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A combinatorial problem that arose in integer B_3 Sets

Un problema combinatorio que surgió en conjuntos B_3 enteros

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Abstract. Let $A = \{a_1, a_2, \dots, a_k\}$ be a set of positive integers with $k \geq 3$, such that $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_k = N$. Our problem is to investigate the number of triplets $(a_r, a_s, a_t) \in A^3$ with $a_r < a_s < a_t$, satisfying

$$a_r + a_s - a_t < 0 \quad \text{and} \quad -a_r + a_s + a_t > N. \tag{1}$$

In this paper we give an upper bound for the maximum number of such a triplets in an arbitrary set of integers with k elements. We also find the number of triplets satisfying () for some families of sets in order to determine lower bounds for the maximum number of such a triplets that a set with k elements can have.

Key words and phrases. B_3 sets, Sidon sets.

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Resumen. Sea $A = \{a_1, a_2, \dots, a_k\}$ un conjunto de enteros positivos con $k \geq 3$, tales que $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_k = N$. Nuestro problema consiste en investigar el número de ternas $(a_r, a_s, a_t) \in A^3$ con $a_r < a_s < a_t$, que satisfacen

$$a_r + a_s - a_t < 0 \quad \text{y} \quad -a_r + a_s + a_t > N.$$

En este artículo presentamos una cota superior para el máximo número de tales ternas en un conjunto de enteros arbitrario con k elementos . Por otro lado, también encontramos el número de ternas que satisfacen las desigualdades () para algunas familias de conjuntos, con el fin de determinar cotas inferiores para el máximo número de tales ternas que un conjunto con k elementos puede tener.

Palabras y frases clave. conjuntos B_3, conjuntos de Sidon.

1. Introduction

A set A of positive integers is called a B_3 set, if all sums $a+b+c$, with $a, b, c \in A$, $a \leq b \leq c$ are different. In order to investigate the maximal possible cardinality of a integer B_3 set contained in $[1, N] = \{1, 2, 3, \ldots, N\}$, the following function is defined

$$F_3(N) := \max\{|A| : A \text{ is a } B_3 \text{ set}, \;\; A \subseteq [1, N]\}.$$

In the study of upper bounds for $F_3(N)$, we try to generalize the idea of Bravo, Ruiz and Trujillo [1] and the following combinatorial problem arose.

Problem 1. Let $A = \{a_1, a_2, a_3, \ldots, a_k = N\}$ be a set of ordered positive integers with $k \geq 3$, and let $[A]^3$ be the set

$$[A]^3 := \{(a_r, a_s, a_t) \in A^3 : a_r < a_s < a_t\}.$$

Determine the number of triplets $(a_r, a_s, a_t) \in [A]^3$ satisfying (). This is equivalent to study the behavior of the function

$$T(A) := |\{(a_r, a_s, a_t) \in [A]^3 : a_r + a_s - a_t < 0 \;\text{ and }\; - a_r + a_s + a_t > N\}|,$$

where $|\cdot|$ denotes the cardinality of a finite set.

Also, we are interested in determining bounds for the value of

$$\mathcal{T}(k) := \max_{|A|=k} T(A). \tag{2}$$

The relationship between Problem 1 and B_3 integer sets is stated by the following property [4].

Property 1. If $A \subset [1, N]$ is a B_3 integer set with $|A| = k$, then

$$2\binom{k}{3} - \mathcal{T}(k) - k \leq N.$$

This property implies that if we know good lower bounds for $\mathcal{T}(k)$, then we will have better upper bounds for $F_3(N)$. For example, we know that $\mathcal{T}(k) \leq \binom{k}{3}$, and so

$$\limsup_{N\to\infty} \left(\frac{F_3(N)}{\sqrt[3]{N}}\right) \leq \sqrt[3]{6}.$$

In this paper we improve the trivial bound by proving that $\mathcal{T}(k) \leq \frac{k^3}{8}$. From this and Property 1 we obtain that

$$\limsup_{N\to\infty} \left(\frac{F_3(N)}{\sqrt[3]{N}}\right) \leq \sqrt[3]{\frac{24}{5}}. \tag{3}$$

And even though the best results known are

$$\begin{aligned} \limsup_{N\to\infty} \left(\tfrac{F_3(N)}{\sqrt[3]{N}}\right) &\leq \sqrt[3]{\tfrac{7}{2}} && \text{(B. Green, [3])} \\ &\leq \sqrt[3]{4}, && \text{(S. Chen, [2]).} \end{aligned} \tag{4}$$

Property 1 implies that it is possible to improve this bounds if we find better upper bounds for $\mathcal{T}(k)$.

