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The formal derivative operator and multifactorial numbers

El operador derivada formal y números multifactoriales

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Abstract. In this paper some properties, examples and counterexamples about the formal derivative operator defined with respect to context-free grammars are presented. In addition, we show a connection between the context-free grammar $G = \left\{a \to ab^r; b \to b^{r+1}\right\}$ and multifactorial numbers. Some identities involving multifactorial numbers will be obtained by grammatical methods.

Key words and phrases. Context-free grammars, formal derivative operator, multifactorial numbers.

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Resumen. En este artículo se presentan algunas propiedades, ejemplos y contraejemplos del operador derivada formal con respecto a gramáticas independientes del contexto. Adicionalmente, se obtiene una relación entre la gramática $G = \left\{a \to ab^r; b \to b^{r+1}\right\}$ y números multifactoriales. Se obtienen algunas identidades sobre números multifactoriales mediante métodos gramaticales.

Palabras y frases clave. Gramáticas independiente del contexto, operador derivada formal, números multifactoriales.

1. Introduction

Let Σ be an alphabet, whose letters are regarded as independent commutative indeterminates. Following [4], a formal function over Σ is defined recursively as follows:

(1) Every letter in Σ is a formal function.

(2) If u, v are formal functions, then $u + v$ and uv are formal functions.

(3) If $f(x)$ is an analytic function in x, and u is a formal function, then $f(u)$ is a formal function.

(4) Every formal function is constructed as above in a finite number of steps.

A context-free grammar G over Σ is defined as a set of substitution rules (called productions) replacing a letter in Σ by a formal function over Σ. For each $a \in \Sigma$, a grammar G contains at most one production of the form $a \to w$. There is here no distinction between terminals and non-terminals, as it is usual in the theory of formal languages.

Definition 1.1. Given a context-free grammar G over Σ, the formal derivative operator D, with respect to G, is defined in the following way:

(1) For u, v formal functions,

$$D(u + v) = D(u) + D(v) \text{ and } D(uv) = D(u)v + uD(v).$$

(2) If $f(x)$ is an analytic function in x and u is a formal function,

$$D(f(u)) = \frac{\partial f(u)}{\partial u} D(u).$$

(3) For $a \in \Sigma$, if $a \to w$ is a production in G, with w a formal function, then $D(a) = w$; in other cases a is called a constant and $D(a) = 0$.

We next define the iteration of the formal derivative operator.

Definition 1.2. For a formal function u, we define $D^{n+1}(u) = D(D^n(u))$ for $n \geq 0$, with $D^0(u) = u$.

For instance, given the context-free grammar $G = \{a \to a + b; b \to b\}$, then $D^0(a) = a$, $D(a) = a + b$, $D(b) = b$, $D(ab) = D(a)b + aD(b) = [a + b]b + a[b] = b^2 + 2ab$, and $D^2(a) = D(D(a))$ so $D^2(a) = D(a + b) = D(a) + D(b) = a + 2b$.

The formal derivative operator, defined with respect to context-free grammars, has been used to study increasing trees [5], triangular arrays [10], permutations [15] and for generating some combinatorial numbers such as Whitney numbers [2], Ramanujan's numbers [7], Stirling numbers [14], among others. In the same way, some families of polynomials such as Bessel polynomials [12], Eulerian polynomials [13], and other polynomials [6], have been studied by grammatical methods.

In section 2 we prove some properties about the formal derivative operator defined with respect to context-free grammars; in section 3 we obtain multifactorial numbers and some identities about them, by means of the context-free grammar $G = \left\{a \to ab^r; b \to b^{r+1}\right\}$. In this paper emphasis is on grammatical methods; consequently, most proofs are carried out by induction rather than by combinatorial arguments.

2. Some properties of the formal derivative operator defined with respect to context-free grammars

The formal derivative operator of Definition 1.1 preserves many of the properties of the differential operator in elementary calculus. In the following propositions we state and prove some of them.

Proposition 2.1. *If v is a formal function, $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$, then $D(\alpha v) = \alpha D(v)$ and $D(v^n) = nv^{n-1}D(v)$.*

Proof. Let $f(x) = \alpha x$. Since $f(x)$ is an analytic function in x and v is a formal function, by Definition 1.1 we get $D(f(v)) = \dfrac{\partial f(v)}{\partial v} D(v) = \alpha D(v)$.

