

Categorical definitions and properties via generators

Definiciones y propiedades categóricas vía generadores

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ABSTRACT. In the present work, we show how the study of categorical constructions does not have to be done with all the objects of the category, but we can restrict ourselves to work with families of generators. Thus, universal properties can be characterized through iterated families of generators, which leads us in particular to an alternative version of the adjoint functor theorem. Similarly, the properties of relations or subobjects algebra can be investigated by this method. We end with a result that relates various forms of compactness through representable functors of generators.

Key words and phrases. Generators, universal property, adjoint functor theorem, relations, subobjects algebra, compactness.

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RESUMEN. En el presente trabajo mostramos cómo el estudio de las construcciones categóricas no tiene por qué realizarse con todos los objetos de la categoría, sino que podemos restringirnos a trabajar con familias de generadores. Así, las propiedades universales pueden ser caracterizadas a través de familias iteradas de generadores, lo que nos lleva en particular a una versión alternativa del teorema del funtor adjunto. De igual forma, las propiedades de las relaciones o del álgebra de subobjetos pueden ser investigadas por este método. Terminamos con un resultado que relaciona diversas formas de compacidad a través de funtores representables de generadores.

Palabras y frases clave. Generadores, propiedad universal, teorema funtor adjunto, relaciones, álgebra de subobjetos, compacidad.

1. Introduction

Category theory studies objects externally, through the relationships they establish with their environment. That is why most of the definitions and categorical theorems are reduced to proving that, given a fixed object, *for all* the other objects of the category that satisfy certain hypotheses there is a morphism between them with certain properties. However, in this article, we show that this verification can be reduced to generator families. Let's review some background of this idea.

Generators were introduced in [11] to simplify the work with the injective objects in the context of the abelian categories. We recall that an object I is injective if for each monomorphism $s : V \hookrightarrow U$ and each morphism $g : V \rightarrow I$, there exists a morphism $h : U \rightarrow I$ such that $hs = g$. In [11, p. 136] Grothendieck proves that an object I in an abelian category AB5 is injective if it satisfies the previous definition only when U is a generator. This theorem allowed him to prove his famous theorem about the existence of enough injectives in abelian categories. These results were generalized in [2] to more general categories, among which are the Grothendieck topos.

The same restriction idea can be applied to study the semantics associated with a topos \mathcal{E} . In 1972 Mitchell discovered that each topos has an internal language, which is a generalization of the first-order classical language, where the variables have types associated with the objects of the topos. A formula φ with free variables x_1, \dots, x_n of respective types A_1, \dots, A_n can be interpreted in \mathcal{E} either by using the subobjects lattice structure (or the presence of a classifier Ω) as a subobject $\llbracket \varphi \rrbracket \hookrightarrow A_1 \times \dots \times A_n$ (or as a morphism $\ulcorner \varphi \urcorner : A_1 \times \dots \times A_n \rightarrow \Omega$). There are two ways to define the validity of the formula φ . In the first place, we can say that φ is valid, written $\models \varphi$, when $\llbracket \varphi \rrbracket = A_1 \times \dots \times A_n$ (in other words, when $\ulcorner \varphi \urcorner$ factors through $\top : \mathbf{1} \rightarrow \Omega$). Secondly, given $X \in ob(\mathcal{E})$ and a family of morphisms $a_i : X \rightarrow A_i$ ($i = 1, \dots, n$), we say that (a_1, \dots, a_n) satisfies or forces φ (denoted $\vdash_X \varphi(a_1, \dots, a_n)$), if $im(a_1, \dots, a_n) \leq \llbracket \varphi \rrbracket$.

$$\begin{array}{ccc}
 & \llbracket \varphi \rrbracket & \longrightarrow \mathbf{1} \\
 & \downarrow & \downarrow \top \\
 X & \xrightarrow{(a_1, \dots, a_n)} A_1 \times \dots \times A_n & \xrightarrow{\ulcorner \varphi \urcorner} \Omega
 \end{array}$$

The relation between both semantics is known since the origins of categorical logic [17]: $\models \varphi$ iff for every $X \in ob(\mathcal{E})$ and every $a_i : X \rightarrow A_i$ ($i = 1, \dots, n$), $\vdash_X \varphi(a_1, \dots, a_n)$. However, it is proved in [4, Volume 3, section 6.6] that the previous result continues to hold if we consider only the case in which X belongs to a family of generators \mathbb{G} .

Finally, generators have been used to characterize the properties of the lattice of subobjects. If \mathcal{C} is a category and \mathbf{A} is a certain subclass of Heyting algebras, we say that \mathcal{C} is \mathbf{A} -Heyting if for every object B in \mathcal{C} , $Sub(B) \in \mathbf{A}$.

It was proved in [5, Lemma 6.3] that a Grothendieck topos \mathcal{E} is bi-Heyting if and only if for every G in a family of generators \mathbb{G} , $Sub(G)$ is bi-Heyting. In section 4 we generalize this result.

Despite this background, this method has not been exploited to its full potential. For example, the important case of universal properties has not been investigated. In this paper, we would like to offer some guidelines in this regard.

2. Generators

Over the years, different variants of the original generator concept have been presented. In the following definition, we mention the most used ones.

Definition 2.1. Let \mathbb{G} be a family of objects in a category \mathcal{C} . We say that \mathbb{G} is a family of

- (1) generators if for each pair of different morphisms $a, b : A \rightrightarrows B$ in \mathcal{C} there is a $G \in \mathbb{G}$ and a morphism $t : G \rightarrow A$ such that $at \neq bt$ [4];
- (2) extremal generators if 1. holds and, furthermore, for all proper subobject $s : U \hookrightarrow A$ there is a $G \in \mathbb{G}$ and a morphism $t : G \rightarrow A$ which don't factor through s [7];
- (3) strong generators if 1. holds and, furthermore, for any morphism $f : A \rightarrow B$ and any subobject $s : U \hookrightarrow B$, if for all $t : G \rightarrow A$, with $G \in \mathbb{G}$, ft factors through s then f factors through s [19];
- (4) iterated generators (colimit-generator in [18]) if every object A in \mathcal{C} is the colimit of $(t_i : N_i \rightarrow A, \delta_{ij} : N_i \rightarrow N_j)$, where each N_i is, in turn, the colimit of a family $(n_i^l : G_i^l \rightarrow N_i, \Delta_i^{lm} : G_i^l \rightarrow G_i^m)$ in \mathbb{G} [22].