On the other hand, regarding to the lower bounds for $\mathcal{T}(k)$, we establish one of the order $\frac{k^3}{24}$ by calculating the exact value of $T(A_k)$ for some families of sets. In fact, the family of sets $C_k = \{1, 2, \dots, k\}$ provides such a bound.

Finally, it is important to remark that for the family of sets given above, the sequence formed by the values of $T(C_k)$ matches with the sequence $A006918$ in [5], which counts the maximum number of squares that can be formed from k lines, for $k \geq 3$.

2. Upper bound for $\mathcal{T}(k)$

In this section we improve the trivial upper bound $\mathcal{T}(k) \leq \binom{k}{3}$. Specifically, we prove the following result.

Theorem 2.1. *The value of $\mathcal{T}(k)$ satisfies*

$$\mathcal{T}(k) \leq \frac{k^3}{8}.$$

Proof. Let $A = \{a_1, a_2, \dots, a_k\}$ be a set of positive integers such that $1 \leq a_1 < a_2 < \cdots < a_k = N$.

The condition () is equivalent to

$$a_t - a_r > \max\{N - a_s, a_s\} = \begin{cases} N - a_s & \text{if } \ a_s \leq N/2, \\ a_s & \text{if } \ a_s > N/2. \end{cases}$$

Let m and M be the number of elements in A less than or equal to $N/2$ and greater than $N/2$, respectively. Note that the elements less than or equal to $N/2$ are $\{a_1, a_2, \dots, a_m\}$ and the elements greater than $N/2$ are $\{a_{m+1}, a_{m+2}, \dots, a_{m+M}\}$.

Now, consider the following cases. First, for all s such that $2 \leq s \leq m$, the number of triplets $(a_r, a_s, a_t) \in [A]^3$ satisfying

$$a_t - a_r > N - a_s = \max\{a_s, N - a_s\},$$

is at most $(s-1)M$, this is because in this case a_t should satisfy $a_t > \frac{N}{2}$. Hence, the total number of triplets in this case is at most

$$\sum_{s=2}^{m} (s-1)M = M\left(\frac{m(m-1)}{2}\right). \tag{5}$$

Second, as above for all s such that $m+1 \leq s \leq m+M-1$, the number of triplets $(a_r, a_s, a_t) \in [A]^3$ satisfying

$$a_t - a_r > a_s = \max\{a_s, N - a_s\},$$

is at most $(k-s)m$, this is because in this case a_r should satisfy $a_r \leq \frac{N}{2}$. Thus, the total number of triplets is at most

$$\sum_{s=m+1}^{k-1}(k-s)m = m\left(\frac{M(M-1)}{2}\right). \tag{6}$$

Using equations (5) and (6), we have

$$\mathcal{T}(k) \leq M\frac{m(m-1)}{2} + m\frac{M(M-1)}{2} = mM\frac{(m+M-2)}{2} = mM\frac{(k-2)}{2}.$$

Finally, note that

$$\mathcal{T}(k) \leq mM\frac{(k-2)}{2} \leq \begin{cases} \frac{k^2}{4}\left(\frac{k-2}{2}\right) < \frac{k^3}{8} & \text{if } k \text{ is even} \\ \left(\frac{k-1}{2}\right)\left(\frac{k+1}{2}\right)\left(\frac{k-2}{2}\right) < \frac{k^3}{8} & \text{if } k \text{ is odd.} \end{cases}$$

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3. Value of $T(A)$ for special sets and lower bounds for $\mathcal{T}(k)$