On the other hand, since $g(x) = x^n$ is an analytic function in x and v is a formal function, by Definition 1.1, we have $D(g(v)) = \dfrac{\partial g(v)}{\partial v} D(v) = nv^{n-1}D(v)$. ☑

Proposition 2.2 (Quotient's rule)**.** *If u, v are formal functions, then $D(uv^{-1}) = [D(u)v - uD(v)]v^{-2}$.*

Proof. By Definition 1.1, $D(uv^{-1}) = D(u)v^{-1} + uD(v^{-1})$. By Proposition 2.1, $D(v^{-1}) = -v^{-2}D(v)$, so

$$D(uv^{-1}) = D(u)v^{-1} - uv^{-2}D(v) = [D(u)v - uD(v)]v^{-2}.$$

☑

The following proposition shows how the formal derivative operator over a product of n formal functions can be calculated.

Proposition 2.3 (Generalized product rule)**.** *If $u_1, u_2, \ldots, u_n$ are formal functions, then*

$$D(u_1u_2 \ldots u_n) = D(u_1)u_2 \ldots u_n + D(u_2)u_1u_3 \ldots u_n + \cdots + D(u_n)u_1u_2 \ldots u_{n-1}.$$

Proof. We argue by induction on n. If $n = 1$, $D(u_1) = D(u_1)$. If $n = 2$, $D(u_1u_2) = D(u_1)u_2 + u_1D(u_2)$, by Definition 1.1. Assuming that $D(u_1u_2 \ldots u_n) = D(u_1)u_2 \cdots u_n + \cdots + D(u_n)u_1 \cdots u_{n-1}$, and considering u_{n+1} a formal function, $D(u_1 \cdots u_{n+1})$ is calculated as follows:

$$\begin{aligned} & D(u_1 \cdots u_n)u_{n+1} + u_1 \cdots u_n D(u_{n+1}) \\ &= [(D(u_1)u_2 \cdots u_n) + \cdots + (D(u_n)u_1 \cdots u_{n-1})]u_{n+1} + [u_1 \cdots u_n D(u_{n+1})] \\ &= [D(u_1)u_2 \cdots u_{n+1}] + \cdots + [D(u_n)u_1 \cdots u_{n-1}u_{n+1}] + [D(u_{n+1})u_1 \cdots u_n]. \end{aligned}$$

☑

Example 2.4. If $G = \{a \to ac; b \to bc; c \to c^2\}$, then $D^n(abc) = \frac{(n+2)!}{2} abc^{n+1}$ for $n \geq 0$.

Since $D^0(abc) = abc$, the formula is true for $n = 0$. Assuming that the formula is true for n, $D^{n+1}(abc)$ is calculated as follows:

$$\begin{aligned}
D^{n+1}(abc) &= D(D^n(abc)) \\
&= D\left(\frac{(n+2)!}{2} abc^{n+1}\right) \\
&= \frac{(n+2)!}{2} \left(D(a)bc^{n+1} + aD(b)c^{n+1} + abD(c^{n+1})\right) \\
&= \frac{(n+2)!}{2} \left(abc^{n+2} + abc^{n+2} + (n+1)abc^n D(c)\right) \\
&= \frac{(n+2)!}{2} (n+3)abc^{n+2} \\
&= \frac{(n+3)!}{2} abc^{n+2}.
\end{aligned}$$

Thus $D^n(abc) = \frac{(n+2)!}{2} abc^{n+1}$.

For the same grammar G it can be similarly proved that $D^n(a) = n!ac^n$, $D^n(b) = n!c^n b$, $D^n(c) = n!c^n$, $D^n(ab) = (n+1)!abc^n$, $D^n(ac) = (n+1)!ac^{n+1}$ and $D^n(bc) = (n+1)!bc^{n+1}$.

Leibniz's formula is also valid for formal functions, which is a result known since the first paper about this topic [4]. It is the main tool used in establishing combinatorial properties of the objects generated through grammars [5]; its proof is not usually given and we present it here for completeness.