The following proposition, known in the literature, justifies the nomenclature used in 2.1.

Proposition 2.2. Let \mathcal{C} be a category with coproducts and \mathbb{G} a family of objects of \mathcal{C} . For every object $A \in \mathcal{C}$, we define

$$\gamma_A : \coprod_{G \in \mathbb{G}, f: \mathcal{C}(G,A)} (\text{domain of } f) \rightarrow A$$

by $\gamma_A \circ \text{inc}_f = f$. Then

- (1) \mathbb{G} is a family of generators iff each γ_A is an epimorphism.
- (2) \mathbb{G} is a family of extremal generators iff each γ_A is an extremal epimorphism.
- (3) \mathbb{G} is a family of strong generators iff each γ_A is a strong epimorphism.

Proposition 2.3. [22] *Every family of iterated generators is strong; every family of strong generators is extremal.*

None of the converse implications holds in general (the singleton is an extremal generator in the category of topological spaces that is not strong [4, Example 4.5.17.f]; the generator given in [6, 4.3] is strong but not is iterated). However,

Proposition 2.4. [22] *Assume that \mathcal{C} has coproducts and every epimorphism is regular. Then every family of generators is iterated.*

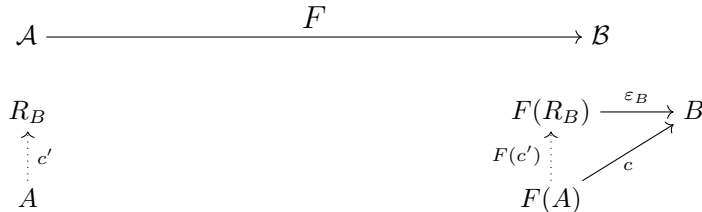
Example 2.5. (1) In every category \mathcal{C} , $ob(\mathcal{C})$ is a family of iterated generators in \mathcal{C} . Therefore, the categorical definitions given in terms of “all objects” of \mathcal{C} constitute a particular case of those given in terms of “generator families.”

- (2) In the category of abelian groups, \mathbb{Z} is an iterated generator. In general, in every category of modules on a ring A , A is an iterated generator. This is a consequence of the fact that every module supports a free presentation.
- (3) In the category of functors, the representable functors constitute a family of iterated generators [4, Volume 1, Teorema 2.15.6]. That a Grothendieck topos has a family of (iterated) generators is one of the conditions of Giraud’s theorem.
- (4) In the category $Sh(\mathcal{L})$ of sheaves on a locale \mathcal{L} , the classifier Ω is a cogenerator [3]; by 2.4 this cogenerator is iterated.
- (5) The singleton is an iterated generator in the category of compact Hausdorff spaces (proposition 2.4).

3. Universal properties

Universal properties are at the heart of category theory since many of the most important categorical constructs are defined in terms of them. The following is its usual definition.

Definition 3.1. [13, Section III.1] Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor and B an object of \mathcal{B} . A universal morphism from F to B is a pair (R_B, ε_B) consisting of an object R_B in \mathcal{A} and a morphism $\varepsilon_B : F(R_B) \rightarrow B$, such that to every pair (A, c) , where $c : F(A) \rightarrow B$, there is a unique $c' : A \rightarrow R_B$ with $c = \varepsilon_B F(c')$.



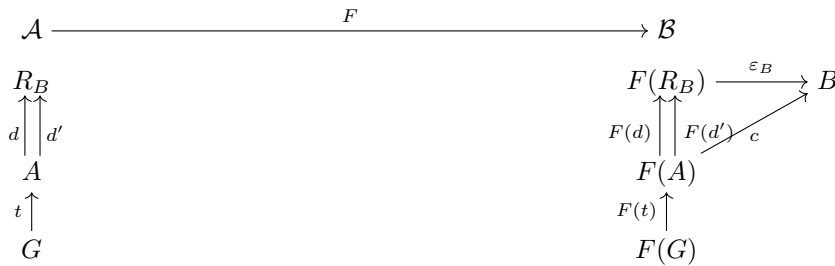
In short, the universal properties are of the form “...for every pair (A, c) ... there is a unique morphism ...”. Therefore, there are basically two ways to soften this notion: we can remove the requirement of uniqueness in the definition 3.1, obtaining the so-called *weak universal properties* [10]; or we can restrict to work with only a certain type of couples (A, c) . This second option will be the one we will take in this paper, where we will consider the case in which A belongs to a family of generators.

Definition 3.2. Let \mathcal{A} be a category with a family of generators \mathbb{G} . A universal morphism with respect to \mathbb{G} (or in short \mathbb{G} -universal) from $F : \mathcal{A} \rightarrow \mathcal{B}$ to $B \in \mathcal{B}$ is a pair (R_B, ε_B) consisting of an object R_B in \mathcal{A} and a morphism $\varepsilon_B : F(R_B) \rightarrow B$, such that to every pair (G, g) , where $G \in \mathbb{G}$ and $g : F(G) \rightarrow B$, there is a unique $g' : A \rightarrow R_B$ with $g = \varepsilon_B F(g')$.

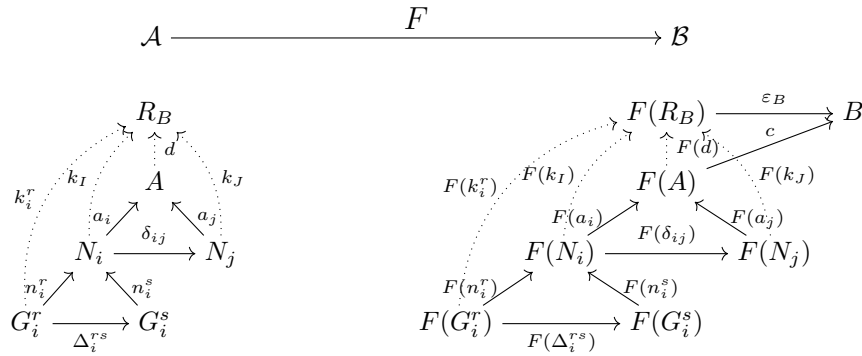
Under certain conditions a weak universal construction becomes universal [15, Theorems 2.24, 7.7], [13, Section X.2]. Similarly, under certain conditions on \mathbb{G} , a \mathbb{G} -universal construction becomes universal.

