := \{g \in G : \sigma_g(\omega) = \omega\}$ is non-confined.

Proof. First notice that elements of G have unique square roots. Indeed, an involution would induce a definable automorphism of order *at most* 2; by assumption and since G is centreless, there are no involutions. Then we use Fact 3. Let $G_1 = G \times \{1\}$, $G_2 = \{1\} \times G$, and $\mathbb{G} = G \times G$ with projections $\pi_1, \pi_2 : \mathbb{G} \to G$.

Let $g_0 \in \Sigma(\omega)$ and $h_0 \in G$ be its unique square root; let $\omega' = (h_0^{-1}, h_0) \cdot \omega$. One easily sees $\Sigma(\omega') = h_0^{-1}\Sigma(\omega)h_0^{-1}$. Clearly $\Sigma(\omega')$ is non-confined as well so up to considering ω' we may assume $1 \in \Sigma(\omega)$. In particular $\omega^* = \omega$. For simplicity let $\Sigma = \Sigma(\omega)$.

Let $\mathbb{H} = \mathrm{Stab}_{\mathbb{G}}(\omega) \leq \mathbb{G}$, which is definable by assumption. Using $\omega^* = \omega$, one sees that for $(a,b) \in \mathbb{G}$, one has $(a,b) \in \mathbb{H}$ iff $(b,a) \in \mathbb{H}$. We say that $\mathbb{H}$ is *swap-invariant*. We shall prove that $\mathbb{H}$ is the graph of an involutive automorphism of G, like $\mathbb{H}$ definable.

Clearly $g \in \Sigma$ iff $(g, g^{-1}) \in \mathbb{H}$, so that $\Sigma \subseteq \pi_i(\mathbb{H})$. Since Σ is non-confined, $\pi_i(\mathbb{H}) = G$. Moreover $\pi_i(\mathbb{H} \cap G_i) \trianglelefteq \pi_i(\mathbb{H}) = G$ which is simple, so $\pi_i(\mathbb{H} \cap G_i) = \{1\}$ or $\pi_i(\mathbb{H} \cap G_i) = G$. In the latter case, swap-invariance implies $\mathbb{H} = \mathbb{G}$, contradicting ω not being fixed under $\mathbb{G}$.

Thus $\mathbb{H}$ is a multiplicative relation with full domain and image, and trivial fibres: it is the graph of an automorphism α of G, which is definable. Since $\mathbb{H}$ is swap-invariant, $\alpha^2 = \mathrm{Id}$; since $\mathbb{H}$ contains a non-trivial set of pairs (g, g^{-1}), one has $\alpha \neq \mathrm{Id}$: a contradiction. ☑

Theorem 2.6 (Frécon, the Poizat–Wagner version)**.** *There is no simple bad group of rank $2n+1$ in which Borel subgroups are abelian of rank n.*

***Proof*.** Let G be such a group. For each $g \in G \setminus \{1\}$, let $H(g) = C_G(g)$ be the only Borel subgroup containing it (Fact 1.1, with abelianity of Borel subgroups). Let $\beta\colon \mathbb{G} \to G$ be the commutator map $\beta(x,y) = [x,y]$; let c_0 be generic in $\beta(\mathbb{G})$. Let $X = \pi_1(\beta^{-1}(\{c_0\}))$.

We first show $\operatorname{rk} X \le 2n$. Otherwise X is generic. Since centralisers have rank n, a trivial fibre computation reveals $\operatorname{rk}\beta^{-1}(\{c_0\}) = \operatorname{rk} X + n$. This is invariant under conjugation, so for each $g \in G$ one has $\operatorname{rk}\beta^{-1}(\{{c_0}^g\}) = \operatorname{rk} X + n$. Taking the disjoint sum,

$$\operatorname{rk}\beta^{-1}(c_0^G) = \operatorname{rk}(c_0^G) + \operatorname{rk} X + n = \operatorname{rk} G + \operatorname{rk} X.$$

Since X is generic in G, the set $U = \beta^{-1}(c_0^G)$ is generic in $\mathbb{G}$.