In this section we study the values of the function $T(A_k)$ for some families of sets $\mathcal{A} = \{A_k : |A_k| = k,\ k \geq 3\}$, such that their sets A_k follow a certain pattern. We also use these values to determine lower bounds for $\mathcal{T}(k)$. Note that if $\mathcal{A}$ is a family of sets A_k, then

$$\mathcal{T}(k) \geq T(A_k).$$

Let $\mathcal{A}$ be the family of sets $A_k = \{1, 2, 4, \ldots, 2^{k-1}\}$. It is not hard to prove that the only triplets satisfying () are those including 2^{k-1} as third element, and thus

$$T(A_k) = \binom{k-1}{2}.$$

In general, the same value is obtained for all the sets $A_k^* = \{a, 2a, 4a, \ldots, 2^{k-1}a\}$, where a is any positive integer. Therefore

$$\mathcal{T}(k) \geq \binom{k-1}{2}.$$

Let $\mathcal{B}$ be the family of sets B_k consisting of the Fibonacci numbers from F_2 to F_{k+1}, that is

$$B_k = \{1, 2, 3, 5, 8, \ldots, F_{k+1}\}.$$

It is easy to prove that the only triplets satisfying () are those including F_{k+1} as the third element, except for the triplet (F_{k-1}, F_k, F_{k+1}), and hence

$$T(B_k) = \binom{k-1}{2} - 1$$

and thus

$$\mathcal{T}(k) \geq \binom{k-1}{2} - 1.$$

Last families of sets give us almost the same lower bound for the value of $\mathcal{T}(k)$. However, since these bounds has order k^2, they are far away from the upper bound obtained in Section 2. Below we present the family of sets which has given us the highest lower bound for $\mathcal{T}(k)$, and whose order is $\frac{k^3}{24}$.

Lemma 3.1. *Let $\mathcal{C}$ be the family of sets $C_k = \{1, 2, \ldots, k\}$.*

(1) If k is even, then

$$T(C_k) = \frac{k(k-1)(k-2)}{24} = \frac{\binom{k}{3}}{4}.$$

(2) If k is odd, then

$$T(C_k) = \frac{(k-3)(k-1)(k+1)}{24}.$$

***Remark* 3.2.** Note that the sequence given by the value of $T(C_k)$ is the same than the sequence $A006918$ in [5]. Specifically, if the sequence $A006918$ is denoted by $a(n)$, then $a(n-3) = T(C_n)$, which is the maximum number of squares that can be formed from n lines for $n \geq 3$.

***Proof*.** (1) Let C_k be the set consisting of the first k positive integers, with $k \geq 4$ an even integer ($k = 2m$). Then the triplets $(a_r, a_s, a_t) \in [C_k]^3$ satisfying () can be grouped as follows.

It is easy to see that there is no triplet (a_r, a_s, a_t) satisfying () with $a_r \geq m$. Furthermore, the only triplet with $a_r = a_{m-1}$ is the one in the set

$$H_1 = \{(m-1, m, 2m)\}.$$

The set of all the triplets with $a_r = a_{m-2}$ is

$$\begin{aligned} H_2 = \{&(m-2, m, 2m-1), (m-2, m, 2m), \\ &(m-2, m-1, 2m), \\ &(m-2, m+1, 2m)\}. \end{aligned}$$

In general, for $1 \leq i \leq m-1$ the set of all the triplets with $a_r = a_{m-i}$ is

$$\begin{aligned}
&H_i = \{(m-i, m, 2m-(i-1)), (m-i, m, 2m-(i-2)), \ldots, (m-i, m, 2m),\\
&(m-i, m-1, 2m-(i-2)), (m-i, m-1, 2m-(i-3)), \ldots, (m-i, m-1, 2m),\\
&(m-i, m+1, 2m-(i-2)), (m-i, m+1, 2m-(i-3)), \ldots, (m-i, m+1, 2m),\\
&(m-i, m-2, 2m-(i-3)), (m-i, m-2, 2m-(i-4)), \ldots, (m-i, m-2, 2m),\\
&(m-i, m+2, 2m-(i-3)), (m-i, m+2, 2m-(i-4)), \ldots, (m-i, m+2, 2m),\\
&\vdots\\
&(m-i, m-(i-1), 2m),\\
&(m-i, m+(i-1), 2m)\}.
\end{aligned}$$

The last sets determine a partition of the set consisting of all the triplets (a_r, a_s, a_t) satisfying (). And since for $1 \leq i \leq m-1$, $|H_i| = i^2$, then

$$T(C_k) = \sum_{i=1}^{m-1} |H_i| = \sum_{i=1}^{m-1} i^2 = \frac{k(k-1)(k-2)}{24} = \frac{\binom{k}{3}}{4}.$$

(2) Let C_k be the set consisting of the first k positive integers, with $k \geq 3$ an odd integer ($k = 2m+1$), clearly $T(C_3) = 0$.