Proposition 3.3. Let \mathbb{G} be a family of iterated generators in \mathcal{A} , $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor that preserves epimorphic families and $B \in \mathcal{B}$. Then every morphism (R_B, ε_B) \mathbb{G} -universal from F to B is also universal.

Proof. Given an object A in \mathcal{A} and a morphism $c : F(A) \rightarrow B$ in \mathcal{B} , there cannot be two different morphisms $d, d' : A \rightarrow R_B$ such that $\varepsilon_B F(d) = \varepsilon_B F(d') = c$ by the generator definition and the uniqueness of the \mathbb{G} -universal.



To see the existence of such d , we consider a family $(a_i : N_i \rightarrow A, \delta_{ij} : N_i \rightarrow N_j)$ whose colimit is A , and families $(n_i^r : G_i^r \rightarrow N_i, \Delta_i^{rs} : G_i^r \rightarrow G_i^s)$ whose colimits are the N_i . For every morphism $cF(a_i)F(n_i^r)$, 3.2 implies the existence of $k_i^r : G_i^r \rightarrow R_B$ such that $\varepsilon_B F(k_i^r) = cF(a_i)F(n_i^r)$; applying twice the notion of colimit and that F preserves epimorphic families we can build d .



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In the rest of this section, we will analyze the most important examples of \mathbb{G} -universal constructions.

3.1. Adjoint functors

A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ has a right adjoint if for each $B \in \mathcal{B}$, there is a universal morphism from F to B [13, p. 81]. Therefore, the above results apply directly in this case.

Definition 3.4. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor, B an object of \mathcal{B} and \mathbb{G} a family of objects in \mathcal{A} . A \mathbb{G} -coreflection of B along F is a pair (R_B, ε_B) , where

- (1) R_B is an object of \mathcal{A} and $\varepsilon_B : F(R_B) \rightarrow B$ is a morphism of \mathcal{B} .
- (2) if $G \in \mathbb{G}$ and $b : F(G) \rightarrow B$ is a morphism of \mathcal{B} , there exists a unique morphism $a : G \rightarrow R_B$ in \mathcal{A} such that $\varepsilon_B F(a) = b$.

Proposition 3.5. Consider a category \mathcal{A} with a family of iterated generators \mathbb{G} and a functor $F : \mathcal{A} \rightarrow \mathcal{B}$. The following conditions are equivalent

- (1) F has a right adjoint functor.
- (2) The following conditions hold:
 - (a) Each object of \mathcal{B} admits a \mathbb{G} -coreflection along F ;
 - (b) F preserves epimorphic families.

Proof. If F has a right adjoint functor, then it satisfies the condition (a) by definition; that it must also satisfy the condition (b) is known in the literature [14, Lemma 2, p. 395]. For the other implication, it is enough to apply the proposition 3.3. ✓

It may be interesting to compare the previous result with the dual of the adjoint functor’s theorem. Both start from weakening the concept “all object of \mathcal{A} has a coreflection” (the latter weakening the concept of coreflection, the former the class of objects that have coreflection), to later restore it by conjugating some kind of glue (cocompleteness, iterated generators) and of continuity of F (preservation of colimits, of epimorphic families). It is also worth noting that the proposition 3.5 is not a consequence of this theorem, as the following examples show.

Example 3.6. (1) Let k be a finite field, $FinSet$ the category of finite sets and $FinVect_k$ the category of finite-dimensional vector spaces over k . The free functor $F : FinSet \rightarrow FinVect_k$ is defined on objects by taking $F(X)$ to be a vector space with basis X . Since $FinSet$ is not cocomplete, we can not use the adjoint functor theorem for proving that F has a right adjoint. However, the proposition 3.5 can be used for this purpose (the singleton is an iterated generator in $FinSet$ and F preserves epimorphic families).

(2) A morphism $f : \mathcal{L} \rightarrow \mathcal{M}$ from a locale \mathcal{L} to a locale \mathcal{M} is a pair of functors $f_* : \mathcal{L} \rightarrow \mathcal{M}$, $f^* : \mathcal{M} \rightarrow \mathcal{L}$ such that $f^* \dashv f_*$ and f^* preserves finite meets [4, Volume 3, Definition 1.3.7]. Given one of such morphisms, we can define the direct image functor $D_f : Sh(\mathcal{L}) \rightarrow Sh(\mathcal{M})$ between the respective categories of sheaves as $D_f(G)(u) = G(f^*(u))$ for each $u \in \mathcal{M}$. It is well-known that D_f has a left adjoint, the inverse image functor. Therefore, one might ask: Under what conditions D_f has a right adjoint? Let’s see how the proposition 3.5 helps us answer this question. By 2.5, representable functors constitute a family of iterated generators in $Sh(\mathcal{L})$. Its values on D_f can be calculated as follows:

$$\begin{aligned} D_f(\mathcal{L}(-, v))(u) &= \mathcal{L}(f^*(u), v) \\ &= \mathcal{M}(u, f_*(v)), \quad \text{because } f^* \dashv f_*. \end{aligned}$$

Given B in $Sh(\mathcal{M})$, the natural transformations $c : \mathcal{M}(-, f_*(v)) \rightarrow B$, $d : \mathcal{L}(-, v) \rightarrow R_B$ correspond, respectively, to elements $\bar{c} : B(f_*(v))$, $\bar{d} : R_B(v)$. This suggests us to define $R_B(v) := B(f_*(v))$. The problem is that when f is arbitrary, the previous definition is not always a sheaf. However, this can be guaranteed by requiring, for example, that the morphism f_* preserves arbitrary joins. If this is the case, the rest of the demonstration is relatively simple. Since that for each $u : \mathcal{M}$, $u \leq f_*f^*(u)$, we define the natural transformation $\epsilon_B : D_f(R_B) \rightarrow B$ by

$$\begin{array}{ccc} D_f(R_B)(u) = B(f_*f^*(u)) & \xrightarrow{\epsilon_B(u)} & B(u) \\ x \vdash & \xrightarrow{\quad} & x|_u. \end{array}$$

We then have that the following diagram commutes

$$\begin{array}{ccc}
 Sh(\mathcal{L}) & \xrightarrow{D_f} & Sh(\mathcal{M}) \\
 \\
 \begin{array}{c} R_B \\ \uparrow \hat{c}' \\ \mathcal{L}(-, v) \end{array} & & \begin{array}{c} D_f(R_B) \xrightarrow{\varepsilon_B} B \\ \uparrow \hat{D}_f(c') \quad \nearrow c \\ \mathcal{M}(-, f_*(v)), \end{array}
 \end{array}$$

where c' is the only natural transformation given by Yoneda lemma on $B(f_*(v)) = R_B(v)$.