Proposition 2.5 (Leibniz's formula)**.** *If u, v are formal functions, then for all $n \geq 0$,*

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

Proof. We argue by induction on n. If u, v are formal functions $D^0(uv) = uv$, then the result is true for $n = 0$. By Definition 1.1 we get $D(uv) = D(u)v + vD(u)$ hence the result is true for $n = 1$. Assuming that $D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v)$, $D^{n+1}(uv)$ is calculated as follows:

$$\begin{aligned} D^{n+1}(uv) &= D\left(\sum_{k=0}^{n}\binom{n}{k}D^k(u)D^{n-k}(v)\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}D\left(D^k(u)D^{n-k}(v)\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}D^{k+1}(u)D^{n-k}(v) + D^k(u)D^{n-k+1}(v). \end{aligned}$$

Expanding the sum, $D^{n+1}(uv)$ is given by

$$\binom{n}{0}uD^{n+1}(v) + \sum_{k=0}^{n-1}\left(\binom{n}{k}D^{k+1}(u)D^{n-k}(v) + \binom{n}{k+1}D^{k+1}(u)D^{n-k}(v)\right) + \binom{n}{n}D^{n+1}(u)v.$$

Since $\binom{n}{0} = \binom{n+1}{0}$, $\binom{n}{n} = \binom{n+1}{n+1}$ and $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$, cf. [3], $D^{n+1}(uv)$ can be written as

$$\begin{aligned} &\binom{n+1}{0}uD^{n+1}(v) + \sum_{k=0}^{n-1}\binom{n+1}{k+1}D^{k+1}(u)D^{n-k}(v) + \binom{n+1}{n+1}D^{n+1}(u)v \\ &= \binom{n+1}{0}uD^{n+1}(v) + \sum_{k=1}^{n}\binom{n+1}{k}D^{k}(u)D^{n+1-k}(v) + \binom{n+1}{n+1}D^{n+1}(u)v \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}D^{k}(u)D^{n+1-k}(v). \end{aligned}$$

Thus $D^n(uv) = \sum_{k=0}^{n}\binom{n}{k}D^k(u)D^{n-k}(v)$. ☑

Given a context-free grammar, if $D(a) \neq D(b)$ then $D^n(a) \neq D^n(b)$ does not necessarily hold for $n \geq 2$. The grammar $G = \{a \to ab\ ;\ b \to ac\ ;\ c \to b^2 + ac - bc\}$ provides a counterexample:

$$\begin{aligned} D^2(a) &= D(D(a)) & D^2(b) &= D(D(b)) \\ &= D(ab) & &= D(ac) \\ &= D(a)b + aD(b) & &= D(a)c + aD(c) \\ &= (ab)b + a(ac) & &= (ab)c + a(b^2 + ac - bc) \\ &= ab^2 + a^2c. & &= ab^2 + a^2c. \end{aligned}$$

In the example above it is clear that $D^n(a) = D^n(b)$ for $n \geq 2$. Actually, in general this is always the case: if $D^k(a) = D^k(b)$, for some k, then $D^n(a) = D^n(b)$ for all $n \geq k$. That is so because n can be written as $n = m + k$, and we have $D^n(a) = D^m(D^k(a)) = D^m(D^k(b)) = D^{m+k}(b) = D^n(b)$.

On the other hand, from $D(a^2) = D(b^2)$ does not necessarily follow that $D(a) = D(b)$. For instance, given the grammar $G = \{a \to ab; b \to a^2\}$, $D(a^2) = 2aD(a) = 2a^2b$ and $D(b^2) = 2bD(b) = 2a^2b$; however $D(a) \neq D(b)$. Similarly, if $D(a^2) = D(ab)$, then $D(a) = D(b)$ does not necessarily hold; for instance, for the grammar $G = \{a \to ab; b \to 2ab - b^2\}$ we have $D(ab) = 2a^2b$ and $D(a^2) = 2a^2b$; however $D(a) \neq D(b)$. These examples provide useful insight and allow us to state the following assertions.

Proposition 2.6. *There is no context-free grammar such that* $D(a^2) = D(b^2) = D(ab)$, *with* $a \neq b$ *and* $D(a), D(b) \neq 0$.