Let $U^{\leftrightarrow} = \{(y,x) : (x,y) \in U\}$, still generic in $\mathbb{G}$; so is $U \cap U^{\leftrightarrow}$ by connectedness of $\mathbb{G}$. Taking a generic pair (x,y), the pair (y,x) is generic as well, so images $g = \beta(x,y) = [x,y]$ and $[y,x] = [x,y]^{-1} = g^{-1}$ are in the same conjugacy class. Thus there is $a \in G$ with $g^a = g^{-1}$; and $a^2 \in C_G(g)$. By unique divisibility in G and $C_G(g)$, we find $a \in C_G(g)$ whence $g^2 = 1$ and $g = 1$, a contradiction. Therefore $\operatorname{rk} X \le 2n$.

For each $a_0 \in X$, say with $[a_0, b_0] = c_0$, let:

$$Y_{a_0} = \bigcup_{b \in H(a_0)b_0} H(b)a_0$$

Then $Y_{a_0} \subseteq X$; indeed for $a \in Y_{a_0}$, say $a \in H(b)a_0$ with $b \in H(a_0)b_0$, one has:

$$[a,b] = [a_0,b] = [a_0,b_0] = c_0.$$

We claim that $\operatorname{rk} Y_{a_0} = 2n$. Indeed the various subgroups $H(b)$ for $b \in H(a_0)b_0$ intersect trivially. Otherwise, by Fact 1.1 there are *commuting* $b_i = d_i b_0$ with $d_i \in H(a_0)$. Then $b_1 b_2^{-1} = d_1 d_2^{-1} \in H(a_0)$, and also equals $b_2^{-1} b_1 = b_0^{-1} d_2^{-1} d_1 b_0 \in H(a_0)^{b_0}$. By disjointness, $H(a_0)^{b_0} = H(a_0)$, implying $b_0 \in N_G(H(a_0)) = H(a_0)$, contradicting $c_0 \ne 1$. Hence the cosets used in the definition of Y_{a_0} are disjoint. Thus $\operatorname{rk} Y_{a_0} = 2\operatorname{rk} H = 2n \ge \operatorname{rk} X \ge \operatorname{rk} Y_{a_0}$.

In the $*$-bi-set $\mathcal{P}_{\mathrm{def}}(G)$ (example 2.2), one has $\sigma_{a_0}(Y_{a_0}) = a_0 Y_{a_0}^{-1} a_0 = Y_{a_0}$. Clearly $\deg Y_{a_0} = 1$; however $\deg X$ is unknown. Let $X = X_1 \sqcup \cdots \sqcup X_d$ be a decomposition of X into degree 1 subsets (there is no notion of 'connected component' in the absence of an algebraic structure). There is j such that for a_0 generic in X_1, the intersection $Y_{a_0} \cap X_j$ has rank $2n$; viz. $Y_{a_0} \sim Y_{a_0} \cap X_j \sim X_j$.

Therefore for a_0 in some subset $\check{X} \subseteq X_1$ of rank $2n$, one has $X_j \sim Y_{a_0} = \sigma_{a_0}(Y_{a_0}) \sim \sigma_{a_0}(X_j)$. Working in the $*$-bi-set $\mathcal{P}_{\mathrm{def}}(G)/\sim$ (example 2.2) and following notation in the Lemma, $\omega := [X_j]_\sim$ satisfies $\check{X} \subseteq \Sigma(\omega)$.

By Fact 4, no rank $2n$ set is confined; so $\Sigma(\omega)$ is non-confined. Finally $\omega \neq [G]_{\sim}, [\varnothing]_{\sim}$, so it is not fixed under $\mathbb{G}$ (example 2.2); use the Lemma to derive a contradiction. ☑

The simplification was found while the second author was visiting the first in Bogotá in the spring of 2017.

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