It remains to prove that D_f preserves epimorphic families. Let $\{t_i : H_i \rightarrow A\}_{i \in I}$ be an epimorphic family in $Sh(\mathcal{L})$, C a sheaf in $Sh(\mathcal{M})$, $x, y : D_f(A) \rightarrow C$ two natural transformations such that $x D_f(t_i) = y D_f(t_i)$ for each $i \in I$. Given $u \in \mathcal{M}$, we have

$$x_u t_i(f^*(u)) = x D_f(t_i)(u) = y D_f(t_i)(u) = y_u t_i(f^*(u))$$

in the category of sets. Thus, $x_u = y_u$ for each u , because $\{t_i\}$ is epimorphic (note that this argument is simpler than to prove that D_f preserves colimits and to apply the dual of the adjoint functor theorem).

3.2. Limits

Let \mathcal{C}, \mathcal{D} be two categories. If \mathcal{D} is small, we denote the category of functors from \mathcal{D} to \mathcal{C} by $\mathcal{C}^{\mathcal{D}}$. We define the diagonal functor $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{D}}$ by:

- for every $c \in ob(\mathcal{C})$, Δc is the constant functor (the functor which sends each $i \in ob(\mathcal{D})$ to c and each $f \in mor(\mathcal{D})$ to the identity $id_c : c \rightarrow c$).
- for every $f : c \rightarrow c'$, Δf is the natural transformation which has the value f at each $i \in ob(\mathcal{D})$.

Given a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, the limit of F is the universal morphism from Δ to F [13, Section III.4]. Thus, in this case, we can express the \mathbb{G} -universality in the following way.

Definition 3.7. Given a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, a \mathbb{G} -cone on F consist of

- (1) an object $G \in \mathbb{G}$,
- (2) for every object $D \in \mathcal{D}$, a morphism $p_D : G \rightarrow FD$ in \mathcal{C} ,

such that for every morphism $d : D \rightarrow D'$ in \mathcal{D} , $p_{D'} = Fd \circ p_D$.

Definition 3.8. Given a functor $F : \mathcal{D} \rightarrow \mathcal{C}$, a \mathbb{G} -limit of F is a cone (in the usual sense) $\{L_{\mathbb{G}}, (p_D)_D\}$ on F such that, for every \mathbb{G} -cone $\{M, (q_D)_D\}$ on F , there exist a unique morphism $m : M \rightarrow L$ such that for every object $D \in \mathcal{D}$, $q_D = p_D \circ m$.

Lemma 3.9. *The functor $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{D}}$ preserves epimorphic families.*

Proof. Suppose that $\{e_k : u_k \rightarrow b\}_{k \in K}$ is an epimorphic family in \mathcal{C} and that $s, t : \Delta b \rightarrow H$ are natural transformations such that $s\Delta(e_k) = t\Delta(e_k)$ for all $k \in K$. Given $d \in \mathcal{D}$, we have

$$s_d e_k = s\Delta(e_k)(d) = t\Delta(e_k)(d) = t_d e_k$$

for all $k \in K$ and hence $s_d = t_d$. ✓

Proposition 3.10. *Let \mathcal{C} be a category with a family of iterated generators \mathbb{G} and $(L_{\mathbb{G}}, (p_D)_D)$ a \mathbb{G} -limit of a functor $F : \mathcal{D} \rightarrow \mathcal{C}$. Then $(L_{\mathbb{G}}, (p_D)_D)$ is the limit of the functor F .*

Proof. Is just the conjunction of 3.9 and 3.3. ✓

Let's see what can be said when the family of generators \mathbb{G} is a little weaker.

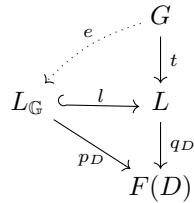
Proposition 3.11. *Let \mathcal{C} be a category with a family of extremal generators \mathbb{G} and $(L_{\mathbb{G}}, (p_D)_D)$ a \mathbb{G} -limit of a functor $F : \mathcal{D} \rightarrow \mathcal{C}$. If F has a limit $(L, (q_D)_D)$, then L is isomorph to $L_{\mathbb{G}}$.*

Proof. The definition of limit implies the existence of a unique morphism $l : L_{\mathbb{G}} \rightarrow L$ such that $q_D l = p_D$ for each object D of \mathcal{D} . Let $a, b : C \rightarrow L_{\mathbb{G}}$ be two different morphisms; then there exists $G \in \mathbb{G}$ and a morphism $m : G \rightarrow C$ such that $am \neq bm$. Since $L_{\mathbb{G}}$ is a \mathbb{G} -limit, there exists D such that $p_D am \neq p_D bm$ and so $q_D lam \neq q_D lbm$. In particular, $la \neq lb$, which proves that l is a monomorphism.

$$\begin{array}{ccccc}
 G & & & \xrightarrow{p_D} & FD \\
 m \downarrow & & & \nearrow & \\
 C & \xrightarrow{a} & L_{\mathbb{G}} & \xrightarrow{l} & L & \xrightarrow{q_D} & FD
 \end{array}$$

Now, if $L_{\mathbb{G}}$ is a proper subobject of L , since \mathbb{G} is a family of extremal generators, there exists a G in \mathbb{G} and a morphism $t : G \rightarrow L$ such that t does not factor through l . Given the family of morphisms $\{q_D t : G \rightarrow F(D)\}$, the notion of \mathbb{G} -limit implies the existence of $e : G \rightarrow L_{\mathbb{G}}$ such that $p_D e = q_D t$ for all D .

Since L is a limit, $le = t$, which contradicts that t does not factor through l .