Moreover, for $m \geq 2$ the triplets $(a_r, a_s, a_t) \in [C_k]^3$ satisfying () can be grouped in the partition given by the following sets.

There is no triplet (a_r, a_s, a_t) satisfying (), with $a_r \geq m$. There are only two triplets with $a_r = a_{m-1}$:

$$\begin{aligned}
H_1 = \{&(m-1, m, 2m+1),\\
&(m-1, m+1, 2m+1)\}.
\end{aligned}$$

The triplets with $a_r = a_{m-2}$ are

$$\begin{aligned}
H_2 = \{&(m-2, m, 2m), (m-2, m, 2m+1),\\
&(m-2, m+1, 2m), (m-2, m+1, 2m+1),\\
&(m-2, m-1, 2m+1),\\
&(m-2, m+2, 2m+1)\}.
\end{aligned}$$

In general, for $1 \leq i \leq m-1$, the set of all the triplets with $a_r = a_{m-i}$ is

$H_i = \{(m-i, m, 2m-(i-2)), (m-i, m, 2m-(i-3)), \ldots, (m-i, m, 2m+1),$
$(m-i, m+1, 2m-(i-2)), (m-i, m+1, 2m-(i-3)), \ldots, (m-i, m+1, 2m+1),$
$(m-i, m-1, 2m-(i-3)), (m-i, m-1, 2m-(i-4)), \ldots, (m-i, m-1, 2m+1),$
$(m-i, m+2, 2m-(i-3)), (m-i, m+2, 2m-(i-4)), \ldots, (m-i, m+2, 2m+1),$
$(m-i, m-2, 2m-(i-4)), (m-i, m-2, 2m-(i-5)), \ldots, (m-i, m-2, 2m+1),$
$(m-i, m+3, 2m-(i-4), (m-i, m+3, 2m-(i-5)), \ldots, (m-i, m+3, 2m+1),$
$\vdots$
$(m-i, m-(i-1), 2m+1),$
$(m-i, m+i), 2m+1)\}.$

Therefore, since for $1 \leq i \leq m-1$, $|H_i| = i(i+1)$, then we have

$$T(C_k) = \sum_{i=1}^{m-1} |H_i| = \sum_{i=1}^{m-1} i(i+1) = \frac{(k-3)(k-1)(k+1)}{24}.$$

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From the last result we obtain the following lower bound for $\mathcal{T}(k)$.

Corollary 3.3. *(1) If k is even, then*

$$\mathcal{T}(k) \geq \frac{k(k-1)(k-2)}{24} = \frac{\binom{k}{3}}{4}.$$

(2) If k is odd, then

$$\mathcal{T}(k) \geq \frac{(k-3)(k-1)(k+1)}{24}.$$

4. Open problems

In this work we prove that

(1) if k is even, then

$$\frac{k(k-1)(k-2)}{24} \leq \mathcal{T}(k) \leq \frac{k^3}{8},$$

(2) if k is odd, then

$$\frac{(k-3)(k-1)(k+1)}{24} \leq \mathcal{T}(k) \leq \frac{k^3}{8}.$$

We are interested in either finding families of sets which give us better lower bounds or improving the counting techniques used to obtain the upper bound.

Intuitively and from our computational evidence we conjecture that the value of $\mathcal{T}(k)$ is much smaller than $\frac{k^3}{8}$. Rather, we think it is very close to $\frac{k^3}{24}$, and if this were the case, from Property 1 we would obtain that

$$\limsup_{N\to\infty}\left(\frac{F_3(N)}{\sqrt[3]{N}}\right)\leq\sqrt[3]{\frac{24}{7}}, \tag{7}$$

which improves the upper bounds in (4).

We are also interested in studying the value of $T(A)$ for other families of sets, for instance if A is the set consisting of the first k squares, cubes, or primes. Further, we proved that $T(A)\leq\frac{k^3}{8}$ for any arbitrary set, we want to investigate if it is possible to improve this bound in the case that A is a B_3 set.

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