Proof. If $D(a^2) = D(ab)$ we get $2aD(a) = D(a)b + aD(b)$, thus obtaining

$$(2a - b)D(a) - aD(b) = 0. \tag{1}$$

Similarly, if $D(b^2) = D(ab)$ we have $2bD(b) = D(a)b + aD(b)$, so

$$-bD(a) + (2b - a)D(b) = 0. \tag{2}$$

From (1) and (2) we obtain the following system of linear equations

$$\begin{bmatrix} 2a - b & -a \\ -b & -a + 2b \end{bmatrix} \begin{bmatrix} D(a) \\ D(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

For the matrix $A = \left(\begin{smallmatrix} 2a-b & -a \\ -b & -a+2b \end{smallmatrix}\right)$, $\det(A) = -2a^2 + 4ab - 2b^2 = -2(a-b)^2$; if $a \neq b$ then $\det(A) \neq 0$. But the system (3) is homogeneous, that is a contradiction. Therefore has a single unique solution $D(a) = D(b) = 0$. ☑

Proposition 2.7. *There is no context-free grammar such that* $D(a) = D(b)$, $D(ac) = D(bc)$ *and* $D(ab) = D(abc)$ *with* $a \neq b$ *and* $D(a), D(b), D(c) \neq 0$.

Proof. Since $D(a) = D(b)$, we get

$$D(a) - D(b) = 0. \tag{4}$$

Since $D(ac) = D(bc)$, we have $D(a)c + aD(c) = D(b)c + bD(c)$, thus obtaining

$$cD(a) - cD(b) + (a - b)D(c) = 0. \tag{5}$$

Similarly, from $D(ab) = D(abc)$ we get $D(a)b + aD(b) = D(a)bc + aD(b)c + abD(c)$, so

$$(b - bc)D(a) + (a - ac)D(b) - abD(c) = 0. \tag{6}$$

From (4), (5) and (6) we obtain the following system of linear equations

$$\begin{bmatrix} 1 & -1 & 0 \\ c & -c & a-b \\ b-bc & a-ac & -ab \end{bmatrix} \begin{bmatrix} D(a) \\ D(b) \\ D(c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

For the matrix $A = \left(\begin{smallmatrix} 1 & -1 & 0 \\ c & -c & a-b \\ b-bc & a-ac & -ab \end{smallmatrix}\right)$, we have

$$\det(A) = a^2c - b^2c - a^2 + b^2 = (a+b)(a-b)(c-1).$$

If $a \neq b$, $\det(A) \neq 0$. But the system (7) is homogeneous, that is a contradiction. Therefore, the system has a single unique solution $D(a) = D(b) = D(c) = 0$. ☑

There are infinitely many context-free grammars such that $D(a)b = aD(b)$, for instance, $G = \{a \to ab^r \ ; \ b \to b^{r+1}\}$ for each r; in section 3 we will use this context-free grammar for generating multifactorial numbers. The following result shows the existence of infinitely many context-free grammars with three variables and some restrictions of the type $D(a)b = aD(b)$.

Proposition 2.8. *There are infinitely many context-free grammars such that $D(a)b = aD(b)$, $D(a)c = aD(c)$ and $acD(b) + abD(c) = 2bcD(a)$, with $D(a)$, $D(b)$, $D(c)$ not simultaneously* 0.

Proof. Since $acD(b) + abD(c) = 2bcD(a)$, we have

$$-2bcD(a) + acD(b) + abD(c) = 0. \tag{8}$$

From $D(a)b = aD(b)$, $D(a)c = aD(c)$ and (8) we obtain the following system of linear equations.

$$\begin{bmatrix} b & -a & 0 \\ c & 0 & -a \\ -2bc & ac & ab-a \end{bmatrix} \begin{bmatrix} D(a) \\ D(b) \\ D(c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the matrix of this system has determinant 0 and the system is homogeneous, we conclude that it has infinitely many solutions. ☑

It is easy to check that the grammar $G = \{a \to ac; b \to bc; c \to c^2\}$ in Example 2.4 satisfies Proposition 2.8.

3. Multifactorial numbers via context-free grammars

The multifactorial numbers $n!_r$ are given by the recurrence relation

$$n!_r = n(n-r)!_r \text{ with } (1-r)!_r = \cdots = (-1)!_r = 0!_r = 1.$$

When $r = 1$ we get factorial numbers i.e., $n!_1 = n!$; when $r = 2$ we get double factorial numbers i.e., $n!_2 = n!!$. As an interesting fact, factorial numbers can be expressed in terms of double factorial numbers, in the form $n! = n!!(n-1)!!$, and double factorial numbers can also be expressed in terms of factorial numbers: $(2n)!! = 2^n n!$, cf. [16]. The following result shows a connection between the context-free grammar $G = \{a \to ab^r; b \to b^{r+1}\}$ and multifactorial numbers.