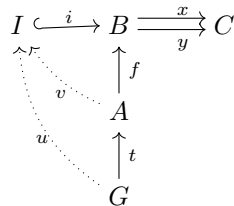


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Unlike 3.10, in the previous proposition, we need to assume the existence of the limit of F . Let's see now how by demanding a little more of the family \mathbb{G} , each \mathbb{G} -equalizer is an equalizer.

Proposition 3.12. *Let \mathcal{C} be a category with a family of strong generators \mathbb{G} . If $i : I \rightarrow B$ is the \mathbb{G} -equalizer of $x, y : B \rightrightarrows C$ then i is the equalizer of x, y .*

Proof. Let $f : A \rightarrow B$ be a morphism such that $xf = yf$. Thus, for all $t : G \rightarrow A$, $xf t = yf t$ and the definition of \mathbb{G} -equalizer implies the existence of $u : G \rightarrow I$ such that $iu = ft$. But then the definition of strong generator implies the existence of $v : A \rightarrow I$ such that $iv = f$.



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For the case of the product, it seems that it is necessary to assume that the family \mathbb{G} is iterated. We present direct proof of this fact.

Proposition 3.13. *Let \mathcal{C} be a category with a family of iterated generators \mathbb{G} . If $(P, (p_k)_{k \in K})$ is the \mathbb{G} -product of $\{B_k\}_{k \in K}$ then $(P, (p_k)_{k \in K})$ is the product of $\{B_k\}_{k \in K}$.*

Proof. Analogous to that of 3.3. ✓

Note that the propositions 3.12, 3.13 provide other demonstration of 3.10, because every limit can be obtained from products and equalizers.

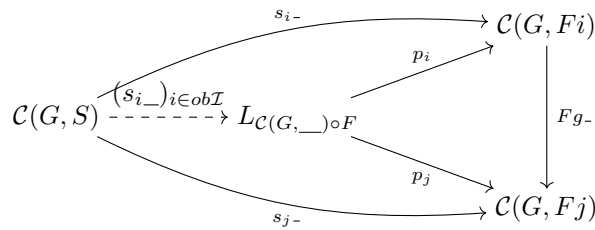
Another way to express the previous results is through the good behavior of representable functors originating from \mathbb{G} with respect to limits. The following definition is well-known in category theory.

Definition 3.14. [4, Volume 1, Definition 2.9.6] A family of functors $(G_k : \mathcal{C} \rightarrow \mathcal{D})_{k \in K}$ collectively reflects limits when, given a $F : \mathcal{I} \rightarrow \mathcal{C}$ and a cone $(S \rightarrow F_i)_{i \in \text{ob}(\mathcal{I})}$ on F if for each k , $G_k(S) \rightarrow G_k(F_i)$ is the limit of $G_k F$, then $(S \rightarrow F_i)_{i \in \text{ob}(\mathcal{I})}$ is the limit of F . When the family is reduced to a single functor $G : \mathcal{C} \rightarrow \mathcal{D}$ we say that G reflects limits.

Proposition 3.15. (1) If \mathbb{G} is a family of extremal generators then the representable functors $\mathcal{C}(G, _) : \mathcal{C} \rightarrow \mathbf{Set}$, with $G \in \mathbb{G}$, collectively reflects all the limits that already existed in \mathcal{C} .

(2) If \mathbb{G} is a family of iterated generators then the functors $\mathcal{C}(G, _) : \mathcal{C} \rightarrow \mathbf{Set}$, with $G \in \mathbb{G}$, collectively reflect limits.

Proof. If $(S \rightarrow F_i)_{i \in \text{ob}(\mathcal{I})}$ is a cone in \mathcal{C} then for each $G \in \mathbb{G}$ we have in the category of sets a diagram of the form



where $L_{\mathcal{C}(G, _) \circ F}$ is the limit of $\mathcal{C}(G, _) \circ F$, p_i are the usual projections in sets and $(s_{i-})_{i \in \text{ob} \mathcal{I}}$ is the induced morphism, which takes the form

$$\begin{array}{ccc} \mathcal{C}(G, S) & \xrightarrow{(s_{i-})_{i \in \text{ob} \mathcal{I}}} & L_{\mathcal{C}(G, _) \circ F} \\ g \vdash & \longrightarrow & (s_i g)_{i \in \text{ob} \mathcal{I}}. \end{array}$$

The hypothesis, that $(s_{i-})_{i \in \text{ob} \mathcal{I}}$ is an isomorphism for each $G \in \mathbb{G}$, means that $(S \rightarrow F_i)_{i \in \text{ob}(\mathcal{I})}$ is a \mathbb{G} -limit of F . Applying 3.11 and 3.10, we get the result. \square

Example 3.16. Let $\{F_k\}_{k \in K}$ be a family of sheaves in $Sh(\mathcal{L})$. We define for each $u \in \mathcal{L}$

$$C(u) = \left\{ \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) \mid u = \bigvee_{k \in K} u_k \text{ disjoint two to two, } x_k \in F_k(u_k) \right\}.$$

Given $v \leq u$, $\left((u_k)_{k \in K}, (x_k)_{k \in K} \right) \mid_v = \left((u_k \wedge v)_{k \in K}, (x_k \mid_{u_k \wedge v})_{k \in K} \right)$. It can be shown from the sheaf definition that $C \in \text{ob}(Sh(\mathcal{L}))$. We prove only that C is

the coproduct of $\{F_k\}_{k \in K}$. Given $i \in K$, $u \in \mathcal{L}$, we define the i -th canonical injection in $I_i(u) : F_i(u) \rightarrow C(u)$ by $I_i(u)(x) = \left((u_k)_{k \in K}, (x_k)_{k \in K} \right)$, where

$$u_k = \begin{cases} u, & \text{if } k = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$x_k = \begin{cases} x \in F_i(u), & \text{if } k = i, \\ ! \in F_k(0), & \text{if } k \neq i. \end{cases}$$

To prove that the previous diagram is a coproduct, by 2.5 (4) and 3.10, it suffices to consider families of natural transformations of the form $\{s_k : F_k \rightarrow \Omega\}$. We define $o : C \rightarrow \Omega$ by $o(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) = \bigvee_{k \in K} s_k(u_k)(x_k)$. Given $v \leq u$, the naturality of the s_k implies that $s_k(u_k \wedge v)(x_k|_{u_k \wedge v}) = v \wedge s_k(u_k)(x_k)$ and thus

$$\begin{aligned} v \wedge o(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) &= v \wedge \bigvee_{k \in K} s_k(u_k)(x_k) \\ &= \bigvee_{k \in K} v \wedge s_k(u_k)(x_k) \\ &= \bigvee_{k \in K} s_k(u_k \wedge v)(x_k|_{u_k \wedge v}) \\ &= o(v) \left((u_k \wedge v)_{k \in K}, (x_k|_{u_k \wedge v})_{k \in K} \right), \end{aligned}$$

which proves that o is a natural transformation. By definition, for all $k \in K$, $oI_k = s_k$. To prove the uniqueness of o , consider another transformation $o' : C \rightarrow \Omega$ such that $o'I_k = s_k$ for every $k \in K$. By the naturality of o' , we have

$$\begin{aligned} o'(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) \wedge u_i &= o'(u) \left((u_k \wedge u_i)_{k \in K}, (x_k|_{u_k \wedge u_i})_{k \in K} \right) \\ &= o'(u_i)I_i(u_i)(x_i) = s_i(u_i)(x_i). \end{aligned}$$

Therefore

$$\begin{aligned} o(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) &= \bigvee_{i \in K} s_i(u_i)(x_i) \\ &= \bigvee_{i \in K} o'(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) \wedge u_i \\ &= o'(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right) \wedge \bigvee_{i \in K} u_i \\ &= o'(u) \left((u_k)_{k \in K}, (x_k)_{k \in K} \right). \end{aligned}$$

3.3. Relations

There are several ways to define relations in a category \mathcal{C} , equivalent to each other provided that \mathcal{C} has good exactness properties (see [4, Volume 2, section 2.5] for a detailed discussion). Here, we will use the most general of them all.

Definition 3.17. [4, Volume 2, pp. 101, 102] A relation (R, r_1, r_2) on an object X is an object R with a monomorphic pair $r_1, r_2 : R \rightrightarrows X$ (i.e. for any $a, b : A \rightrightarrows R$, $a = b$ iff $r_1a = r_1b$ and $r_2a = r_2b$).

For every $A \in \text{Ob}(\mathcal{C})$, we define a relation (in the usual sense) generated by (R, r_1, r_2) on $\mathcal{C}(A, X)$ as:

$$R_A = \{(r_1a, r_2a) \mid a \in \mathcal{C}(A, R)\}.$$

Since that the R_A are usual relations, we can say that (R, r_1, r_2) has a certain property if all R_A have it. So, we say that (R, r_1, r_2) is reflexive (respectively, symmetric, antisymmetric, transitive, ...) if for every object A in \mathcal{C} , the relation R_A is reflexive (respectively, symmetric, antisymmetric, transitive, ...).

We will show that we can restrict the verification of the properties of a relation of all the objects in the category to work with only families of generators. The idea is that these can be seen as universal properties.

Proposition 3.18. *Let \mathcal{C} be a category with a family of iterated generators \mathbb{G} and (R, r_1, r_2) a relation on X . If for all $G \in \mathbb{G}$, R_G is reflexive (respectively, symmetric, transitive) then (R, r_1, r_2) is reflexive (respectively, symmetric, transitive).*

Proof. (1) Reflexivity.

We must prove that for each $A \in \text{ob}(\mathcal{C})$ and each $f : A \rightarrow X$ there exists $b : A \rightarrow R$ such that $r_1b = f$, $r_2b = f$. Note that this b is unique because r_1, r_2 is monomorphic. As this is valid by hypothesis when $A \in \mathbb{G}$, the proposition 3.3 implies that (R, r_1, r_2) is reflexive.

(2) Symmetry.

This can be established as the following universal property: for every $A \in \text{ob}(\mathcal{C})$ and every pair of morphisms $f_1, f_2 : A \rightrightarrows X$ such that $(f_1, f_2) \in R_A$, there exists a unique $b : A \rightarrow R$ such that $r_1b = f_2$, $r_2b = f_1$.

(3) Transitivity.

The universal property, in this case, is: for every $A \in \text{ob}(\mathcal{C})$ and every pair of morphisms $f_1, f_2 : A \rightrightarrows X$ such that there exists $f_3 : A \rightarrow X$ with $(f_1, f_3), (f_3, f_2) \in R_A$, then there exists a unique $b : A \rightarrow R$ such that $r_1b = f_1$, $r_2b = f_2$.

□

The case of antisymmetry is much simpler since the Proposition 3.3 does not need to be applied nor the hypothesis that \mathbb{G} is iterated.

Proposition 3.19. *Let \mathcal{C} be a category with a family of generators \mathbb{G} and (R, r_1, r_2) a relation on X . If for each $G \in \mathbb{G}$, R_G is antisymmetric then (R, r_1, r_2) is antisymmetric.*

Proof. We consider $(f_1, f_2), (f_2, f_1) \in R_A$. Then there exists $a, b \in \mathcal{C}(A, R)$ such that

$$r_1a = f_1, r_2a = f_2, r_1b = f_2, r_2b = f_1.$$

Given an arbitrary generator G and a arbitrary morphism $g : G \rightarrow A$, the equations

$$r_1ag = f_1g, r_2ag = f_2g, r_1bg = f_2g, r_2bg = f_1g$$

imply that $(f_1g, f_2g), (f_2g, f_1g) \in R_G$ and thus $f_1g = f_2g$. Since G, g are arbitrary, the generator definition implies that $f_1 = f_2$. \square

For an application of this method to a concrete example see [1, Proposition 2.3.5].

4. Subobjects lattice

We will show that to prove that \mathcal{C} is \mathbf{A} -Heyting, it is enough to see that for all G in a family of generators \mathbb{G} , $Sub(G) \in \mathbf{A}$. This procedure is a generalization of [5, Section 6].

Each morphism $f : Y \rightarrow X$ in \mathcal{C} induces the inverse image function $f^* : Sub(X) \rightarrow Sub(Y)$. The least that we demand to the operations in $Sub(_)$ is that they respect this function.