Proposition 3.1. *If* $G = \left\{a \to ab^r; b \to b^{r+1}\right\}$, *then for integers* $n \geq 0$ *and* $m, r \geq 1$ *it holds*

(1) $D^n(a^m) = \dfrac{(m+(n-1)r)!_r}{(m-r)!_r} a^m b^{nr}$.

(2) $D^n(b^m) = \dfrac{(m+(n-1)r)!_r}{(m-r)!_r} b^{m+nr}$.

(3) $D^n(a^m b^m) = \dfrac{(2m+(n-1)r)!_r}{(2m-r)!_r} a^m b^{m+nr}$.

Proof. Here we prove (2); the other results can be proved similarly.

Since $D^0(b^m) = b^m$, the proposition is true for $n = 0$. Assuming that $D^n(b^m) = \dfrac{[m+(n-1)r]!_r}{[m-r]!_r} b^{m+nr}$, $D^{n+1}(b^m)$ is calculated as follows

$$\begin{aligned}
D^{n+1}(b^m) &= D(D^n(b^m)) \\
&= D\left(\frac{(m+(n-1)r)!_r}{(m-r)!_r} b^{m+nr}\right) \\
&= \frac{(m+(n-1)r)!_r}{(m-r)!_r} D(b^{m+nr}) \\
&= \frac{(m+(n-1)r)!_r}{(m-r)!_r} [m+nr] b^{m+nr-1} D(b) \\
&= \frac{(m+(n-1)r)!_r}{(m-r)!_r} [m+nr] b^{m+nr-1} [b^{r+1}] \\
&= \frac{(m+nr)!_r}{(m-r)!_r} b^{m+(n+1)r}.
\end{aligned}$$

Hence $D^n(b^m) = \dfrac{(m+(n-1)r)!_r}{(m-r)!_r} b^{m+nr}$. ☑

For the following identity for $(2n+1)!!$ we give a proof by means of context-free grammars.

Corollary 3.2. $(2n+1)!! = \sum_{k=0}^{n} \binom{n}{k} (2k-1)!!(2(n-k))!!$, *for all* $n > 0$.

Proof. If $r = 2$ in Proposition 3.1 we obtain the context-free grammar $G = \{a \to ab^2; b \to b^3\}$ for which $D^n(b^m) = \frac{(m+2(n-1))!!}{(m-2)!!} b^{m+2n}$; then by Leibniz's formula we get

$$D^n(b^3) = \sum_{k=0}^{n} \binom{n}{k} D^k(b) D^{n-k}(b^2). \tag{9}$$

By Proposition 3.1, $D^k(b) = (2k-1)!!b^{2k+1}$, $D^{n-k}(b^2) = (2(n-k))!!b^{2(n-k)+2}$ and $D^n(b^3) = (2n+1)!!b^{2n+3}$; replacing in (9) we obtain

$$\begin{aligned}(2n+1)!!b^{2n+3} &= \sum_{k=0}^{n} \binom{n}{k} \left((2k-1)!!b^{2k+1}\right) \left((2(n-k))!!b^{2(n-k)+2}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} (2k-1)!!(2(n-k))!!b^{2n+3}.\end{aligned}$$

By equating the coefficients, $(2n+1)!! = \sum_{k=0}^{n} \binom{n}{k} (2k-1)!!(2(n-k))!!$. ☑

The next corollary is proved in [1] by combinatorial arguments; here a proof can be obtained by rewriting some terms in Corollary 3.2.

Corollary 3.3 ([1], result 4.5)**.** $(2n-1)!! = \sum_{k=1}^{n} \frac{(2n-2)!!(2k-3)!!}{(2k-2)!!}$ *for all* $n \geq 1$.

The following proposition is an identity about multifactorial numbers.