Definition 4.1. [16] Let $T_X : \prod_I Sub(X) \rightarrow Sub(X)$ be a family of I -ary operations on $Sub(X)$, for each X in a category \mathcal{C} . We say that $\{T_X\}_{X \in \mathcal{C}}$ is natural if for every $f : Y \rightarrow X$ and every $\{S_i\}_I \subseteq Sub(X)$, the function $f^* : Sub(X) \rightarrow Sub(Y)$ satisfies that $f^*(T_X(S_i)_I) = T_Y(f^*S_i)_I$.

Proposition 4.2. *Let \mathcal{C} be a regular category with disjoint coproducts; \mathbf{A} be a subclass of Heyting algebras closed under isomorphisms, subalgebras and arbitrary products; and \mathbb{G} be a family of generators in \mathcal{C} . Suppose that all language operations of \mathbf{A} are natural in \mathcal{C} . If $Sub(G) \in \mathbf{A}$ for each $G \in \mathbb{G}$ then $Sub(X) \in \mathbf{A}$ for each $X \in \mathcal{C}$.*

Proof. Given an arbitrary $X \in \mathcal{C}$, there exists an epimorphism $\gamma : \coprod_f Dom(f) \rightarrow X$, where $f \in \bigcup_{G \in \mathbb{G}} \mathcal{C}(G, X)$ [4, Volume 1, Proposition 4.5.2]. The function

$\gamma^* : Sub(X) \rightarrow Sub(\coprod_f Dom(f))$ is injective: indeed, given $A, B \in Sub(X)$ if $\gamma^*(A) = \gamma^*(B)$ then we have the pullbacks

$$\begin{array}{ccc} \gamma^*(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \coprod_f Dom(f) & \xrightarrow{\gamma} & X \end{array}$$

$$\begin{array}{ccc} \gamma^*(B) & \longrightarrow & B \\ \downarrow & & \downarrow \\ \coprod_f Dom(f) & \xrightarrow{\gamma} & X \end{array}$$

and, by the uniqueness of the epi-mono factorization, we conclude that $A = B$.

On the other hand, we have $Sub(\coprod_f Dom(f)) = \prod_f Sub(Dom(f))$. Indeed, if we denote with $i_f : Dom(f) \hookrightarrow \coprod_f Dom(f)$ the injections of the coproduct, we can define

$$\begin{array}{ccc} Sub(\coprod_f Dom(f)) & \xrightarrow{\alpha} & \prod_f Sub(Dom(f)) \\ A \vdash & \longrightarrow & (i_f^* A)_f \end{array}$$

$$\begin{array}{ccc} \prod_f Sub(Dom(f)) & \xrightarrow{\beta} & Sub(\coprod_f Dom(f)) \\ (A_f)_f \vdash & \longrightarrow & \bigcup_f i_f(A_f) \end{array}$$

where $i_f(A_f)$ is the direct image of A_f through i_f . Let's see that α and β are inverse functions. First, given any $g \in \bigcup_{G \in \mathbb{G}} \mathcal{C}(G, X)$ we have $Dom(g) \cap \bigcup_f i_f(A_f) = \bigcup_f Dom(g) \cap i_f(A_f) = Dom(g) \cap i_g(A_g) = i_g(A_g)$, because the coproducts are disjoint. So, the diagram

$$\begin{array}{ccc} A_g & \hookrightarrow & \bigcup_f A_f \\ \downarrow & & \downarrow \\ Dom(g) & \xrightarrow{i_g} & \coprod_f Dom(f) \end{array}$$

is a pullback and $\alpha\beta(A_f)_f = (A_f)_f$. On the other hand, given any $A \in Sub(\coprod_f Dom(f))$, we have that $Dom(f) \cap A = i_f^* A$ and thus $A = A \cap \coprod_f Dom(f) = A \cap \bigcup_f Dom(f) = \bigcup_f A \cap Dom(f) = \bigcup_f i_f^* A = \beta\alpha(A)$. \checkmark

Lemma 4.3. *Let X be an object of a category \mathcal{C} such that $Sub(X)$ is a Heyting algebra and $T_X : \prod_I Sub(X) \rightarrow Sub(X)$ is a natural operation along monomorphism. Then T_X is a compatible operation on $Sub(X)$.*

Proof. Given $\{S_i\}_I \subseteq \text{Sub}(X)$ and $i : A \hookrightarrow X$, since $i^*(S_i) = A \wedge S_i$, we have $T_X(S_i)_I \wedge A = i^*(T_X(S_i)_I) = T_X(i^*(S_i)_I) = T_X(S_i \wedge A)_I$ and thus the result follows from [9, Lemma 2.1]. \square

Proposition 4.4. *Let \mathbf{A} be a subvariety of Heyting algebras, \mathcal{L} be a locale, $\text{Sh}(\mathcal{L})$ be the topos of sheaves over \mathcal{L} , and suppose that all the operations of \mathbf{A} are natural in $\text{Sh}(\mathcal{L})$. Then for every $X \in \text{Sh}(\mathcal{L})$, $\text{Sub}(X) \in \mathbf{A}$ if and only if $\mathcal{L} \in \mathbf{A}$.*

Proof. \Rightarrow Immediate because $\text{Sub}(1) \cong \mathcal{L}$.

\Leftarrow Since $\text{Sub}(1) \cong \mathcal{L}$, we have $\text{Sub}(1)$ is of \mathbf{A} -Heyting. Thus, if $i : U \hookrightarrow 1$ then $i^* : \text{Sub}(1) \rightarrow \text{Sub}(U)$ is a surjective homomorphism (given $a : A \hookrightarrow U$, a preimage is $ia : U \hookrightarrow 1$) and by 4.3 $\text{Sub}(U)$ is also of \mathbf{A} -Heyting. Since $\text{Sh}(\mathcal{L})$ is a localic topos, it is generated by subobject of 1 and by the proposition 4.2 $\text{Sh}(\mathcal{L})$ is of \mathbf{A} -Heyting. \square

5. Compactness

We say that a family of generators $\mathbb{G}_{\mathbb{P}}$ is projective if each $G \in \mathbb{G}_{\mathbb{P}}$ is projective, i.e. when for any morphism $f : G \rightarrow B$ and any epimorphism $q : A \rightarrow B$, f factors through q by some morphism $G \rightarrow A$. The representable functors indexed by families of projective generators allow us to capture forms of compactness in some categories.

Proposition 5.1. *Let \mathcal{C} be a category with a family of projective generators $\mathbb{G}_{\mathbb{P}}$ and $F : \mathcal{I} \rightarrow \mathcal{C}$ a functor with limit $\{l_i : L \rightarrow Fi\}_{i \in \mathcal{I}}$. Given a cone $\{s_i : S \rightarrow Fi\}_{i \in \mathcal{I}}$, we consider the following statements*

- (1) *the canonical morphism $s : S \rightarrow L$ is epimorphic;*
- (2) *for any $G \in \mathbb{G}_{\mathbb{P}}$ and for any choice of $x_i \in \mathcal{C}(G, Fi)$ if*

$$\bigcap_I (s_{i-})^{-1}(x_i) = \emptyset$$

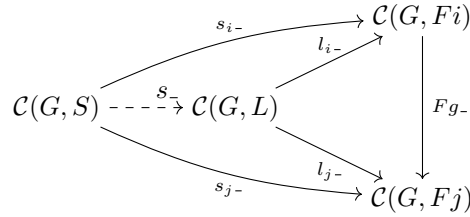
then there exists $J \subset I$ finite such that

$$\bigcap_J (s_{j-})^{-1}(x_j) = \emptyset,$$

where $s_{i-} : \mathcal{C}(G, S) \rightarrow \mathcal{C}(G, Fi)$ is induced by s_i .

Then 1. always implies 2., and 2. implies 1. in the case where \mathcal{I} is cofiltered and the s_i epimorphic.

Note that in statement 1. objects live in the category \mathcal{C} while in 2. they all live in the category of sets. In effect, we have that in the diagram



objects are sets of morphisms and arrows are functions that send morphisms in morphisms.

Proof. 1. \Rightarrow 2. Suppose that every finite intersection of

$$\{(s_{i-})^{-1}(x_i)\}_{i \in ob(\mathcal{I})}$$

is not empty. Given any $Fg : Fj \rightarrow Fk$, since $(s_{j-})^{-1}(x_j) \cap (s_{k-})^{-1}(x_k) \neq \emptyset$ then there exists $y : G \rightarrow S$ such that $s_j y = x_j$ and $s_k y = x_k$, and thus $Fg x_j = Fg s_j y = s_k y = x_k$. Since representable functors preserve limits and by the form that limits have in the category of sets, we conclude that $(x_i)_{i \in ob(\mathcal{I})} \in \mathcal{C}(G, L)$. By hypothesis $s : S \rightarrow L$ is epimorphic and G is projective so that $s_- : \mathcal{C}(G, S) \rightarrow \mathcal{C}(G, L)$ is also an epimorphism. Therefore, there exists $x : G \rightarrow S$ such that $s x = (x_i)_{i \in \mathcal{I}}$ and by construction $x \in \bigcap_I (s_{i-})^{-1}(x_i)$.

2. \Rightarrow 1. Suppose by contradiction that $s : S \rightarrow L$ is not epimorphic. Then there is some $g : G \rightarrow L$ that can not be factored through s . We consider $l_i g \in \mathcal{C}(G, Fi)$. Since G is projective and the s_i are epimorphic, we have that the s_{i-} are also epimorphic and thus $(s_{i-})^{-1}(p_i g) \neq \emptyset$. Let $J \subset I$ be any finite subset. By cofiltered hypothesis, there exists Fk with $g_{kj} : k \rightarrow j$ for all $j \in J$. If $y \in (s_{k-})^{-1}(l_k g)$ then $s_j y = Fg_{kj} s_k y = Fg_{kj} l_k g = l_j g$ and thus $y \in \bigcap_J (s_{j-})^{-1}(l_j g)$. By 2. there exists $x \in \bigcap_I (s_{i-})^{-1}(x_i)$. Then $s x = (s_-) x = (l_i g)_{i \in ob(\mathcal{I})} = g$, seen as elements of $\mathcal{C}(G, L)$, which contradicts that g is not factored through s . \checkmark

Example 5.2. (1) In the category **HComp** of compact Hausdorff spaces, the singleton is a projective generator. In this case, **HComp** $(\{*\}, S)$ agrees with the elements $x_i \in S$ and since $\{x_i\}$ is closed we have that $(s_{i-})^{-1}(x_i)$ is closed and compact. Since the closed subsets of a compact space satisfy the finite intersection property, the proposition 5.1 is reduced to the fact:

“Let $\{s_i : S \rightarrow Fi\}$ be a family of continuous and surjective functions indexed by a cofiltered set I , where S and the Fi are compact Hausdorff

spaces. Then the induced function $s : S \rightarrow \lim(Fi)$ is continuous and surjective.”

This result is known in the literature [8, p. 89].

- (2) In the category M_R of right R -modules, the ring R is a projective generator. Since that any surjective linear mapping is a quotient $M \rightarrow M/M_i$, with $M_i \subseteq M$, any $f \in M_R(R, M/M_i)$ is characterized by its image in $1 \in R$, which is of the form $f(1) = x_i + M_i$ for some $x_i \in M_i$. Thus the proposition 5.1 can be written as:

“Given a R -module M the following conditions are equivalent:

- (a) for each cofiltered family of submodules, the homomorphism $M \rightarrow \lim(M/M_i)$ is surjective;
- (b) for each family of cosets $\{x_i + M_i\}_{i \in I}$, where $x_i \in M$ and $M_i \subseteq M$, if the intersection of any finite many of these cosets is not empty, then also $\bigcap_{i \in I} (x_i + M_i) \neq \emptyset$.”

This proposition is known in the literature [21, p. 243] and the modules that satisfy these conditions are called linearly compact. They coincide with those that satisfy the AB5* axiom of [11].

- (3) Since in the category of sets the functor $\mathbf{Set}(\{*\}, _)$ does not generally satisfy the finite intersection property, the proposition 5.1 also helps to understand why there exists functors $F : \mathcal{I} \rightarrow \mathbf{Set}$, with

- \mathcal{I} cofiltered,
- $F(i) \neq \emptyset$, for all $i \in \text{ob}(\mathcal{I})$,
- $F(g)$ surjective, for all $g \in \text{mor}(\mathcal{I})$,

such that $\lim F = \emptyset$ (for some examples of this type of functors, see [12], [20]).

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