Proposition 3.4. *For integers* $n \geq 0$ *and* $m, r \geq 1$ *we have*

$$\frac{(2m+(n-1)r)!_r}{(2m-r)!_r} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(m+(k-1)r)!_r}{(m-r)!_r} \frac{(m+(n-k-1)r)!_r}{(m-r)!_r} \right).$$

Proof. Let G be the grammar $\{a \to ab^r; b \to b^{r+1}\}$. Applying Leibniz's formula in $D^n(a^m b^m)$ we get

$$D^n(a^m b^m) = \sum_{k=0}^{n} \binom{n}{k} D^k(a^m) D^{n-k}(b^m).$$

By Proposition 3.1 we have $D^k(a^m) = \frac{(m+(k-1)r)!_r}{(m-r)!_r} a^m b^{kr}$ and $D^{n-k}(b^m) = \frac{(m+(n-k-1)r)!_r}{(m-r)!_r} b^{m+(n-k)r}$, then $D^n(a^m b^m)$ is given by

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} \left(\frac{(m+(k-1)r)!_r}{(m-r)!_r} a^m b^{kr} \right) \left(\frac{(m+(n-k-1)r)!_r}{(m-r)!_r} b^{m+(n-k)r} \right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(m+(k-1)r)!_r}{(m-r)!_r} \frac{(m+(n-k-1)r)!_r}{(m-r)!_r} a^m b^{m+nr}.\end{aligned}$$

On the other hand, by Proposition 3.1 we have

$$D^n(a^m b^m) = \frac{(2m+(n-1)r)!_r}{(2m-r)!_r} a^m b^{m+nr},$$

therefore by equating coefficients of b^{m+nr} we get

$$\frac{(2m+(n-1)r)!_r}{(2m-r)!_r} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(m+(k-1)r)!_r}{(m-r)!_r} \frac{(m+(n-k-1)r)!_r}{(m-r)!_r} \right).$$

☑

By taking $r = m$ in Proposition 3.4 we have the following identity for multifactorial numbers.

Corollary 3.5. $((n+1)r)!_r = r \sum_{k=0}^{n} \binom{n}{k} (kr)!_r ((n-k)r)!_r$.

Additionally, by taking $r = 1$ in Proposition 3.4 we get

$$\frac{(2m+n-1)!}{(2m-1)!} = \sum_{k=0}^{n} \binom{n}{k} \frac{(m+k-1)!}{(m-1)!} \frac{(m+n-k-1)!}{(m-1)!}. \tag{10}$$

Identity (10) can be expressed in terms of rising factorial numbers,

$$m^{\overline{n}} = m(m+1)\cdots(m+n-1),$$

also known as Pochhammer upper factorial $(m)_n$, cf. [11]. Since $m^{\overline{n}} = \binom{m+n-1}{n}$, (10) can also be expressed in terms of binomial coefficients as stated in the following corollary.

Corollary 3.6. *For $n \geq 0$, $m \geq 1$ we have:*

(1) $(2m)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} m^{\overline{k}} m^{\overline{n-k}}$.

$$(2)\quad \binom{2m+n-1}{n} = \sum_{k=0}^{n} \binom{m+k-1}{k}\binom{m+n-k-1}{n-k}.$$

By taking $r = 2$ in Proposition 3.4 we obtain a property relating binomial coefficients, double factorial and rising factorial numbers.

Corollary 3.7. $2^n m^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(m+2(k-1))!!}{(m-2)!!} \frac{(m+2(n-k-1))!!}{(m-2)!!} \right).$

Proof. If $r = 2$ in Proposition 3.4, we obtain

$$\frac{(2m+2(n-1))!!}{(2m-2)!!} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(m+2(k-1))!!}{(m-2)!!} \frac{(m+2(n-k-1))!!}{(m-2)!!} \right).$$

Since $2^t t! = (2t)!!$, we have

$$\frac{(2m+2(n-1))!!}{(2m-2)!!} = \frac{2^{m+n-1}(m+n-1)!}{2^{m-1}(m-1)!} = 2^n m^{\overline{n}},$$

thus

$$2^n m^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(m+2(k-1))!!}{(m-2)!!} \frac{(m+2(n-k-1))!!}{(m-2)!!} \right).$$

☑

By taking $m = 1$ in Corollary 3.7 we get the next result, presented as a problem in [8], which is proved in [9] by combinatorial methods.

Corollary 3.8 ([9], Theorem 3)**.** $(2n)!! = \sum_{k=0}^{n} \binom{n}{k} (2(n-k)-1)!!(2k-1)!!,$ *for all* $n \geq 0$.

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